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By Asma BENCHEKCHEK

Dissertation committee :

President: Dr. BENDOUKHA Berrabah Full professor at UMAB
Supervisor: Dr. BELAÏDI Benharrat Full professor at UMAB
Examiner: Dr. FETTOUCH Houari Associate professor at UMAB

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MASTER

Résumé :

Dans ce travail, nous étudions la croissance des solutions de certaines équations différentielles linéaires complexes dont les coefficients sont des fonctions analytiques dans $\mathbb{C} \cup \{\infty\} \setminus \{z_0\}$, où $z_0 \in \mathbb{C}$ est un point singulier isolé. Des estimations de la borne inférieure de la croissance des solutions de ces équations différentielles sont obtenues en utilisant le concept du φ –ordre inférieur.

Abstract :

In this work, we investigate the growth of solutions of some complex linear differential equations having coefficients which are analytic functions in $\mathbb{C} \cup \{\infty\} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$ is an isolated singular point. Some estimations of lower bound of the growth of solutions of such differential equations are obtained by using the concept of lower φ –order.

الملخص

في هذا العمل، ندرس نمو حلول بعض المعادلات التفاضلية الخطية المعقدة ذات معاملات دوال تحليلية في $\mathbb{C} \cup \{\infty\} \setminus \{z_0\}$ حيث $z_0 \in \mathbb{C}$ نقطة مفردة معزولة. يتم الحصول على بعض تقديرات الحد الأدنى لنمو حلول هذه المعادلات التفاضلية باستخدام مفهوم φ –رتبة أقل.

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Finally, I dedicate this work to myself, who has been a profound source of inspiration, and to myself for believing in the dream and never surrendering, no matter how challenging the journey became. As the poet once said: "He who fears climbing the mountains... shall forever live among the pits."

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INTRODUCTION

" If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is "

John Louis Von Neumann

Complex analysis is one of the fundamental branches of mathematics, focusing on the study of complex numbers and analytic functions that depend on complex variables. This field is based on the imaginary unit (i), which is defined as the square root of (-1) , a concept that revolutionized our understanding of numbers and mathematical functions. **Complex analysis** has wide-ranging applications in various fields, such as algebraic geometry, analytic number theory, and analytic combinatorics, making it a cornerstone of many modern mathematical advancements.

The Nevanlinna Theory was discovered in 1925 when **Rolf Nevanlinna** studied the asymptotic behavior of **meromorphic functions**, building on earlier ideas in complex analysis, particularly **Charles Émile Picard** Exceptional Value Theory. His research focused on the distribution of values taken by these functions in the complex plane, leading him to develop new mathematical tools such as the counting function $N(r, a)$, which measures the frequency of a given value up to a radius r , and the growth function $T(r, f)$, which describes how the function increases toward infinity. Based on these concepts, he **formulated Nevanlinna's two fundamental equations**, which became the cornerstone of the value distribution theory of complex functions. This theory brought a major transformation to complex analysis, providing a systematic framework for studying meromorphic functions in depth and influencing other fields such as dynamical systems and approximation theory, or more one can checks ([17] and [25]).

Today, many concepts in complex analysis can be expressed using the language of applied mathematics, where equations and mathematical models, especially in the form of complex differential systems, are used to study the behavior of functions and analyze the distribution of their values.

In this work, we focus on studying the following complex differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z)f = 0, \quad k \geq 2, \quad (0.0.1)$$

where the functions A_j ($0 \leq j \leq k-1$) are analytic in a complex domain, and f is a meromorphic function in the punctured plane $\bar{\mathbb{C}} - \{z_0\}$, where z_0 represents a singular point. Extensive research has been conducted on this equation, particularly on analyzing the order of growth of its solutions near singular points. One of the major breakthroughs in this area was Nevanlinna's discovery, which introduced fundamental properties and theorems that significantly advanced the field.

To further investigate the behavior of solutions to this equation, we introduce new concepts related to the classical order of growth, as well as the φ -order and φ -type of holomorphic functions. Here, f belongs to a special function class that has been previously studied. A well-known result in this context is the following generalized equation:

$$g^{(k)} + A_{k-1}(z) g^{(k-1)} + \cdots + A_0(z)g = 0, \quad k \geq 2 \quad (0.0.2)$$

where the coefficients A_j (for $0 \leq j \leq k-1$) are analytic, while the leading coefficient A_0 has an infinite φ -order. Studying such differential equations is crucial for understanding value distribution, asymptotic behavior of solutions, and their applications in modern complex analysis.

Nevanlinna Theory Near an Essential Singular Point

1.1 Motivation

The growth of meromorphic functions is a fundamental topic in complex analysis, providing insight into how these functions behave over different domains. Some meromorphic functions exhibit rapid and unbounded growth, while others remain more controlled. Understanding these variations is crucial for applications in differential equations and value distribution theory. Early contributions to the study of function growth were made by **Picard** and **Montel** (see ([22]) and ([21])), who explored fundamental aspects of meromorphic function theory. Later, **A. Goldberg** and **I. Ostrovskii** ([10]) introduced refined techniques to classify growth based on the distribution of zeros and poles. A key question in this field is: How can we compare two holomorphic functions when one dominates the other in terms of growth? To address this, researchers have examined models of the form

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z)f = 0,$$

where the coefficients A_j (for $(0 \leq j \leq k-1)$) without imposing any specific conditions and determine the order and type of growth, respectively. **Chung-Chun Yang** ([24]) introduced alternative methods using generalized growth indicators, expanding upon classical order and type classifications. Some studies have also employed iterated logarithmic transformations or special function classes that exhibit similar asymptotic properties to traditional logarithmic scaling. This

study aims to analyze the growth of solutions to a specific differential equation (0.0.1) by comparing its coefficients for A_j (for $(0 \leq j \leq k-1)$). By investigating how these coefficients influence function behavior, we can gain deeper insights into the structural properties of differential equations. Our analysis builds upon results by **Bank** and **Laine**([1]), who explored connections between meromorphic growth and differential equations through the framework of generalized order and type.

1.2 Concepts near a singular point

Let f be a meromorphic function defined in the punctured plane $\bar{\mathbb{C}} - \{z_0\}$ for some complex singular points.

The Nevanlinna's fundamentals are the most important tool for our study. For that reason, we dedicate the entire chapter to fully developing the theory for a function with singular point z_0 .

In order to develop our study, we firstly recall some related notations. Let f be a meromorphic function in $\bar{\mathbb{C}} - \{z_0\}$, where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the whole extended complex plane, $z_0 \in \mathbb{C}$ is some essential singularity.

Define the counting function of f near z_0 by the following formula

$$N_{z_0}(r, f) = - \int_{\infty}^r \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r,$$

where $n(t, f)$ denote the number of poles of f in the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ counting its multiplicities, we also define the proximity function near z_0 by

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi.$$

Summing up together, the characteristic function of f near z_0 will be

$$T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f).$$

Definition 1.2.1 *Let f be a nonconstant meromorphic function in $\bar{\mathbb{C}} - \{z_0\}$. The function f is called a transcendental or admissible meromorphic function in $\bar{\mathbb{C}} - \{z_0\}$ provided that*

$$\limsup_{r \rightarrow 0} \frac{T_{z_0}(r, f)}{\log \frac{1}{r}} = +\infty$$

and f is a rational function in $\overline{\mathbb{C}} - \{z_0\}$ provided that

$$\liminf_{r \rightarrow 0} \frac{T_{z_0}(r, f)}{\log \frac{1}{r}} < +\infty.$$

Recently, I. Chyzhykov and N. Semochko, ([7]) have given general definition of growth for an entire function in the complex plane by introducing a new class of functions. So, as in ([7]), let ϕ be the class of positive and unbounded increasing functions φ on $[1, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.,

$$\forall c > 0 : \lim_{t \rightarrow +\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1.$$

Here some useful properties of a function $\varphi \in \phi$.

Proposition 1.2.1 ([7]) *If $\varphi \in \phi$, then the following hold:*

- (i) $\forall \delta > 0 : \lim_{x \rightarrow +\infty} \frac{\log \varphi^{-1}((1 + \delta)x)}{\log \varphi^{-1}(x)} = +\infty,$
- (ii) $\forall m > 0, k \geq 0 : \lim_{x \rightarrow +\infty} \frac{\varphi^{-1}(\log x^m)}{x^k} = +\infty,$
- (iii) $\forall c > 0, \varphi(ct) \leq \varphi(t^c) \leq (1 + o(1))\varphi(t), \quad t \rightarrow +\infty.$

Now, we will define the φ -orders of growth for a meromorphic function of f in whole complex plane.

Definition 1.2.2 ([7]) *Let φ be an increasing unbounded function on $[1, +\infty)$. The φ -orders of a meromorphic function f are defined by*

$$\rho_{\varphi}^0(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(e^{T(r, f)})}{\log r}, \quad \rho_{\varphi}^1(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(T(r, f))}{\log r}.$$

If f is an entire function, then the φ -orders are defined by

$$\tilde{\rho}_{\varphi}^0(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(M(r, f))}{\log r}, \quad \tilde{\rho}_{\varphi}^1(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(\log M(r, f))}{\log r}.$$

Proposition 1.2.2 ([3]) *Let $\varphi \in \Phi$ and f be an entire function. Then*

$$\rho_{\varphi}^i(f) = \tilde{\rho}_{\varphi}^i(f), \quad i = 0, 1.$$

Here, we will define the φ -orders of growth for a meromorphic function of f near a singular point.

Definition 1.2.3 Let φ be an increasing and unbounded function on $[1, +\infty)$. Then, the φ -orders of the growth of a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ are given by

$$\rho_{\varphi}^0(f, z_0) = \limsup_{r \rightarrow 0} \frac{\varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}}, \quad \rho_{\varphi}^1(f, z_0) = \limsup_{r \rightarrow 0} \frac{\varphi(T_{z_0}(r, f))}{\log \frac{1}{r}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then the φ -orders are defined by

$$\tilde{\rho}_{\varphi}^0(f, z_0) = \limsup_{r \rightarrow 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}}, \quad \tilde{\rho}_{\varphi}^1(f, z_0) = \limsup_{r \rightarrow 0} \frac{\varphi(\log M_{z_0}(r, f))}{\log \frac{1}{r}},$$

here $M_{z_0}(r, f) = \max\{|f(z)| : |z - z_0| = r\}$.

Remark 1.2.1 ([16]) A motivational observation of the above definition is that $\varphi(r) = \log \log r \in \Phi$, it's also obvious that $\tilde{\rho}_{\varphi}^i(f, z_0) = \rho_{\varphi}^i(f, z_0)$ ($i = 1, 2$) for an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. In indeed, by Lemma 2.2 in ([9]), we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Proposition 1.2.2, we get the conclusion.

Definition 1.2.4 Let φ be an increasing and unbounded function on $[1, +\infty)$. Then, the φ -types of the growth of an analytic and a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\tilde{\rho}_{\varphi}^0(f, z_0) \in (0, +\infty)$ and $\tilde{\rho}_{\varphi}^1(f, z_0) \in (0, +\infty)$ are defined by

$$\begin{aligned} \tilde{\tau}_{\varphi}^0(f, z_0) &= \limsup_{r \rightarrow 0} \frac{\exp\{\varphi(M_{z_0}(r, f))\}}{\frac{1}{r \tilde{\rho}_{\varphi}^0(f, z_0)}}, \quad \tilde{\tau}_{\varphi}^0(f, z_0) = \limsup_{r \rightarrow 0} \frac{\exp\{\varphi(\exp(T_{z_0}(r, f)))\}}{\frac{1}{r \tilde{\rho}_{\varphi}^0(f, z_0)}}, \\ \tilde{\tau}_{\varphi}^1(f, z_0) &= \limsup_{r \rightarrow 0} \frac{\exp\{\varphi(\log M_{z_0}(r, f))\}}{\frac{1}{r \tilde{\rho}_{\varphi}^1(f, z_0)}}, \quad \tilde{\tau}_{\varphi}^1(f, z_0) = \limsup_{r \rightarrow 0} \frac{\exp\{\varphi(T_{z_0}(r, f))\}}{\frac{1}{r \tilde{\rho}_{\varphi}^1(f, z_0)}}. \end{aligned}$$

Definition 1.2.5 Let φ be an increasing unbounded function on $[1, +\infty)$. The Lower φ -orders of a meromorphic function f are defined by

$$\mu_{\varphi}^0(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(e^{T(r, f)})}{\log r}, \quad \mu_{\varphi}^1(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(T(r, f))}{\log r}.$$

If f is an entire function, then the lower φ -orders are defined by

$$\tilde{\mu}_{\varphi}^0(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(M(r, f))}{\log r}, \quad \tilde{\mu}_{\varphi}^1(f) := \limsup_{r \rightarrow +\infty} \frac{\varphi(\log M(r, f))}{\log r}.$$

Proposition 1.2.3 ([3]) Let $\varphi \in \Phi$ and f be an entire function. Then

$$\mu_{\varphi}^i(f) = \tilde{\mu}_{\varphi}^i(f), \quad i = 0, 1.$$

Definition 1.2.6 Let φ be an increasing unbounded function on $[1, +\infty)$. The lower φ -orders of a meromorphic function f are defined by

$$\mu_{\varphi}^0(f, z_0) := \liminf_{r \rightarrow 0} \frac{\varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}}, \quad \mu_{\varphi}^1(f, z_0) := \liminf_{r \rightarrow 0} \frac{\varphi(T_{z_0}(r, f))}{\log \frac{1}{r}}.$$

If f is an entire function, then the lower φ -orders are defined by

$$\tilde{\mu}_{\varphi}^0(f, z_0) = \liminf_{r \rightarrow 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}}, \quad \tilde{\mu}_{\varphi}^1(f, z_0) = \liminf_{r \rightarrow 0} \frac{\varphi(\log M_{z_0}(r, f))}{\log \frac{1}{r}}.$$

Remark 1.2.2 A motivational observation of the above definition is that $\varphi(r) = \log \log r \in \phi$, it's also obvious that $\tilde{\mu}_{\varphi}^i(f, z_0) = \mu_{\varphi}^i(f, z_0)$ ($i = 1, 2$) for an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. In indeed, by Lemma 2.2 in ([9]), we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Proposition 1.2.2, we get the conclusion.

Definition 1.2.7 Let φ be an increasing and unbounded function on $[1, +\infty)$. Then, the lower φ -types of the growth of an analytic and a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\tilde{\mu}_{\varphi}^0(f, z_0) \in (0, +\infty)$ and $\tilde{\mu}_{\varphi}^1(f, z_0) \in (0, +\infty)$ are defined by

$$\begin{aligned} \tilde{\tau}_{\varphi}^0(f, z_0) &= \liminf_{r \rightarrow 0} \frac{\exp \{ \varphi(M_{z_0}(r, f)) \}}{\frac{1}{r^{\tilde{\mu}_{\varphi}^0(f, z_0)}}}, \quad \tilde{\tau}_{\varphi}^0(f, z_0) = \liminf_{r \rightarrow 0} \frac{\exp \{ \varphi(\exp(T_{z_0}(r, f))) \}}{\frac{1}{r^{\tilde{\mu}_{\varphi}^0(f, z_0)}}}, \\ \tilde{\tau}_{\varphi}^1(f, z_0) &= \liminf_{r \rightarrow 0} \frac{\exp \{ \varphi(\log M_{z_0}(r, f)) \}}{\frac{1}{r^{\tilde{\mu}_{\varphi}^1(f, z_0)}}}, \quad \tilde{\tau}_{\varphi}^1(f, z_0) = \liminf_{r \rightarrow 0} \frac{\exp \{ \varphi(T_{z_0}(r, f)) \}}{\frac{1}{r^{\tilde{\mu}_{\varphi}^1(f, z_0)}}}. \end{aligned}$$

Proposition 1.2.4 ([16]) Let $\varphi \in \Phi$ and f, g be two analytic functions in $\mathbb{C} - \{z_0\}$. Then

(i)

$$\tilde{\rho}_{\varphi}^j(f + g, z_0) \leq \max \{ \tilde{\rho}_{\varphi}^j(f, z_0), \tilde{\rho}_{\varphi}^j(g, z_0) \} \quad \text{for } j = 0, 1,$$

$$\tilde{\rho}_{\varphi}^j(fg, z_0) \leq \max \{ \tilde{\rho}_{\varphi}^j(f, z_0), \tilde{\rho}_{\varphi}^j(g, z_0) \} \quad \text{for } j = 0, 1.$$

(ii) If $\tilde{\rho}_{\varphi}^j(g, z_0) < \tilde{\rho}_{\varphi}^j(f, z_0)$, ($j = 0, 1$), then

$$\tilde{\rho}_{\varphi}(f + g, z_0) = \tilde{\rho}_{\varphi}(fg, z_0) = \tilde{\rho}_{\varphi}(f, z_0) \quad \text{for } j = 0, 1.$$

(iii)

$$\tilde{\rho}_{\varphi}^j(f', z_0) = \tilde{\rho}_{\varphi}^j(f, z_0) \quad \text{for } j = 0, 1.$$

For $k \geq 2$ a positive integer, consider the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \quad (1.2.1)$$

where the coefficients are analytic in some complex domain. Since it's hard to find some general forms for the solutions of (1.2.1), many searchers are interested on the study of the behavior of such solutions and specially the notion of the growth. The strongest tool they used for establishing their results is the Nevanlinna theory which can be found in ([2]), ([10]) ([12]) ([17]), and ([25]).

Recently in ([18]), Liu, Long and Zeng, by using the concept of lower order have studied the relationship between lower-order of coefficients and the growth of solutions of second order complex differential equations near a singular point of the form

$$f'' + A(z)f' + B(z)f = 0, \quad (1.2.2)$$

where $A(z)$ and $B(z)$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ and established some growth properties of the solutions of equation (1.2.2).

Theorem 1.2.1 ([18]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\mu(A, z_0) < \mu(B, z_0) < +\infty$. Then, every nontrivial solution f of (1.2.2), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies*

$$\rho(f, z_0) \geq \mu(B, z_0).$$

Theorem 1.2.2 ([18]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying the following conditions:*

$$(i) \quad \mu(B, z_0) = \mu(A, z_0),$$

$$(ii) \quad \tau(A, z_0) < \tau(B, z_0).$$

Then, every nontrivial solution f of (1.2.2), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\rho_2(f, z_0) \geq \mu(B, z_0).$$

Theorem 1.2.3 ([18]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying*

$$\mu(B, z_0) < \mu(A, z_0) < \infty.$$

Then, every transcendental solution f of (1.2.2) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\rho(f, z_0) \geq \mu(A, z_0).$$

In this work, by making use of the concept of lower φ -order, we obtain some theorems which improve and generalize the results of Liu, Long and Zeng.

Theorem 1.2.4 ([5]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\tilde{\mu}_\varphi^0(A, z_0) \leq \tilde{\mu}_\varphi^0(B, z_0) < +\infty$. Then, every nontrivial solution f of (1.2.2), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies*

$$\tilde{\rho}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(B, z_0).$$

Theorem 1.2.5 ([5]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying the following conditions:*

- (i) $\tilde{\mu}_\varphi^0(B, z_0) = \tilde{\mu}_\varphi^0(A, z_0) < +\infty$,
- (ii) $\underline{\tau}_1 = \tilde{\tau}_\varphi^0(A, z_0) < \tilde{\tau}_\varphi^0(B, z_0) = \underline{\tau} < +\infty$.

Then, every nontrivial solution f of (1.2.2), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\tilde{\rho}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(B, z_0).$$

We get the two results above when the coefficient $B(z)$ is a dominant coefficient. It is a natural problem: what happen the growth of solutions of (1.2.2) provided $A(z)$ is a dominant coefficient in terms of the concept of lower φ -order. We consider the problem and get the following result.

Theorem 1.2.6 ([5]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Suppose that*

$$\tilde{\mu}_\varphi^0(B, z_0) < \tilde{\mu}_\varphi^0(A, z_0) < \infty.$$

Then, every transcendental solution f of (1.2.2) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\tilde{\mu}_\varphi^0(A, z_0) \leq \tilde{\rho}_\varphi^0(f, z_0).$$

Recently in ([16]), by introducing the concept of the φ -order of growth near an essential singularity, Khedim and Belaïdi have investigated the φ -order of growth of solutions of some given linear differential equations of type (1.2.1) with analytic coefficients in $\overline{\mathbb{C}} - \{z_0\}$, where $z_0 \in \mathbb{C}$ represents an essential singularity. They obtained some results which are generalization of the work of Long and Zeng ([19]).

Theorem 1.2.7 ([16]) Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$, all together satisfying $\max \{\tilde{\rho}_\varphi^0(A_j, z_0) : j \neq 0\} < \tilde{\rho}_\varphi^0(A_0, z_0) < \infty$. Then, every nontrivial solution f of (1.2.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)$.

Theorem 1.2.8 ([16]) Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ all together satisfying the following conditions

$$\max \{\tilde{\rho}_\varphi^0(A_j, z_0) : j \neq 0\} \leq \tilde{\rho}_\varphi^0(A_0, z_0) < \infty,$$

$$\max \{\tilde{\tau}_\varphi^0(A_j, z_0) : \tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0) > 0, j \neq 0\} < \tilde{\tau}_\varphi^0(A_0, z_0) < \infty.$$

Then, every nontrivial solution f of (1.2.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

Now, you will find our contribution, which includes a comprehensive generalization of the work Dahmani and Belaïdi ([8]) and an extension of results of Khedim and Belaïdi ([16]).

Theorem 1.2.9 ([6]) Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that

$$\max \{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, \dots, k-1\} < \tilde{\mu}_\varphi^0(A_0, z_0) \leq \tilde{\rho}_\varphi^0(A_0, z_0) < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.2.1) satisfies

$$\tilde{\mu}_\varphi^0(A_0, z_0) = \tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

Theorem 1.2.10 ([6]) Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that

$$\max \{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, \dots, k-1\} \leq \tilde{\mu}_\varphi^0(A_0, z_0) \leq \tilde{\rho}_\varphi^0(A_0, z_0) = \rho < +\infty$$

and

$$\tau_1 = \max \{\tilde{\tau}_\varphi^0(A_j, z_0) : \tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0) > 0; j \neq 0\} < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.2.1) satisfies

$$\tilde{\mu}_\varphi^0(A_0, z_0) = \tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

Theorem 1.2.11 ([6]) Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that

$$\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, \dots, k-1\} \leq \tilde{\mu}_\varphi^0(A_0, z_0) \leq \tilde{\rho}_\varphi^0(A_0, z_0) < +\infty$$

and

$$\limsup_{r \rightarrow 0} \frac{\sum_{j=1}^{k-1} m_{z_0}(r, A_j)}{m_{z_0}(r, A_0)} < 1.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.2.1) satisfies

$$\tilde{\mu}_\varphi^0(A_0, z_0) = \tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

Remark 1.2.3 The condition that f is analytic in $\overline{\mathbb{C}} - \{z_0\}$ is necessary. The following example shows that there exists a solution $f(z)$ of (1.2.1) such that $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$ provided that all coefficients $A_j(z)$ ($j = 0, \dots, k-1$) of (1.2.1) are analytic in $\overline{\mathbb{C}} - \{z_0\}$. For instance, we consider the equation

$$f'' + \exp_2 \left\{ \frac{1}{(z_0 - z)^n} \right\} f' + \frac{n}{z_0 - z} \left(\exp_2 \left\{ \frac{1}{(z_0 - z)^n} \right\} + \frac{n-1}{z_0 - z} \right) f = 0. \quad (1.2.3)$$

The function $f(z) = (z_0 - z)^n$ solves (1.2.3), and $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$. So, in our results, we suppose always that $f(z)$ is analytic in $\overline{\mathbb{C}} - \{z_0\}$.

1.3 Preliminary results

Now, we are going to focus on the main preliminaries needed for establishing the proofs of our results. We firstly clarify some notations. Denote, the logarithmic measure of a set $E \subset (0, 1)$ by

$$m_l(E) = \int_E \frac{dt}{t}.$$

We also denote by $\nu_g(r)$ the central index of an entire function g in $\mathbb{C}$, for more properties, see ([13], p. 33-35). Finally, denote the central index of an analytic function f in $\overline{\mathbb{C}} - \{z_0\}$ by $\nu_{z_0}(r, f)$, reader may check ([11], p. 996).

Lemma 1.3.1 Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\tilde{\mu}_\varphi^0(f, z_0) = \mu \in (0, \infty)$ and $\tilde{\tau}_\varphi^0(f, z_0) = \tau \in (0, \infty)$. Then, for any given $\beta > \tau$, there exists a set $E \subset (0, 1)$ of infinite logarithmic measure such that for $|z - z_0| = r \in E$, we have

$$\varphi(M_{z_0}(r, f)) \leq \log \left(\frac{\beta}{r^\mu} \right).$$

Proof. By Definition [1.2.7](#), there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to zero satisfying

$$r_{n+1} < \frac{n}{n+1} r_n$$

and

$$\hat{\tau}_\varphi^0(f, z_0) = \lim_{n \rightarrow +\infty} \frac{\exp\{\varphi(M_{z_0}(r_n, f))\}}{\frac{1}{r_n^{\mu_\varphi^0(f, z_0)}}} < \beta.$$

Then, for any given $\varepsilon > 0$, there exists an $n_0 \geq 1$ ($n_0 \in \mathbb{N}$) such that for all $n \geq n_0$,

$$\varphi(M_{z_0}(r_n, f)) \leq \log\left(\beta\left(\frac{1}{r_n^{\mu-\varepsilon}}\right)\right). \quad (1.3.1)$$

For ε given above, there exists an $n_1 \geq 1$ ($n_1 \in \mathbb{N}$) such that for all $n \geq n_1$ and $r \in \left[\frac{n}{n+1}r_n, r_n\right]$,

$$\frac{1}{\left(\frac{n}{n+1}\right)^{\mu-\varepsilon}} \leq \frac{1}{r^\varepsilon}. \quad (1.3.2)$$

By [\(1.3.1\)](#) and [\(1.3.2\)](#), for all $n \geq n_2 = \max\{n_0, n_1\}$ and for any $r \in \left[\frac{n}{n+1}r_n, r_n\right]$,

$$\begin{aligned} \varphi(M_{z_0}(r, f)) &\leq \varphi\left(M_{z_0}\left(\frac{n}{n+1}r_n, f\right)\right) \leq \log\left(\beta\left(\frac{1}{\left(\frac{n}{n+1}r_n\right)^{\mu-\varepsilon}}\right)\right) \\ &\leq \log\left(\beta\left(\frac{1}{\left(\frac{n}{n+1}\right)^{\mu-\varepsilon}} \cdot \frac{1}{r^{\mu-\varepsilon}}\right)\right) \leq \log\left(\frac{\beta}{r^\mu}\right). \end{aligned}$$

Set

$$E = \bigcup_{n=n_2}^{+\infty} \left[\frac{n}{n+1}r_n, r_n\right],$$

we get

$$m_1(E) = \int_E \frac{1}{t} dt = \sum_{n=n_2}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{1}{t} dt = \sum_{n=n_2}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty.$$

□

Lemma 1.3.2 ([\[16\]](#)) *Let f_1, f_2 be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\rho_\varphi^0(f_1, z_0) = \rho_1 > 0$, $\rho_\varphi^0(f_2, z_0) = \rho_2 < \infty$ and $\rho_2 < \rho_1$. Then, there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$ one has*

$$\lim_{r \rightarrow 0} \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} = 0.$$

Lemma 1.3.3 *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\mu_\varphi^0(f, z_0) = \mu_0 < \infty$. Then there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$, we have*

$$\mu_0 = \lim_{r \rightarrow 0} \frac{\varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}} = \lim_{r \rightarrow 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}},$$

and for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E$

$$M_{z_0}(r, f) \leq \varphi^{-1} \left((\mu_0 + \varepsilon) \log \frac{1}{r} \right).$$

Proof. By the definition of the lower φ -order, there exists a sequence $\{r_n\}_{n=1}^{+\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1}r_n$ and

$$\lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{r_n}} = \mu_0.$$

Therefore, there exists an integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for any $r \in \left[\frac{n}{n+1}r_n, r_n \right]$, we get

$$\frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1}r_n}} \leq \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}} \leq \frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n, f))}{\log \frac{1}{r_n}}.$$

Since

$$\lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1}r_n}} = \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{r_n} + \log \frac{n+1}{n}} = \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{r_n}} = \mu_0,$$

and

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1}r_n}} &= \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n, f)) \log \frac{1}{\frac{n}{n+1}r_n}}{\log \frac{1}{\frac{n}{n+1}r_n} \log \frac{1}{r_n}}, \\ \lim_{n \rightarrow +\infty} \frac{\log \frac{1}{\frac{n}{n+1}r_n}}{\log \frac{1}{r_n}} &= 1, \quad \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n, f))}{\log \frac{1}{\frac{n}{n+1}r_n}} = \mu_0, \end{aligned}$$

then

$$\lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1}r_n}} = \lim_{n \rightarrow +\infty} \frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n, f))}{\log \frac{1}{r_n}} = \mu_0,$$

so for any $r \in \left[\frac{n}{n+1}r_n, r_n \right]$, we get

$$\lim_{r \rightarrow 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}} = \mu_0.$$

Set $E = \bigcup_{n=n_0}^{+\infty} \left[\frac{n}{n+1}r_n, r_n \right]$, with

$$m_l(E) = \sum_{n=n_0}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_0}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty.$$

□

Similarly, we can prove the following two lemmas.

Lemma 1.3.4 *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\tilde{\mu}_\varphi^1(f, z_0) = \mu_1 < \infty$. Then there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$, we have*

$$\mu_1 = \lim_{r \rightarrow 0} \frac{\varphi(T_{z_0}(r, f))}{\log \frac{1}{r}} = \lim_{r \rightarrow 0} \frac{\varphi(\log M_{z_0}(r, f))}{\log \frac{1}{r}},$$

and for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E$

$$T_{z_0}(r, f) \leq \varphi^{-1} \left((\mu_1 + \varepsilon) \log \frac{1}{r} \right).$$

Lemma 1.3.5 *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\rho_\varphi^0(f, z_0) = \rho$. Then, there exists a set $E \subset (0, 1)$ with $m_l(E) = +\infty$ such that for all $|z - z_0| = r \in E$, we have*

$$\lim_{r \rightarrow 0} \frac{\varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}} = \rho$$

and for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E$

$$T_{z_0}(r, f) \geq \log(\varphi^{-1} \left((\rho - \varepsilon) \log \frac{1}{r} \right)).$$

Lemma 1.3.6 *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $0 < \tilde{\mu}_\varphi^0(f, z_0) = \mu < \infty$ and $0 < \tilde{\tau}_\varphi^0(f, z_0) = \tau < \infty$. Then there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$, we have*

$$M_{z_0}(r, f) < \varphi^{-1} \left\{ (\tau + \varepsilon) \log \left(\frac{1}{r} \right)^\mu \right\}.$$

Proof. By the definition of lower φ -order and lower φ -type, there exists a sequence $\{r_m\}_{m=1}^\infty$ tending to 0 satisfying $r_{m+1} < \frac{r_m}{m+1}$ and

$$\lim_{m \rightarrow \infty} \frac{\exp(\varphi(M_{z_0}(r_m, f)))}{\left(\frac{1}{r_m}\right)^\mu} = \tau.$$

For any $r \in \left[\frac{m}{m+1}r_m, r_m\right]$, we have

$$\frac{\exp(\varphi(M_{z_0}(r_m, f)))}{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu} \leq \frac{\exp(\varphi(M_{z_0}(r, f)))}{\left(\frac{1}{r}\right)^\mu} \leq \frac{\exp(\varphi(M_{z_0}(\frac{m}{m+1}r_m, f)))}{\left(\frac{1}{r_m}\right)^\mu}.$$

Since

$$\frac{\exp(\varphi(M_{z_0}(r_m, f)))}{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu} = \frac{\exp(\varphi(M_{z_0}(r_m, f)))}{\left(\frac{1}{r_m}\right)^\mu} \cdot \frac{\left(\frac{1}{r_m}\right)^\mu}{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu}$$

and

$$\frac{\exp(\varphi(M_{z_0}\left(\frac{m}{m+1}r_m, f\right)))}{\left(\frac{1}{r_m}\right)^\mu} = \frac{\exp(\varphi(M_{z_0}\left(\frac{m}{m+1}r_m, f\right)))}{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu} \cdot \frac{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu}{\left(\frac{1}{r_m}\right)^\mu},$$

then

$$\lim_{n \rightarrow +\infty} \frac{\exp(\varphi(M_{z_0}(r_m, f)))}{\left(\frac{1}{\frac{m}{m+1}r_m}\right)^\mu} = \lim_{n \rightarrow +\infty} \frac{\exp(\varphi(M_{z_0}\left(\frac{m}{m+1}r_m, f\right)))}{\left(\frac{1}{r_m}\right)^\mu} = \mathcal{I},$$

so for any $r \in \left[\frac{m}{m+1}r_m, r_m\right]$, we get

$$\lim_{r \rightarrow 0} \frac{\exp(\varphi(M_{z_0}(r, f)))}{\left(\frac{1}{r}\right)^\mu} = \mathcal{I}.$$

Then, for any given $\varepsilon > 0$, there exists a positive integer m_0 such that for all $m \geq m_0$ and for all $r \in \left[\frac{m}{m+1}r_m, r_m\right]$, we have

$$M_{z_0}(r, f) < \varphi \left\{ (\mathcal{I} + \varepsilon) \left(\frac{1}{r} \right)^\mu \right\}.$$

Set $E = \bigcup_{m=m_0}^{+\infty} \left[\frac{m}{m+1}r_m, r_m\right]$. Then for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E$, we have

$$M_{z_0}(r, f) < \varphi \left\{ (\mathcal{I} + \varepsilon) \left(\frac{1}{r} \right)^\mu \right\},$$

where

$$m_l(E_2) = \sum_{m=m_0}^{+\infty} \int_{\frac{m}{m+1}r_m}^{r_m} \frac{dt}{t} = \sum_{m=m_0}^{+\infty} \log \left(1 + \frac{1}{m} \right) = +\infty.$$

□

Lemma 1.3.7 ([9]) *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, let $\alpha > 1$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$. Then there exist a set $E_3 \subset (0, r_0]$, $r_0 \in (0, 1)$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on α and k such that for all $|z - z_0| = r \in (0, r_0] \setminus E$, we have*

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{r}{\alpha}, f \right) \log T_{z_0} \left(\frac{r}{\alpha}, f \right) \right]^j.$$

Lemma 1.3.8 ([11], Theorem 8) *Let f be nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then, there exists a set $E \subset (0, 1)$ that has finite logarithmic measure, such that for all $j = 0, 1, \dots, k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu_{z_0}(r, f)}{z_0 - z_r} \right)^j (1 + o(1)),$$

as $r \rightarrow 0$, $r \notin E$, where z_r is a point in the circle $|z - z_0| = r$ that satisfies $|f(z_r)| = \max\{|f(z)| : |z - z_0| = r\}$.

Lemma 1.3.9 ([19], Lemma 2.5) *Let $g : (0, 1) \rightarrow \mathbb{R}$, $h : (0, 1) \rightarrow \mathbb{R}$ be monotone decreasing functions such that $g(r) \geq h(r)$ possibly outside an exceptional set $E \subset (0, 1)$ that has finite logarithmic measure. Then, for any given $\beta > 1$, there exists a constant $0 < r_1 < 1$ such that for all $r \in (0, r_1)$, we have $g(r^\beta) \geq h(r)$.*

Lemma 1.3.10 ([19]) *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. Then f satisfies the following two properties:*

- (i) $T_{z_0}(r, \frac{1}{f}) = T_{z_0}(r, f) + O(1)$,
- (ii) $T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right)$, $r \in (0, r_2] \setminus E$, where $E \subset (0, r_2]$ with $m_l(E) < \infty$.

Lemma 1.3.11 [13] *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function. Let $\mu(r)$ and $\nu_f(r)$ denote respectively the maximum term and the central index of f , i.e., $\mu(r) = \max\{|a_n| r^n; n = 0, 1, \dots\}$ and $\nu_f(r) = \max\{n : \mu(r) = |a_n| r^n\}$. Then, we have*

$$\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \quad (|a_0| \neq 0), \quad (1.3.3)$$

$$M(r, f) < \mu(r) \left\{ \nu_f(R) + \frac{R}{R-r} \right\} \quad (R > r). \quad (1.3.4)$$

Lemma 1.3.12 *Let f be a nonconstant analytic function in $\mathbb{C}$ with $\mu_\varphi^0(f) = \mu_0 < \infty$. Then, we have*

$$\mu_0 = \liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r)})}{\log r}.$$

Proof. Denote $\mu := \liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r)})}{\log r}$. Then, by the definition of the lower limit, there exists a sequence $\{r_n\}_{n=1}^{+\infty}$ tending to $+\infty$ such that for any given $\varepsilon > 0$, and sufficiently large r , we have

$$\nu_f(r_n) \leq \log \varphi^{-1}(\log r_n^{\mu+\varepsilon}). \quad (1.3.5)$$

By setting $R = 2r_n$ in (1.3.4), we get

$$M(r_n, f) < \mu(r_n)(\nu_f(2r_n) + 2) = |a_{\nu_f(r_n)}| r^{\nu_f(r_n)} (\nu_f(2r_n) + 2). \quad (1.3.6)$$

Since $\{a_n\}_{n \geq 0}$ is a bounded sequence, then by using Proposition (1.2.1) case (i), (1.3.5) and (1.3.6), we obtain

$$\begin{aligned} \log M(r_n, f) &\leq \nu_f(r_n) \log r_n + \log \nu_f(2r_n) + c_1 \\ &\leq \log \varphi^{-1}(\log r_n^{\mu+\varepsilon}) \log r_n + \log \log (\varphi^{-1}\{\log(2r_n)^{\mu+\varepsilon}\}) + c_1 \\ &\leq \log \varphi^{-1}(\log r_n^{\mu+2\varepsilon}) + \log \varphi^{-1}(\log(2r_n)^{\mu+\varepsilon}) \\ &\leq \log \varphi^{-1}(\log r_n^{\mu+3\varepsilon}), \end{aligned} \quad (1.3.7)$$

where $c_1 > 0$ is a real constant. From (1.3.7), by the monotonicity of φ , we get

$$\frac{\varphi(M(r_n, f))}{\log r_n} \leq \mu + 3\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$ and Proposition (1.2.3), we obtain

$$\mu_\varphi^0(f) \leq \mu := \liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r)})}{\log r}. \quad (1.3.8)$$

Now, we prove the reverse inequality. Without loss of generality, we may assume $|a_0| \neq 0$. It follows from (1.3.4) that

$$\begin{aligned} \log \mu(2r) &= \log |a_0| + \int_0^{2r} \frac{\nu_f(t)}{t} dt \geq \log |a_0| + \nu_f(r) \int_r^{2r} \frac{dt}{t} \\ &= \log |a_0| + \nu_f(r) \log 2. \end{aligned}$$

By Cauchy's inequality we have $\mu(2r) \leq M(2r, f)$ and then

$$\nu_f(r) \leq \frac{\log M(2r, f)}{\log 2} - \frac{\log |a_0|}{\log 2} \leq c_2 \log M(2r, f), \quad (1.3.9)$$

where $c_2 > 2$ is a real constant. It follows from (1.3.9) and Proposition (1.2.1) case (iii) that

$$\begin{aligned} \frac{\varphi(e^{\nu_f(r)})}{\log r} &\leq \frac{\varphi(e^{c_2 \log M(2r, f)})}{\log r} = \frac{\varphi(e^{(\log M(2r, f))^{c_2}})}{\log r} \\ &= \frac{\varphi(M(2r, f)^{c_2})}{\log r} \\ &= \frac{(1 + o(1))\varphi(M(2r, f))}{\log 2r} \cdot \frac{\log 2r}{\log r}. \end{aligned}$$

Then

$$\frac{\varphi(e^{\nu_f(r)})}{\log r} \leq \frac{(1 + o(1))\varphi(M(2r, f))}{\log 2r} \cdot \frac{\log 2r}{\log r}. \quad (1.3.10)$$

Hence

$$\liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r)})}{\log r} \leq \liminf_{r \rightarrow +\infty} \frac{\varphi(M(2r, f))}{\log 2r} = \mu_\varphi^0(f). \quad (1.3.11)$$

We deduce from (1.3.8) and (1.3.11) that

$$\mu_\varphi^0(f) = \liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r)})}{\log r}.$$

□

Lemma 1.3.13 *Let f be an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. The φ -order of f near z_0 is defined by*

$$\mu_\varphi^0 = \liminf_{r \rightarrow +\infty} \frac{\varphi(e^{\nu_f(r, z_0)})}{\log \frac{1}{r}},$$

where $M_{z_0}(r, f) = \max\{|f(z)| : |z - z_0| = r\}$ and $\nu_{z_0}(r, f)$ is the central index of f near z_0 .

Proof. Set $g(w) = f(z_0 - \frac{1}{w})$. As the function g is entire ([11], Remark 7), it turns out that

$$\nu_{z_0}(r, f) = \nu_g(R), \quad R = \frac{1}{r}.$$

By Lemma 1.3.6, we have

$$\mu_\varphi^0(g) = \liminf_{R \rightarrow +\infty} \frac{\varphi(e^{\nu_g(R)})}{\log R}.$$

That gives

$$\mu_\varphi^0(f, z_0) = \mu_\varphi^1(g) = \liminf_{r \rightarrow 0} \frac{\varphi(e^{\nu_{z_0}(r, f)})}{\log \frac{1}{r}}.$$

□

Lemma 1.3.14 *Let $\varphi \in \Phi$ and f, g be two analytic function in $\mathbb{C}$, with $\tilde{\rho}_\varphi^j$ as φ -order of a function f and $\tilde{\mu}_\varphi^j$ as the lower φ -order of a function g . Then,*

i)

$$\tilde{\mu}_\varphi^j(f + g) \leq \max\{\tilde{\rho}_\varphi^j(f), \tilde{\mu}_\varphi^j(g)\} \text{ for } j = 0, 1,$$

$$\tilde{\mu}_\varphi^j(fg) \leq \max\{\tilde{\rho}_\varphi^j(f), \tilde{\mu}_\varphi^j(g)\} \text{ for } j = 0, 1.$$

ii) If $\tilde{\rho}_\varphi^j(f) < \tilde{\mu}_\varphi^j(g)$, ($j = 0, 1$), then we obtain $\tilde{\mu}_\varphi^j(f + g) = \tilde{\mu}_\varphi^j(fg) = \tilde{\mu}_\varphi^j(g)$ for $j = 0, 1$.

Proof. Using Remark 1.2.1 and Remark 1.2.2, it is sufficient to prove as follows. Without loss of generality, we assume that $\tilde{\rho}_\varphi^j < \infty$ and $\tilde{\mu}_\varphi^j < \infty$. We will choose $j = 1$ to prove it, and apply the same proof for $j = 0$. From the definition of the lower φ -order there exists a sequence $r_n \rightarrow +\infty$ ($n \rightarrow +\infty$) such that

$$\mu_\varphi^1(g) = \lim_{n \rightarrow +\infty} \frac{\varphi(T(r_n, g))}{\log r_n}.$$

Then, for any given $\varepsilon > 0$ and all $r_n \rightarrow +\infty$ ($n \rightarrow +\infty$)

$$T(r, g) \leq \varphi^{-1}((\mu_\varphi^1(g) + \varepsilon) \log r_n)$$

holds for $n > N_1$. From the definition of φ -order, for any given $\varepsilon > 0$, there exists a positive number R such that,

$$T(r, f) \leq \varphi^{-1}((\rho_\varphi^1(f) + \varepsilon) \log r), \quad r \geq R,$$

and there exists a positive integer N_2 such that $r_n > R$ and thus

$$T(r_n, f) \leq \varphi^{-1}((\rho_\varphi^1(f) + \varepsilon) \log r_n)$$

holds for $n > N_2$. Note that

$$T(r, f + g) \leq T(r, f) + T(r, g) + \log 2, \text{ and } T(r, fg) \leq T(r, f) + T(r, g).$$

Then, for any given $\varepsilon > 0$, we have for $n > \max\{N_1, N_2\}$

$$\begin{aligned} T(r_n, f + g) &\leq T(r_n, f) + T(r_n, g) + \log 2 \\ &\leq \varphi^{-1}((\rho_\varphi^1(f) + \varepsilon) \log r_n) + \varphi^{-1}((\mu_\varphi^1(g) + \varepsilon) \log r_n) + \log 2 \\ &\leq 3\varphi^{-1}((\max(\rho_\varphi^1(f), \mu_\varphi^1(g)) + \varepsilon) \log r_n) \end{aligned} \tag{1.3.12}$$

and

$$\begin{aligned} T(r_n, fg) &\leq T(r_n, f) + T(r_n, g) \\ &\leq \varphi^{-1}((\rho_\varphi^1(f) + \varepsilon) \log r_n) + \varphi^{-1}((\mu_\varphi^1(g) + \varepsilon) \log r_n) \\ &\leq 2\varphi^{-1}((\max(\rho_\varphi^1(f), \mu_\varphi^1(g)) + \varepsilon) \log r_n). \end{aligned} \tag{1.3.13}$$

Since $\varepsilon > 0$ is arbitrary, then from (1.3.12) and (1.3.13), we easily obtain

$$\mu_\varphi^1(f + g) \leq \max\{\rho_\varphi^1(f), \mu_\varphi^1(g)\}, \text{ and } \mu_\varphi^1(fg) \leq \max\{\rho_\varphi^1(f), \mu_\varphi^1(g)\}.$$

Suppose now that $\rho_\varphi^j(f) < \mu_\varphi^j(g)$. Considering that

$$T(r, g) \leq T(r, f + g - f) \leq T(r, f + g) + T(r, f) + \log 2 \quad (1.3.14)$$

and

$$T(r, g) \leq T(r, fg \frac{1}{f}) \leq T(r, fg) + T(r, f) + o(1). \quad (1.3.15)$$

By (1.3.14), (1.3.15) and the same method as above, we obtain that

$$\mu_\varphi^1(g) \leq \max \{ \rho_\varphi^1(f), \mu_\varphi^1(f + g) \} = \mu_\varphi^1(f + g)$$

and

$$\mu_\varphi^1(g) \leq \max \{ \rho_\varphi^1(f), \mu_\varphi^1(f + g) \} = \mu_\varphi^1(f + g).$$

□

Lemma 1.3.15 *Let $\varphi \in \Phi$ and f, g be two analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then*

i)

$$\tilde{\mu}_\varphi^j(f + g, z_0) \leq \min \{ \tilde{\mu}_\varphi^j(f, z_0), \tilde{\mu}_\varphi^j(g, z_0) \} \text{ for } j = 0, 1,$$

$$\tilde{\mu}_\varphi^j(fg, z_0) \leq \min \{ \tilde{\mu}_\varphi^j(f, z_0), \tilde{\mu}_\varphi^j(g, z_0) \} \text{ for } j = 0, 1.$$

ii) If $\tilde{\rho}_\varphi^j(g, z_0) < \tilde{\mu}_\varphi^j(f, z_0)$, ($j = 0, 1$), then $\tilde{\mu}_\varphi^j(f + g, z_0) = \tilde{\mu}_\varphi^j(fg, z_0) = \tilde{\mu}_\varphi^j(f, z_0)$ for $j = 0, 1$.

Proof. By Lemma 2.2 in [9], we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Lemma 1.3.14, we get the conclusions of Lemma 1.3.15. □

Lemma 1.3.16 *Let $\varphi \in \Phi$ and f, g be two analytic function in $\mathbb{C}$.*

i) If $0 < \tilde{\rho}_\varphi^j(f) < \tilde{\mu}_\varphi^j(g) < \infty$ and $\tilde{\tau}_\varphi^j(f) < \tilde{\tau}_\varphi^j(g)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g) = \tilde{\tau}_\varphi^j(fg) = \tilde{\tau}_\varphi^j(g), (j = 0, 1).$$

ii) If $0 < \tilde{\rho}_\varphi^j(f) = \tilde{\mu}_\varphi^j(g) = \tilde{\mu}_\varphi^j(f + g) < \infty$, ($j = 0, 1$) and $\tilde{\tau}_\varphi^j(f) < \tilde{\tau}_\varphi^j(g)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g) < \max(\tilde{\tau}_\varphi^j(f), \tilde{\tau}_\varphi^j(g)), (j = 0, 1).$$

Furthermore, if $\tilde{\tau}_\varphi^j(f) \neq \tilde{\tau}_\varphi^j(g)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g) = \max(\tilde{\tau}_\varphi^j(f), \tilde{\tau}_\varphi^j(g)), (j = 0, 1).$$

iii) If $0 < \tilde{\rho}_\varphi^j(f) = \tilde{\mu}_\varphi^j(g) = \tilde{\mu}_\varphi^j(fg) < \infty$, $(j = 0, 1)$ and $\tilde{\tau}_\varphi^j(f) < \tilde{\tau}_\varphi^j(g)$, $(j = 0, 1)$, then

$$\tilde{\tau}_\varphi^j(fg) < \max(\tilde{\tau}_\varphi^j(f), \tilde{\tau}_\varphi^j(g)), (j = 0, 1).$$

Furthermore, if $\tilde{\tau}_\varphi^j(f) \neq \tilde{\tau}_\varphi^j(g)$, $(j = 0, 1)$, then

$$\tilde{\tau}_\varphi^j(fg) = \max(\tilde{\tau}_\varphi^j(f), \tilde{\tau}_\varphi^j(g)), (j = 0, 1).$$

Proof. We will prove the theorem for $j = 0$, the proofs for $j = 1$ are analogous.

i) The definition of the $\tilde{\tau}_\varphi^0$ -type implies that for any given $\varepsilon > 0$,

$$M(r, g) \geq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) - \varepsilon) \left(\frac{1}{r} \right)^{\tilde{\mu}_\varphi^0(g)}) \right\}. \quad (1.3.16)$$

and for sufficiently large r

$$M(r, f) \leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r} \right)^{\tilde{\rho}_\varphi^0(g)}) \right\}. \quad (1.3.17)$$

We know that $M(r, f + g) \geq |f(z) + g(z)| \geq |g(z)| - |f(z)|$. Then by using Proposition [1.2.1](#)

$$\begin{aligned} M(r, f + g) &\geq |M(z, g)| - |M(z, f)| \geq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) - \varepsilon) \left(\frac{1}{r} \right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \\ &\quad - \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r} \right)^{\tilde{\rho}_\varphi^0(g)}) \right\} \geq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) - 2\varepsilon) \left(\frac{1}{r} \right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \end{aligned} \quad (1.3.18)$$

provided $\varepsilon \in (0, \frac{\tilde{\tau}_\varphi^0(g) - \tilde{\tau}_\varphi^0(f)}{2})$. It follows from Lemma [1.3.14](#) that $\tilde{\mu}_\varphi^0(f + g) = \tilde{\mu}_\varphi^0(g)$ and by the monotonicity of φ and [\(1.3.18\)](#), we get

$$\frac{\exp \{ \varphi (M (r, f + g)) \}}{\frac{1}{r^{\tilde{\mu}_\varphi^0(f+g)}}} \geq \tilde{\tau}_\varphi^0(g) + 2\varepsilon,$$

since ε can be arbitrarily chosen such that $\varepsilon \in (0, \frac{\tilde{\tau}_\varphi^0(g) - \tilde{\tau}_\varphi^0(f)}{2})$, thus

$$\tilde{\tau}_\varphi^0(f + g) \geq \tilde{\tau}_\varphi^0(g). \quad (1.3.19)$$

It remains to prove the converse inequality. Indeed, by applying [\(1.3.19\)](#) and since

$$\tilde{\mu}_\varphi^0(f + g) = \tilde{\mu}_\varphi^0(g) > \tilde{\rho}_\varphi^0(f) = \tilde{\rho}_\varphi^0(-f),$$

we obtain

$$\tilde{\tau}_\varphi^0(g) = \tilde{\tau}_\varphi^0(f + g - f) \geq \tilde{\tau}_\varphi^0(f + g). \quad (1.3.20)$$

We deduce from (1.3.19) and (1.3.20) that $\tilde{\tau}_\varphi^0(f + g) = \tilde{\tau}_\varphi^0(g)$.

Now, we prove that $\tilde{\tau}_\varphi^0(fg) \leq \tilde{\tau}_\varphi^0(g)$. We have

$$\begin{aligned} M(r, fg) &\leq M(r, f)M(r, g) \leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) - \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(f)}) \right\} \\ &\leq \left(\varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \right)^2. \end{aligned}$$

By the monotonicity of φ and Proposition 1.2.1

$$\varphi(M(r, fg)) \leq (1 + o(1)) \log((\tilde{\tau}_\varphi^0(g) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(g)}) \leq \log((\tilde{\tau}_\varphi^0(g) + 2\varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(g)}).$$

Since $\tilde{\mu}_\varphi^0(f + g) = \tilde{\mu}_\varphi^0(g)$, one can deduce that $\tilde{\tau}_\varphi^0(fg) \leq \tilde{\tau}_\varphi^0(g)$.

ii) The definition of the $\tilde{\tau}_\varphi^0$ -type implies that for any given $\varepsilon > 0$, there exists a sequence $\{r_n, n \geq 1\}$ tending to infinity such that

$$M(r_n, g) \leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) + \varepsilon) \left(\frac{1}{r_n}\right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \quad (1.3.21)$$

and for sufficiently large r

$$M(r, f) \leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(f)}) \right\}. \quad (1.3.22)$$

By the assumption $0 < \tilde{\rho}_\varphi^0(f) = \tilde{\mu}_\varphi^0(g) = \tilde{\mu}_\varphi^0(f + g) < \infty$, we get by using Proposition 1.2.1

$$\begin{aligned} M(r, f + g) &\leq M(r, f) + M(r, g) \\ &\leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(f)}) \right\} + \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \\ &\leq \varphi^{-1} \left\{ \log((\max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g) + 2\varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(f+g)}) \right\}. \end{aligned}$$

By the monotonicity of φ , we obtain

$$\frac{\exp\{\varphi(T(r, f + g))\}}{\frac{1}{r^{\tilde{\mu}_\varphi^0(f+g)}}} \leq \max\{\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)\} + 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily, then we get

$$\tilde{\tau}_\varphi^0(f+g) \leq \max\{\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)\}. \quad (1.3.23)$$

Without loss of generality, we may suppose $\tilde{\tau}_\varphi^0(f) < \tilde{\tau}_\varphi^0(g)$ then by (1.3.23) and since $\tilde{\mu}_\varphi^0(f+g) = \tilde{\rho}_\varphi^0(f) = \tilde{\rho}_\varphi^0(-f)$, it follows

$$\tilde{\tau}_\varphi^0(g) = \tilde{\tau}_\varphi^0(f+g-f) \leq \max\{\tilde{\tau}_\varphi^0(f+g), \tilde{\tau}_\varphi^0(f)\} = \tilde{\tau}_\varphi^0(f+g). \quad (1.3.24)$$

We deduce from (1.3.23) and (1.3.24) that $\tilde{\tau}_\varphi^0(f+g) = \max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g))$.

iii) Since $0 < \tilde{\rho}_\varphi^0(f) = \tilde{\mu}_\varphi^0(g) = \tilde{\mu}_\varphi^0(f+g) < \infty$, then

$$\begin{aligned} M(r, fg) &\leq M(r, f)M(r, g) \\ &\leq \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(f) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(f)}) \right\} \varphi^{-1} \left\{ \log((\tilde{\tau}_\varphi^0(g) + \varepsilon) \left(\frac{1}{r_n}\right)^{\tilde{\mu}_\varphi^0(g)}) \right\} \\ &\leq \left(\varphi^{-1} \left\{ \log((\max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\mu}_\varphi^0(fg)}) \right\} \right)^2. \end{aligned}$$

By the monotonicity of φ and Proposition 1.2.1

$$\begin{aligned} \varphi(M(r, fg)) &\leq (1 + o(1)) \log((\max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)) + \varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(fg)}) \\ &\leq \log((\max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)) + 2\varepsilon) \left(\frac{1}{r}\right)^{\tilde{\rho}_\varphi^0(fg)}) \\ &\frac{\exp\{\varphi(M(r, fg))\}}{\frac{1}{r^{\tilde{\mu}_\varphi^0(fg)}}} \leq \max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)) + 2\varepsilon. \end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we deduce that

$$\tilde{\tau}_\varphi^0(fg) < \max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g)). \quad (1.3.25)$$

On the other hand, if we suppose that $\tau_\varphi^0(f) < \tilde{\tau}_\varphi^0(g)$, then by (1.3.25) and since $\tilde{\mu}_\varphi^0(fg) = \tilde{\rho}_\varphi^0(f) = \tilde{\rho}_\varphi^0(\frac{1}{f})$, we get

$$\tilde{\tau}_\varphi^0(g) = \tilde{\tau}_\varphi^0(fg \frac{1}{f}) \leq \max\{\tilde{\tau}_\varphi^0(fg), \tilde{\tau}_\varphi^0(f)\} = \tilde{\tau}_\varphi^0(fg). \quad (1.3.26)$$

It follows from (1.3.25) and (1.3.26) that $\tilde{\tau}_\varphi^0(fg) = \max(\tilde{\tau}_\varphi^0(f), \tilde{\tau}_\varphi^0(g))$. $\square$

Lemma 1.3.17 *Let $\varphi \in \Phi$ and f, g be two analytic function in $\overline{\mathbb{C}} - \{z_0\}$.*

i) If $0 < \tilde{\rho}_\varphi^j(f, z_0) < \tilde{\mu}_\varphi^j(g, z_0) < \infty$, and $\tilde{\tau}_\varphi^j(f, z_0) < \tilde{\tau}_\varphi^j(g, z_0)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g, z_0) = \tilde{\tau}_\varphi^j(fg, z_0) = \tilde{\tau}_\varphi^j(g, z_0), (j = 0, 1).$$

ii) If $0 < \tilde{\rho}_\varphi^j(f, z_0) = \tilde{\mu}_\varphi^j(g, z_0) = \tilde{\mu}_\varphi^j(fg, z_0) < \infty$, ($j = 0, 1$), and $\tilde{\tau}_\varphi^j(f, z_0) < \tilde{\tau}_\varphi^j(g, z_0)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g, z_0) < \max(\tilde{\tau}_\varphi^j(f, z_0), \tilde{\tau}_\varphi^j(g, z_0)), (j = 0, 1).$$

Furthermore, if $\tilde{\tau}_\varphi^j(f, z_0) \neq \tilde{\tau}_\varphi^j(g, z_0)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(f + g, z_0) = \max(\tilde{\tau}_\varphi^j(f, z_0), \tilde{\tau}_\varphi^j(g, z_0)), (j = 0, 1).$$

iii) If $0 < \tilde{\rho}_\varphi^j(f, z_0) = \tilde{\mu}_\varphi^j(g, z_0) = \tilde{\mu}_\varphi^j(fg, z_0) < \infty$, ($j = 0, 1$) and $\tilde{\tau}_\varphi^j(f, z_0) < \tilde{\tau}_\varphi^j(g, z_0)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(fg, z_0) < \max(\tilde{\tau}_\varphi^j(f, z_0), \tilde{\tau}_\varphi^j(g, z_0)), (j = 0, 1).$$

Furthermore, if $\tilde{\tau}_\varphi^j(f, z_0) \neq \tilde{\tau}_\varphi^j(g, z_0)$, ($j = 0, 1$), then

$$\tilde{\tau}_\varphi^j(fg, z_0) = \max(\tilde{\tau}_\varphi^j(f, z_0), \tilde{\tau}_\varphi^j(g, z_0)), (j = 0, 1).$$

Proof. By Lemma 2.2 in [9], we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Lemma 1.3.16, we get the conclusions of Lemma 1.3.17. $\square$

Growth of Solutions of Complex Linear Differential Equations Near an Essential Singular Point

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty-cold and austere, like that of sculpture-without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show."

Bertrand Russell

The proofs of Theorems 1.2.4-1.2.11 play a key role in understanding the behavior of non-trivial solutions to differential equations in the complex plane near isolated singularities. These proofs allow us to obtain precise information about the growth of solutions without explicitly solving the equation, using analytic indicators such as $\tilde{\mu}_\varphi$ and $\tilde{\rho}_\varphi$. They also highlight the importance of linking the properties of the coefficients with the nature of the solutions, contributing to their classification and enriching complex analysis techniques. These results open doors to future theoretical and applied developments in both mathematics and physics.

2.1 Proof of Theorems

Proof of Theorem 1.2.4

Proof. Set $\tilde{\mu}_\varphi^0(A, z_0) = \mu_A$ and $\tilde{\mu}_\varphi^0(B, z_0) = \mu_B$. By Definition 1.2.6, for any given $\varepsilon \in (0, \frac{\mu_B - \mu_A}{2})$, there exists $r_1 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_1)$, we have

$$|B(z)| \geq \varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right). \quad (2.1.1)$$

By Lemma 1.3.3 for all ε given above, we conclude the existence a set $E_1 \subset (0, 1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_1$ and $|A_0(z)| = M_{z_0}(r, A_0)$

$$|A(z)| \leq \varphi^{-1} \left((\mu_A + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.2)$$

By Lemma 1.3.5, there exist a set $E_2 \subset (0, r_2]$ ($r_2 \in (0, 1)$) that has finite logarithmic measure and a constant $\lambda > 0$ that depends on $\gamma > 1$ such that for all $|z - z_0| = r \in (0, r_2] \setminus E_2$ the following occurs

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq \lambda \left(\frac{1}{r^2} T_{z_0} \left(\frac{r}{\gamma}, f \right) \log T_{z_0} \left(\frac{r}{\gamma}, f \right) \right)^j, \quad j = 1, 2. \quad (2.1.3)$$

By (1.2.2), we get

$$|B(z)| \leq \left| \frac{f''}{f} \right| + |A(z)| \left| \frac{f'}{f} \right|. \quad (2.1.4)$$

As the last step, let $E_3 = (0, r_2] \cap (0, r_1) \cap E_1 \setminus E_2$, obviously E_3 has infinite logarithmic measure. Consequently, the combination between (2.1.1), (2.1.2), (2.1.3) and (2.1.4) gives for any given $\varepsilon \in (0, \frac{\mu_B - \mu_A}{2})$ and for all $|z - z_0| = r \in E_3$,

$$\varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right) \leq \lambda \left(\frac{1}{r} T_{z_0} \left(\frac{r}{\gamma}, f \right) \right)^4 \varphi^{-1} \left((\mu_A + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.5)$$

Now, we prove that $\rho_\varphi^1(f, z_0) \geq \mu_B$. By contradiction, we suppose that

$$\rho_1 = \rho_\varphi^1(f, z_0) < \mu_B.$$

Then, for any given $\varepsilon \in (0, \min \left\{ \frac{\mu_B - \mu_A}{2}, \frac{\mu_B - \rho_1}{2} \right\})$, there exists $r_3 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_3)$, we have

$$T_{z_0}(r, f) \leq \varphi^{-1} \left((\rho_1 + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.6)$$

By (2.1.5) and (2.1.6), for any given $\varepsilon \in \left(0, \min \left\{ \frac{\mu_B - \mu_A}{2}, \frac{\mu_B - \rho_1}{2} \right\} \right)$ and $|z - z_0| = r \in (0, r_3) \cap E_3$, we obtain

$$\varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right) \leq \frac{\lambda}{r^4} \left(\varphi^{-1} \left((\rho_1 + \varepsilon) \log \frac{\gamma}{r} \right) \right)^4 \varphi^{-1} \left((\mu_A + \varepsilon) \log \frac{1}{r} \right).$$

Since $\mu_B - \varepsilon > \max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \}$, we get

$$\varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right) \leq \frac{\lambda}{r^4} \left(\varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right) \right)^5. \quad (2.1.7)$$

Applying the logarithm on the both sides of (2.1.7), we find

$$\begin{aligned} & \frac{4 \log r}{5 \log \varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right)} \\ & + \frac{\log \varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right)}{5 \log \varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right)} \\ & \leq \frac{\log \lambda}{5 \log \varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right)} + 1. \end{aligned} \quad (2.1.8)$$

Set $\varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right) = x$. Then $\log r = -\frac{\varphi(x)}{\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \}} + \log \gamma$ and by using Karamata's Theorem [23], $\varphi(e^t) = t^{o(1)}$, $t \rightarrow +\infty$, we immediately get

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{\log r}{\log \varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right)} \\ & = \lim_{x \rightarrow +\infty} \left(-\frac{\varphi(x)}{\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log x} + \frac{\log \gamma}{\log x} \right) \\ & = \lim_{x \rightarrow +\infty} \left(-\frac{(\log x)^{o(1)}}{\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log x} \right) = 0. \end{aligned} \quad (2.1.9)$$

As we did earlier in Lemma 2.12 in ([16]), we have

$$\lim_{r \rightarrow 0} \frac{\log \varphi^{-1} \left((\mu_B - \varepsilon) \log \frac{1}{r} \right)}{\log \varphi^{-1} \left(\max \{ \rho_1 + \varepsilon, \mu_A + \varepsilon \} \log \frac{\gamma}{r} \right)} = +\infty. \quad (2.1.10)$$

The right hand side in (2.1.8) is finite, while the left hand side is infinite, thus a contradiction holds, i.e.,

$$\rho_\varphi^1(f, z_0) \geq \mu_B.$$

Thus, by Remark 1.2.1 and as $\varphi \in \phi$, we obtain

$$\tilde{\rho}_\varphi^1(f, z_0) \geq \mu_B.$$

□

Proof of Theorem 1.2.5

Proof. Set $\tilde{\tau}_\varphi^0(B, z_0) = \tau_B$, $\tilde{\tau}_\varphi^0(A, z_0) = \tau_A$ and $\tilde{\mu}_\varphi^0(B, z_0) = \tilde{\mu}_\varphi^0(A, z_0) = \mu$. From Definition 1.2.7, there exists $r_4 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_4)$

$$|B(z)| \geq \varphi^{-1} \left(\log \frac{\tau_B}{r^\mu} \right). \quad (2.1.11)$$

By Lemma 1.3.1, there exists a set $E_4 \subset (0, 1)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in E_4$

$$|A(z)| \leq \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right). \quad (2.1.12)$$

We now turn to Lemma 1.3.5, it claims the existence of a set $E_5 \subset (0, r_5]$ ($r_5 \in (0, 1)$) having finite logarithmic measure and a constant $\alpha > 0$ that depends on some given $\gamma > 1$ such that for all $|z - z_0| = r \in (0, r_5] \setminus E_5$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq \lambda \left(\frac{1}{r^2} T_{z_0} \left(\frac{r}{\alpha}, f \right) \log T_{z_0} \left(\frac{r}{\alpha}, f \right) \right)^j, \quad j = 1, 2. \quad (2.1.13)$$

Set $E_6 = E_4 \cap (0, r_4) \cap (0, r_5] \setminus E_5$, for sure E_6 has infinite logarithmic measure. Combining (2.1.31), (2.1.11) and (2.1.33) with (2.1.4) we get for all $|z - z_0| = r \in E_6$,

$$\varphi^{-1} \left(\log \frac{\tau_B}{r^\mu} \right) \leq \lambda \left(\frac{1}{r} T_{z_0} \left(\frac{r}{\alpha}, f \right) \right)^4 \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right). \quad (2.1.14)$$

The last inequality implies $\rho_\varphi^1(f, z_0) \geq \mu$. To see why, we assume that $\rho_\varphi^1(f, z_0) < \mu$. Then, there exists $r_6 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_6)$, we have

$$T_{z_0}(r, f) \leq \varphi^{-1} \left(\rho_2 \log \frac{1}{r} \right) \quad (2.1.15)$$

for some $\rho_2 < \mu$. Consequently, by (2.1.34) and (2.1.35), for all $|z - z_0| = r \in E_6 \cap (0, r_6)$, we get

$$\begin{aligned} \varphi^{-1} \left(\log \frac{\tau_B}{r^\mu} \right) &\leq \frac{\lambda}{r^4} \left(\varphi^{-1} \left(\log \frac{\alpha \rho_2}{r^{\rho_2}} \right) \right)^4 \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right) \\ &\leq \frac{\lambda k}{r^4} \left(\varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right) \right)^5. \end{aligned}$$

Applying the logarithm on both sides, we find

$$\frac{4 \log r}{5 \log \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right)}$$

$$+ \frac{\log \varphi^{-1} \left(\log \frac{\tau_B}{r^\mu} \right)}{5 \log \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right)} \leq \frac{\log(\lambda)}{5 \log \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right)} + 1. \quad (2.1.16)$$

As we did before

$$\lim_{r \rightarrow 0} \frac{\log r}{\log \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right)} = 0$$

and

$$\lim_{r \rightarrow 0} \frac{\log \varphi^{-1} \left(\log \frac{\tau_B}{r^\mu} \right)}{5 \log \varphi^{-1} \left(\log \frac{\tau_A}{r^\mu} \right)} = +\infty$$

because $\tau_B > \tau_A$. Since the right hand side of the inequality (2.1.36) is bounded by 1, thus taking limits yielding $+\infty \leq 1$ which is a contradiction. Hence, by Remark 1.2.1 and as $\varphi \in \phi$, we get $\rho_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^1(f, z_0) \geq \mu$. $\square$

Proof of Theorem 1.2.6

Proof. Assume that $f(z) \neq 0$ is a rational solution of (1.2.2) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. Then $f'(z) \neq 0$ and by (1.2.2), we have

$$A(z) f'(z) = -f''(z) - B(z) f(z).$$

By Lemma 1.3.7, it follows that

$$\begin{aligned} \tilde{\mu}_\varphi^0(A, z_0) &= \tilde{\mu}_\varphi^0(Af', z_0) = \tilde{\mu}_\varphi^0(-f'' - Bf, z_0) \\ &\leq \tilde{\mu}_\varphi^0(B, z_0), \end{aligned}$$

which is a contradiction. Hence, f must be a transcendental. Suppose that f is a transcendental solution of equation (1.2.2) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. The equation (1.2.2) yields

$$m_{z_0}(r, A) \leq m_{z_0} \left(r, \frac{f''}{f'} \right) + m_{z_0} \left(r, \frac{f}{f'} \right) + m_{z_0}(r, B) + \log 2. \quad (2.1.17)$$

By Lemma 1.3.8, there exists a set $E_7 \subset (0, r_7]$ for fixed $r_7 \in (0, 1)$ which has finite logarithmic measure such that for all $|z - z_0| = r \in (0, r_7] \setminus E_7$, we have

$$m_{z_0} \left(r, \frac{f''}{f'} \right) + m_{z_0} \left(r, \frac{f}{f'} \right) \leq O \left(T_{z_0}(r, f) + \log \frac{1}{r} \right). \quad (2.1.18)$$

By Lemma 1.3.2, there exists a set $E_8 \subset (0, 1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_8$

$$\lim_{r \rightarrow 0} \frac{T_{z_0}(r, B)}{T_{z_0}(r, A)} = 0,$$

so for any given $\varepsilon \in (0, \frac{1}{2})$

$$m_{z_0}(r, B) \leq \varepsilon m_{z_0}(r, A). \quad (2.1.19)$$

By (2.1.37), (2.1.38) and (2.1.39), we conclude that for all $|z - z_0| = r \in E_8 \cap (0, r_7] \setminus E_7$,

$$m_{z_0}(r, A) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right) + \varepsilon m_{z_0}(r, A) + \log 2.$$

So

$$(1 - \varepsilon) m_{z_0}(r, A) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right). \quad (2.1.20)$$

Since $\varepsilon \in (0, \frac{1}{2})$, then $1 - \varepsilon > 1 - \frac{1}{2} = \frac{1}{2}$. From (2.1.40), we get

$$\frac{1}{2} m_{z_0}(r, A) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right),$$

that is by f is a transcendental

$$m_{z_0}(r, A) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right) \leq O(T_{z_0}(r, f)).$$

Using this, we get

$$\liminf_{r \rightarrow 0} \frac{\varphi(\exp(m_{z_0}(r, A)))}{\log \frac{1}{r}} \leq \limsup_{r \rightarrow 0} \frac{\varphi(\exp(cT_{z_0}(r, f)))}{\log \frac{1}{r}},$$

where $c > 0$ is some constant. By using Proposition 1.2.1 case (iii), we obtain

$$\liminf_{r \rightarrow 0} \frac{\varphi(\exp(m_{z_0}(r, A)))}{\log \frac{1}{r}} \leq \limsup_{r \rightarrow 0} \frac{(1 + o(1)) \varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}} = \rho_\varphi^0(f, z_0).$$

Therefore,

$$\mu_\varphi^0(A, z_0) \leq \rho_\varphi^0(f, z_0)$$

and by Remark 1.2.1 this leads

$$\tilde{\mu}_\varphi^0(A, z_0) \leq \tilde{\rho}_\varphi^0(f, z_0).$$

□

Proof of Theorem 1.2.9

Proof. We only need to prove that every solution $f \neq 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.2.1) satisfies $\tilde{\mu}_\varphi^0(A_0, z_0) = \tilde{\mu}_\varphi^1(f, z_0)$, because we already have from Theorem (1.2.7) $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)$. We rewrite (1.2.1) as

$$|A_0(z)| \leq \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \cdots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right|. \quad (2.1.21)$$

Set $\rho = \max \{ \tilde{\rho}_\varphi^0(A_j, z_0) : j \neq 0 \} < \mu = \tilde{\mu}_\varphi^0(A_0, z_0)$. By Definition 1.2.3 and Definition 1.2.6, for any given $\varepsilon \in (0, \frac{\mu - \rho}{2})$, there exists $r_0 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_0)$, we have

$$M_{z_0}(r, A_j) \leq \varphi^{-1} \left((\rho + \varepsilon) \log \frac{1}{r} \right), \quad j = 1, 2, \dots, k-1 \quad (2.1.22)$$

and

$$M_{z_0}(r, A_0) \geq \varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right). \quad (2.1.23)$$

By Lemma 1.3.7, there exist a set $E_1 \subset (0, r_1]$ ($r_1 \in (0, 1)$) that has a finite logarithmic measure and a constant $\lambda > 0$ that depends on $\alpha > 1$ and $j = 1, 2, \dots, k$ such that for all $r = |z - z_1|$ satisfying $r \in (0, r_1] \setminus E_1$, we obtain

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{1}{\alpha} r, f \right) \log T_{z_0} \left(\frac{1}{\alpha} r, f \right) \right]^j, \quad (j = 1, 2, \dots, k). \quad (2.1.24)$$

Substituting (2.1.22)-(2.1.24) into (2.1.21), for the above ε and $r \in (0, r_0) \cap (0, r_1] \setminus E_1$, we have

$$\begin{aligned} \varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right) &\leq \lambda k \left(\frac{1}{r^2} T_{z_0} \left(\frac{r}{\alpha}, f \right) \log T_{z_0} \left(\frac{r}{\alpha}, f \right) \right)^k \\ &\quad \times \varphi^{-1} \left((\rho + \varepsilon) \log \frac{1}{r} \right). \end{aligned} \quad (2.1.25)$$

By (2.1.25) we get

$$\varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right) \leq \lambda k \left(\frac{1}{r} T_{z_0} \left(\frac{r}{\alpha}, f \right) \right)^{2k} \varphi^{-1} \left((\rho + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.26)$$

Now, we prove that $\tilde{\mu}_\varphi^1(f, z_0) \geq \mu$. By contradiction, we suppose that

$$\mu_1 = \tilde{\mu}_\varphi^1(f, z_0) < \mu.$$

By Lemma 1.3.4, there exists a set $E_2 \subset (0, 1)$ having infinite logarithmic measure such that for all and all $|z - z_0| = r \in E_2$, and for any given $\varepsilon > 0$ we have

$$T_{z_0}(r, f) \leq \varphi^{-1} \left((\mu_1 + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.27)$$

By (2.1.26) and (2.1.27), for any given $\varepsilon \in (0, \min\{\frac{\rho - \alpha}{2}, \frac{\rho - \mu_1}{2}\})$ and $|z - z_0| = r \in (0, r_0) \cap (0, r_1] \cap E_2 \setminus E_1$, we obtain

$$\varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right) \leq \frac{\lambda k}{r^{2k}} \left(\varphi^{-1} \left((\mu_1 + \varepsilon) \log \frac{\alpha}{r} \right) \right)^{2k}$$

$$\times \varphi^{-1} \left((\rho + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.28)$$

Since $\mu - \varepsilon > \max\{\mu_1 + \varepsilon, \rho + \varepsilon\}$, we get

$$\varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right) \leq \frac{\lambda k}{r^{2k}} \left(\varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right) \right)^{2k+1}. \quad (2.1.29)$$

Applying the logarithm on the both sides of (2.1.29), we find

$$\begin{aligned} & \frac{2k \log r}{(2k+1) \log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right)} \\ & + \frac{\log \varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right)}{(2k+1) \log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right)} \\ & \leq \frac{\log \lambda k}{(2k+1) \log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \mu + \varepsilon\} \log \frac{\alpha}{r} \right)} + 1. \end{aligned} \quad (2.1.30)$$

Set $\varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right) = x$. Then $\log r = -\frac{\varphi(x)}{\max\{\mu_1 + \varepsilon, \rho + \varepsilon\}} + \log \alpha$ and by using Karata's theorem ([23]), $\varphi(e^t) = t^{o(1)}$, $t \rightarrow +\infty$, we immediately get

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{\log r}{\log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right)} \\ & = \lim_{x \rightarrow +\infty} \left(-\frac{\varphi(x)}{\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log x} + \frac{\log \alpha}{\log x} \right). \end{aligned} \quad (2.1.31)$$

Since $\varphi(x) = (\log x)^{o(1)}$, we have

$$\lim_{x \rightarrow +\infty} \left(\frac{(\log x)^{o(1)}}{\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log x} \right) = 0$$

and

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{\log \varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right)}{\log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right)} \\ & \geq \lim_{r \rightarrow 0} \frac{\log \varphi^{-1} \left(\frac{(\mu - \varepsilon)}{\max\{\mu_1 + \varepsilon, \rho + \varepsilon\}} \max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{1}{r} \right)}{\log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{1}{r} \right)} \\ & = \lim_{y \rightarrow +\infty} \frac{\log \varphi^{-1} \left((1 + \delta)y \right)}{\log \varphi^{-1} (y)} = +\infty, \end{aligned}$$

then

$$\lim_{r \rightarrow 0} \frac{\log \varphi^{-1} \left((\mu - \varepsilon) \log \frac{1}{r} \right)}{\log \varphi^{-1} \left(\max\{\mu_1 + \varepsilon, \rho + \varepsilon\} \log \frac{\alpha}{r} \right)} = +\infty. \quad (2.1.32)$$

The right hand side in (2.1.30) is finite, while the left hand side is infinite, thus a contradiction holds, i.e.,

$$\tilde{\mu}_\varphi^1(f, z_0) \geq \mu.$$

By (1.2.1), we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |A_{k-1}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \cdots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| + |A_0(z)|. \quad (2.1.33)$$

By Lemma 1.3.8, there exists a set $E_3 \subset (0, 1)$ that has a finite logarithmic measure, such that for all $j = 0, 1, \dots, k$ and $r \notin E_3$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = |1 + o(1)| \left(\frac{V_{z_0}(r, f)}{z_0 - z_r} \right)^j, \quad (2.1.34)$$

where z is a point in the circle $|z - z_0| = r$ that satisfies $|f(z)| = M_{z_0}(r, f)$. By Lemma 1.3.3, there exists a set $E_4 \subset (0, 1)$ having infinite logarithmic measure, such that for any given $\varepsilon > 0$ and for all $|z - z_0| = r \in E_4$, we have

$$|A_0(z)| \leq M_{z_0}(r, A_0) \leq \varphi^{-1} \left((\mu + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.35)$$

By substituting (2.1.22), (2.1.34) and (2.1.35) into (2.1.33), for any given $\varepsilon > 0$ and for all $|z - z_0| = r \in E_4 \cap (0, r_0) \setminus E_3$ and $|f(z)| = M_{z_0}(r, f)$, we get

$$\begin{aligned} |1 + o(1)| (V_{z_0}(r, f))^k &\leq kr\varphi^{-1} \left((\mu + \varepsilon) \log \frac{1}{r} \right) \\ &\times |1 + o(1)| (V_{z_0}(r, f))^{k-1}, \end{aligned} \quad (2.1.36)$$

then we obtain

$$V_{z_0}(r, f) \leq kr\varphi^{-1} \left((\mu + \varepsilon) \log \frac{1}{r} \right) |1 + o(1)|, \quad r \in E_4 \cap (0, r_0) \setminus E_3. \quad (2.1.37)$$

By Lemma 1.3.3, Lemma 1.3.9 and (2.1.37), we get $\tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\mu}_\varphi^0(A_0, z_0) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\mu}_\varphi^0(A_0, z_0), \quad (2.1.38)$$

from (2.1.38) and $\tilde{\mu}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(A_0, z_0)$, we obtain

$$\tilde{\mu}_\varphi^1(f, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0).$$

□

Proof of Theorem 1.2.10

Proof. By Theorem (1.2.8) we have $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)$. We only need to prove that $\tilde{\mu}_\varphi^1(f, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0)$.

Suppose that $(f \not\equiv 0)$ is a solution of equation (1.2.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. By an analogous progress, set $\tilde{\mu}_\varphi^0(A_0, z_0) = \mu$, $\tilde{\tau}_\varphi^0(A_0, z_0) = \tau$. If $\rho_2 = \max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, \dots, k-1\} < \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$, then by Theorem 1.2.7, we obtain $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0)$. Suppose that

$$\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, 2, \dots, k-1\} = \tilde{\mu}_\varphi^0(A_0, z_0) = \mu \quad (0 < \mu < +\infty)$$

and

$$\max\{\tilde{\tau}_\varphi^0(A_j, z_0) : \tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0) > 0\} < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau \quad (0 < \tau < +\infty).$$

Then, there exists a set $I \subseteq \{1, 2, \dots, k-1\}$ such that $\tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$ ($j \in I$) and $\tilde{\tau}_\varphi^0(A_j, z_0) < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau$ ($j \in I$). For any given $\varepsilon \in \left(0, \frac{\mu - \rho_2}{2}\right)$, there exists $r_4 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_4)$, we have

$$|A_j(z)| \leq \varphi^{-1} \left((\mu + \varepsilon) \log \frac{1}{r} \right) \quad (j \in I) \quad (2.1.39)$$

and

$$|A_j(z)| \leq \varphi^{-1} \left((\rho_2 + \varepsilon) \log \frac{1}{r} \right) \leq \varphi^{-1} \left((\mu + \varepsilon) \log \frac{1}{r} \right), \quad (j \in \{1, \dots, k-1\} \setminus I). \quad (2.1.40)$$

If $\tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0)$; $\tilde{\tau}_\varphi^0(A_1, z_0) \leq \tau_1 < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau$, then for any given $\varepsilon \in \left(0, \frac{\tau - \tau_1}{2}\right)$, there exist $r_5 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_5)$, we have

$$M_{z_0}(r, A_j) \leq \varphi^{-1} \left\{ \log \left((\tau_1 + \varepsilon) \left(\frac{1}{r} \right)^\mu \right) \right\} \quad (2.1.41)$$

and

$$M_{z_0}(r, A_0) \geq \varphi^{-1} \left\{ \log \left((\tau - \varepsilon) \left(\frac{1}{r} \right)^\mu \right) \right\}. \quad (2.1.42)$$

By substituting (2.1.24) and (2.1.39)-(2.1.42) into (2.1.21), then for any given

$$\varepsilon \in \left(0, \frac{\min(\mu - \rho_2, \tau - \tau_1)}{2}\right)$$

and for all $|z - z_0| = r \in (0, r_0] \cap (0, r_4) \cap (0, r_5) \setminus E_3$, has infinite logarithmic measure, we obtain

$$\begin{aligned} \varphi^{-1} \left\{ \log((\underline{\tau} - \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\} &\leq \lambda k \left(\frac{1}{r} T_{z_0} \left(\frac{r}{\alpha}, f \right) \right)^{2k} \\ &\varphi^{-1} \left\{ \log((\tau_1 + \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\}. \end{aligned} \quad (2.1.43)$$

The last inequality implies $\tilde{\mu}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$. To see why, we assume that $\tilde{\mu}_\varphi^1(f, z_0) < \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$. Then, there exists $r_6 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_6)$, we have

$$T_{z_0}(r, f) \leq \varphi^{-1} \left(\rho_2 \log \frac{1}{r} \right), \quad (2.1.44)$$

for some $\rho_2 < \mu$. Consequently, by (2.1.53) and (2.1.44) for all $|z - z_0| = r \in (0, r_0] \cap (0, r_4) \cap (0, r_5) \cap (0, r_6) \setminus E_3$, we get

$$\begin{aligned} \varphi^{-1} \left\{ \log((\underline{\tau} - \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\} &\leq \frac{\lambda k}{r^{2k}} \left(\varphi^{-1} \left(\rho_2 \log \frac{1}{r} \right) \right)^{2k} \varphi^{-1} \left\{ \log((\tau_1 + \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\} \\ &\leq \frac{\lambda k}{r^{2k}} \left(\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\} \right)^{2k+1}. \end{aligned}$$

Applying the logarithm on both sides, we find

$$\begin{aligned} &\frac{2k \log r}{(2k+1)\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\}} + \frac{\log(\varphi^{-1} \left\{ \log((\underline{\tau} - \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\})}{(2k+1)\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\}} \\ &\leq \frac{\log \lambda k}{(2k+1)\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\}} + 1. \end{aligned} \quad (2.1.45)$$

As we did before

$$\lim_{r \rightarrow 0} \frac{\log r}{\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\}} = 0$$

and

$$\lim_{r \rightarrow 0} \frac{\log(\varphi^{-1} \left\{ \log((\underline{\tau} - \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\})}{\varphi^{-1} \left\{ \log((\max(\rho_2, \tau_1 + \varepsilon)) \left(\frac{1}{r} \right)^\mu) \right\}} = +\infty$$

because $\underline{\tau} - \varepsilon > \max(\rho_2, \tau_1 + \varepsilon)$. Since the right hand side of the inequality (2.1.45) is bounded by 1, thus taking limits yielding $+\infty \leq 1$ which is a contradiction. Hence, we get $\tilde{\mu}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$.

By Lemma (1.3.6), there exists a set $E_2 \subset (0, 1)$ having infinite logarithmic measure, such that for all $|z - z_0| = r \in E_2$, we have

$$|A_0(z)| < \varphi^{-1} \left\{ \log((\underline{\tau} + \varepsilon) \left(\frac{1}{r} \right)^\mu) \right\}. \quad (2.1.46)$$

By combining (2.1.33), (2.1.34), (2.1.39), (2.1.40), (2.1.41) and (2.1.46) for all $|z - z_0| = r \in E_2 \cap (0, r_4) \cap (0, r_5) \setminus E_4$, $r \rightarrow 0$ and $|f(z)| = M_{z_0}(r, f)$, we have

$$|1 + o(1)| (V_{z_0}(r, f))^k \leq kr\varphi^{-1} \left\{ \log((\underline{\tau} + \varepsilon) \left(\frac{1}{r}\right)^\mu) \right\} \\ |1 + o(1)| (V_{z_0}(r, f))^{k-1},$$

so

$$V_{z_0}(r, f) \leq kr\varphi^{-1} \left\{ \log((\underline{\tau} + \varepsilon) \left(\frac{1}{r}\right)^\mu) \right\} |1 + o(1)|. \quad (2.1.47)$$

By Lemma (1.3.3), Lemma (1.3.9) and (2.1.47), we obtain $\tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\mu}_\varphi^0(A_0, z_0) = \mu$.

□

Proof of Theorem 1.2.11

Proof. By (1.2.1), we have

$$m_{z_0}(r, A_0) \leq \sum_{j=1}^{k-1} m_{z_0}(r, A_j) + \sum_{j=1}^k m_{z_0} \left(r, \frac{f^{(j)}(z)}{f(z)} \right) + O(1). \quad (2.1.48)$$

By Lemma (1.3.10), for a constant $r_2 \in (0, 1)$, there is a set $E_6 \subset (0, r_2]$ with $m_l(E_6) < +\infty$ such that for all $|z - z_0| = r \in (0, r_2] \setminus E_6$, we have

$$\sum_{j=1}^k m_{z_0} \left(r, \frac{f^{(j)}(z)}{f(z)} \right) \leq O \left(T_{z_0}(r, f) + \log \frac{1}{r} \right). \quad (2.1.49)$$

Setting $\limsup_{r \rightarrow 0} \frac{\sum_{j=1}^{k-1} m_{z_0}(r, A_j)}{m_{z_0}(r, A_0)} < \beta < 1$. Then for $r \rightarrow 0$, we have

$$\sum_{j=1}^{k-1} m_{z_0}(r, A_j) < \beta m_{z_0}(r, A_0). \quad (2.1.50)$$

By substituting (2.1.49), (2.1.50) into (2.1.48) we obtain for all $|z - z_0| = r \in (0, r_2] \setminus E_6$, $r \rightarrow 0$

$$(1 - \beta)m_{z_0}(r, A_0) \leq O \left(T_{z_0}(r, f) + \log \frac{1}{r} \right). \quad (2.1.51)$$

By the definition of lower φ -order, for any given $\varepsilon > 0$, there exists $r_6 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_6)$, we have

$$m_{z_0}(r, A_0) = T_{z_0}(r, A_0) \geq \log(\varphi^{-1}((\mu - \varepsilon) \log(\frac{1}{r}))). \quad (2.1.52)$$

By (2.1.51) and (2.1.52), for any given $\varepsilon > 0$ and $|z - z_0| = r \in (0, r_2] \cap (0, r_6) \setminus E_6, r \rightarrow 0$, we obtain

$$(1 - \beta) \log(\varphi^{-1}((\mu - \varepsilon) \log(\frac{1}{r}))) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right). \quad (2.1.53)$$

By Definition (1.2.6), Lemma (1.3.9) and (2.1.53), we have $\tilde{\mu}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(A_0, z_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\tilde{\mu}_\varphi^1(f, z_0) \geq \tilde{\mu}_\varphi^0(A_0, z_0). \quad (2.1.54)$$

Set $\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, \dots, k-1\} = \rho_3 \leq \tilde{\mu}_\varphi^0(A_0, z_0) \leq \tilde{\rho}_\varphi^0(A_0, z_0)$. Then for any given $\varepsilon > 0$, there exists $r_7 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_7)$, we have

$$M_{z_0}(r, A_j) \leq \varphi^{-1}\left((\rho_3 + \varepsilon) \log \frac{1}{r}\right) \leq \varphi^{-1}\left((\mu + \varepsilon) \log \frac{1}{r^\rho}\right), \quad (2.1.55)$$

$$(j \in \{1, \dots, k-1\} \setminus I).$$

By substituting (2.1.34), (2.1.35) and (2.1.55) into (2.1.33), for any given $\varepsilon > 0$ and for all $|z - z_0| = r \in E \cap (0, r_7) \setminus E_4$ and $|f(z)| = M_{z_0}(r, f)$, we get

$$|1 + o(1)| (V_{z_0}(r, f))^k \leq kr\varphi^{-1}\left((\mu + \varepsilon) \log \frac{1}{r}\right) \\ \times |1 + o(1)| (V_{z_0}(r, f))^{k-1}, \quad (2.1.56)$$

which implies

$$V_{z_0}(r, f) \leq kr\varphi^{-1}\left((\mu + \varepsilon) \log \frac{1}{r}\right) |1 + o(1)|, \quad r \in E \cap (0, r_3) \setminus E_4, \quad (2.1.57)$$

where $|z - z_0| = r \in E \cap (0, r_7) \setminus E_4$ and $|f(z)| = M_{z_0}(r, f)$. By (2.1.57), Lemma (1.3.9) and Lemma (1.3.3), we obtain

$$\tilde{\mu}_\varphi^1(f, z_0) \leq \tilde{\mu}_\varphi^0(A_0, z_0). \quad (2.1.58)$$

Thus, from (2.1.58) and (2.1.54), we have

$$\tilde{\mu}_\varphi^1(f, z_0) = \tilde{\mu}_\varphi^0(A_0, z_0).$$

By using similar method, from (2.1.51), we have for $|z - z_0| = r \in (0, r_2] \setminus E_6, r \rightarrow 0$

$$(1 - \beta)m_{z_0}(r, A_0) = (1 - \beta)T_{z_0}(r, A_0) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right). \quad (2.1.59)$$

By Lemma 1.3.5, there exists a set $E_1 \subset (0, 1)$ having infinite logarithmic measure such that for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1$

$$T_{z_0}(r, A_0) \geq \varphi^{-1} \left((\rho_\varphi^0(A_0, z_0) - \varepsilon) \log \frac{1}{r} \right). \quad (2.1.60)$$

By substituting (2.1.60) into (2.1.59), we obtain for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1 \cap (0, r_2] \setminus E_6$, $r \rightarrow 0$

$$\begin{aligned} (1 - \beta) \varphi^{-1} \left((\rho_\varphi^0(A_0, z_0) - \varepsilon) \log \frac{1}{r} \right) &\leq (1 - \beta) T_{z_0}(r, A_0) \\ &\leq O \left(T_{z_0}(r, f) + \log \frac{1}{r} \right). \end{aligned} \quad (2.1.61)$$

Making use of Lemma 1.3.9 and Lemma 1.3.3, from (2.1.61) by Remark 1.2.1, we get

$$\tilde{\rho}_\varphi^1(f, z_0) \geq \tilde{\rho}_\varphi^0(A_0, z_0). \quad (2.1.62)$$

By the definition of the φ -order of $\tilde{\rho}_\varphi^0(A_0, z_0)$ for any given $\varepsilon > 0$, there exists $r_8 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_8)$, we have

$$M_{z_0}(r, A_0) \leq \varphi^{-1} \left((\tilde{\rho}_\varphi^0(A_0, z_0) + \varepsilon) \log \frac{1}{r} \right). \quad (2.1.63)$$

Also, by substituting (2.1.34), (2.1.55), (2.1.63) into (2.1.33), for any given $\varepsilon > 0$ and for all $|z - z_0| = r \in (0, r_7) \cap (0, r_8) \setminus E_4$ and $|f(z)| = M_{z_0}(r, f)$, we can find that

$$V_{z_0}(r, f) \leq kr \varphi^{-1} \left((\rho_\varphi^0(A_0, z_0) + \varepsilon) \log \frac{1}{r} \right) |1 + o(1)|. \quad (2.1.64)$$

By using lemma 1.3.13, Lemma 1.3.9 and (2.1.64), we get

$$\tilde{\rho}_\varphi^1(f, z_0) \leq \tilde{\rho}_\varphi^0(A_0, z_0). \quad (2.1.65)$$

Thus, from (2.1.62) and (2.1.65), we conclude that

$$\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

The proof is complete □

Examples

Here, we provide some examples that illustrate all what we did before.

Example 3.0.1 Consider the equation

$$f'' + \left(\frac{3}{z - z_0} + \frac{2}{(z - z_0)^3}\right)f' - \frac{4}{(z - z_0)^9}e^{\frac{2}{(z - z_0)^2}}f = 0, \quad (3.0.1)$$

It is not hard to see that $f(z) = \exp(\exp(\frac{1}{(z - z_0)^2}))$ which is analytic in $\mathbb{C} - \{z_0\}$ is a solution for (3.0.1). Notice that, the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\tilde{\mu}_\varphi^0(A_1, z_0) = 0, \quad \tilde{\mu}_\varphi^0(A_0, z_0) = 2,$$

where

$$A_0(z) = \frac{4}{(z - z_0)^9}e^{\frac{2}{(z - z_0)^2}}, \text{ and } A_1(z) = \frac{3}{z - z_0} + \frac{2}{(z - z_0)^3}.$$

We have $\tilde{\mu}_\varphi^0(A_1, z_0) = 0 \leq \tilde{\mu}_\varphi^0(A_2, z_0) = 2 < +\infty$. On the other hand, a simple computation gives

$$M_{z_0}(r, f) = e^{e^{\frac{1}{r^2}}}.$$

Therefore

$$\tilde{\rho}_\varphi^1(f, z_0) = \limsup_{r \rightarrow 0} \frac{\varphi(\log M_{z_0}(r, f))}{\log \frac{1}{r}} = \limsup_{r \rightarrow 0} \frac{\varphi(\log \log \log e^{e^{\frac{1}{r^2}}})}{\log \frac{1}{r}} = 2.$$

This emphasizes the conclusion of the first Theorem 1.2.4

Example 3.0.2 Consider the equation

$$f'' + \left(\frac{1}{z} + \frac{2}{z^3}\right)e^{\frac{1}{z^2}} f' - \left[\left(\frac{6}{z^4} + \frac{4}{z^9}\right)e^{\frac{1}{z^2}} - \frac{2}{z^4}e^{\frac{2}{z^2}}\right]f = 0, \quad (3.0.2)$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{z^2}\right)$ which is analytic in $\overline{\mathbb{C}} \setminus \{0\}$ is a solution for (3.0.2). Notice that, the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\tilde{\mu}_\varphi^0(A_1, 0) = \tilde{\mu}_\varphi^0(A_0, 0) = 2.$$

By using Lemma 1.3.17 we find

$$\tilde{\tau}_\varphi^0(A_1, 0) = 1 < \tilde{\tau}_\varphi^0(A_0, 0) = 2 < +\infty,$$

where

$$A_0(z) = -\left[\left(\frac{6}{z^4} + \frac{4}{z^9}\right)e^{\frac{1}{z^2}} - \frac{2}{z^4}e^{\frac{2}{z^2}}\right], \text{ and } A_1(z) = \left(\frac{1}{z} + \frac{2}{z^3}\right)e^{\frac{1}{z^2}}.$$

It is clear that the conditions of Theorem 1.2.5 are satisfied and we see that

$$\tilde{\rho}_\varphi^1(f, 0) = 2 \geq 2 = \tilde{\mu}_\varphi^0(A_0, 0).$$

Example 3.0.3 Consider the equation

$$f'' - \left(\frac{2}{z}e^{-\frac{1}{\alpha z}} + \frac{1}{\alpha z^2}\right)f' - \frac{2}{\alpha z^3}f = 0, \quad (3.0.3)$$

It is not hard to see that $f(z) = e^{\frac{1}{\alpha z}} - 1$, where $\alpha \in \mathbb{R}^*$ which is analytic in $\overline{\mathbb{C}} \setminus \{0\}$ is a solution for (3.0.3). Notice that, the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\tilde{\mu}_\varphi^0(A_0, 0) = 0 < \tilde{\mu}_\varphi^0(A_1, 0) = 1 < \infty,$$

where

$$A_0(z) = -\frac{2}{\alpha z^3}, \text{ and } A_1(z) = -\left(\frac{2}{z}e^{-\frac{1}{\alpha z}} + \frac{1}{\alpha z^2}\right).$$

It is clear that the conditions of Theorem 1.2.6 are satisfied and we see that

$$\tilde{\mu}_\varphi^0(A_1, 0) = 1 \leq 1 = \tilde{\rho}_\varphi^0(f, 0).$$

While simple calculations give

$$\begin{aligned}\tilde{\rho}_\varphi^0(f, 0) &= \limsup_{r \rightarrow 0} \frac{\varphi\left(\log \log e^{e^{\frac{1}{\alpha r}-1}}\right)}{\log \frac{1}{r}} = 1, \\ \tilde{\rho}_\varphi^1(f, 0) &= \limsup_{r \rightarrow 0} \frac{\varphi(\log_3 e^{e^{\frac{1}{\alpha r}-1}})}{\log \frac{1}{r}} = 0.\end{aligned}$$

Example 3.0.4 Consider the equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0, \quad (3.0.4)$$

where

$$\begin{aligned}A_0(z) &= \left(\frac{24}{z^7} + \frac{16}{z^9} + \frac{8}{z^9}e^{\frac{1}{z^2}}\right)e^{\frac{1}{z^2}}, \\ A_1(z) &= -\left(\frac{3}{z^2} + \frac{6}{z^4}\right), \\ A_2(z) &= \frac{3}{z^2} + \frac{2}{z^3}.\end{aligned}$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{z^2}\right)$, which is analytic in $\overline{\mathbb{C}} \setminus \{0\}$ is a solution for (3.0.4). Notice that the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that is, φ is unbounded, increasing, and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand-wavy calculation gives

$$\max\{\tilde{\rho}_\varphi^0(A_1, 0), \tilde{\rho}_\varphi^0(A_2, 0)\} = 0 < \tilde{\mu}_\varphi^0(A_0, 0) = 2 \leq 2 = \tilde{\rho}_\varphi^0(A_0, 0) < +\infty.$$

We see that A_0 is a dominant coefficient, so by Theorem 1.2.9, we conclude that

$$\tilde{\rho}_\varphi^1(f, 0) = \tilde{\mu}_\varphi^0(A_0, 0) = 2.$$

On the other hand, a simple computation gives

$$M_0(r, f) = e^{e^{\frac{1}{r^2}}}.$$

Therefore

$$\tilde{\rho}_\varphi^1(f, 0) = \limsup_{r \rightarrow 0} \frac{\varphi(\log M_0(r, f))}{\log \frac{1}{r}} = \limsup_{r \rightarrow 0} \frac{\varphi(\log \log \log e^{e^{\frac{1}{r^2}}})}{\log \frac{1}{r}} = 2.$$

This emphasizes the conclusion of Theorem 1.2.9.

Example 3.0.5 Consider the equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0, \quad (3.0.5)$$

where

$$\begin{aligned} A_0(z) &= \left(\frac{12}{z^7} + \frac{8}{z^9}\right)e^{\frac{2}{z^2}} + \frac{8}{z^9}e^{\frac{3}{z^2}}, \\ A_1(z) &= -\left(\frac{3}{z^2} + \frac{6}{z^4} + \left(\frac{6}{z^4} + \frac{4}{z^6}\right)e^{\frac{1}{z^2}}\right), \\ A_2(z) &= \frac{3}{z} + \frac{2}{z^2}. \end{aligned}$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{z^2}\right)$, which is analytic in $\overline{\mathbb{C}} \setminus \{0\}$ is a solution for (3.0.5).

Notice that the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that is, φ is unbounded, increasing, and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand-wavy calculation gives

$$\max\{\tilde{\rho}_\varphi^0(A_1, 0), \tilde{\rho}_\varphi^0(A_2, 0)\} = 2 \leq 2 = \tilde{\mu}_\varphi^0(A_0, 0) \leq \tilde{\rho}_\varphi^0(A_0, 0) = 2 < +\infty.$$

By using Lemma 1.3.17 we find

$$\max\{\tilde{\tau}_\varphi^0(A_1, 0), \tilde{\tau}_\varphi^0(A_2, 0)\} = 1 < \tilde{\tau}_\varphi^0(A_0, 0) = 3 < +\infty.$$

Lossly speaking A_0 is a dominant coefficient so by Theorem 1.2.10, we conclude that

$$\tilde{\rho}_\varphi^1(f, 0) = \tilde{\mu}_\varphi^0(A_0, 0) = 2.$$

This confirm the conclusion of Theorem 1.2.10.

Example 3.0.6 Consider the equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0, \quad (3.0.6)$$

where

$$\begin{aligned} A_0(z) &= \frac{1}{\alpha^3 z^6} e^{\frac{3}{\alpha z}}, \\ A_1(z) &= -\left(\frac{1}{z^2} + \frac{2}{\alpha z^3} + \left(\frac{3}{\alpha z^3} + \frac{2}{\alpha^2 z^4}\right)e^{\frac{1}{\alpha z}}\right), \\ A_2(z) &= \frac{1}{z} + \frac{1}{\alpha z^2}. \end{aligned}$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{\alpha z}\right)$, where $\alpha \in \mathbb{R}^*$ which is analytic in $\overline{\mathbb{C}} \setminus \{0\}$ is a solution for (3.0.6). Notice that, the function $\varphi(t) = \log \log t = \log_2 t$ is a function of ϕ , that

is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\max\{\tilde{\rho}_\varphi^0(A_1, 0), \tilde{\rho}_\varphi^0(A_2, 0)\} = 1 \leq 1 = \tilde{\mu}_\varphi^0(A_0, 0) = \tilde{\rho}_\varphi^0(A_0, 0) < +\infty$$

and

$$\limsup_{r \rightarrow 0} \frac{m_0(r, A_1) + m_0(r, A_2)}{m_0(r, A_0)} = \frac{1}{3} < 1.$$

Obviously the conditions of Theorem [1.2.11](#) are verified and we see that

$$\tilde{\mu}_\varphi^0(A_0, 0) = \tilde{\mu}_\varphi^1(f, 0) = 1 \leq 1 = \tilde{\rho}_\varphi^1(f, 0) = \tilde{\rho}_\varphi^0(A_0, 0).$$

CONCLUSION

Throughout this master thesis, by using the concepts of lower φ -order and lower φ -type, we obtain some growth properties of solutions of second order and higher order linear differential equations in which the coefficients are analytic functions in the whole extended punctured plane denoted by $\overline{\mathbb{C}} - \{z_0\}$, where z_0 is an essential singularity. We improve and extend some results recently obtained by Khedim and Belaïdi ([16]). Inspired by the results already established by Long, Qin and Tao ([20]), one may explore analogous theorems for differential equations in which the coefficients are analytic functions in the whole extended punctured plane around singular point of $[p, q]_{\varphi}$ -order. Furthermore, we can also study differential polynomials generated by solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0.$$

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الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي والبحث العلمي

جامعة عبد الحميد بن باديس - مستغانم -
كلية العلوم الدقيقة والإعلام الآلي

تصريح شرفي خاص بالالتزام بقواعد النزاهة العلمية
لإنجاز البحث

أنا الممضي أدناه،
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شعبة : / التخصص : تحليل دالي

والمكلف بإنجاز مذكرة ماستر بعنوان:

..... Complex Linear Differential Equations with Domain-
..... Coefficient of Lower Order Around a Singular
..... Point

أصرح بشرفي أنني ألتزم بمراعاة المعايير العلمية والمنهجية ومعايير الأخلاقيات العلمية
والنزاهة الأكاديمية المطلوبة في إنجاز البحث ، وأتحمل المسؤولية الشخصية عن كل
المحتوى المتضمن في البحث المذكور أعلاه .

التاريخ:/...../.....

إمضاء المعني



* ملحق القرار الوزاري رقم 933 المؤرخ في 28 جويلية 2016 الذي يحدد القواعد المتعلقة بالوقاية من السرقة العلمية ومكافحتها.