Applied Mathematics Letters 22 (2009) 268–275

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Multiple positive solutions for elliptic equations involving a concave term and critical Sobolev–Hardy exponent

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ARTICLE INFO

Article history:
Received 7 March 2007
Received in revised form 18 September 2007
Accepted 24 March 2008

Keywords:
Positive solutions
Variational methods
Concave term
Critical Sobolev–Hardy exponent
Singularity

ABSTRACT

In this paper, we establish the existence of multiple positive solutions for elliptic equations involving a concave term and critical Sobolev–Hardy exponent.

1. Introduction

This paper deals with the existence and multiplicity of weak solutions to the following problem

$$(\mathcal{P}_{\mu,s})\begin{cases}-\Delta u-\dfrac{\mu}{|x|^2}u=\lambda u^{q-1}+\dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega\backslash\{0\} \quad (\mathrm{a})_\mu\\ u>0 & \text{in } \Omega\backslash\{0\} \quad (\mathrm{b})\\ u=0 & \text{on } \partial\Omega \quad (\mathrm{c})\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is an open bounded domain with smooth boundary, $0\in\Omega$, $0\leq\mu<\overline{\mu}:=\left(\frac{N-2}{2}\right)^2$ which is the best constant in the Hardy inequality, $1<q<2$, $0\leq s<2$, $2^*(s)=\frac{2(N-s)}{N-2}$ is the so-called critical Sobolev–Hardy exponent and λ is a positive parameter.

We start by giving a brief historic.

Ambrosetti et al. [1] have studied problem $(\mathcal{P}_{0,0})$. They proved that there exists $\Lambda>0$ such that $\left(\mathcal{P}_{0,0}\right)$ has at least two positive solutions for all $\lambda\in(0,\Lambda)$. To obtain a first positive solution, they used sub-super solutions method and applied the Mountain Pass Theorem to obtain a second positive solution.

In the case $q=2$, $s=0$. If $0\leq\mu<(\frac{N-2}{2})^2-4(N\geq 7)$, Cao and Peng [4] established a pair of sign-changing solutions for problem $(\mathrm{a})_0$–(c) with $0<\lambda<\lambda_1$, here λ_1 is the first eigenvalue of $-\Delta-\frac{\mu}{|x|^2}$. Subsequently, Cao and Han [3] proved that if $0\leq\mu<(\frac{N-2}{2})^2-(\frac{N+2}{N})^2$ $(N\geq 5)$, then, for all $\lambda>0$ there exists a nontrivial solution for problem $(\mathrm{a})_0$–(c) with critical level in the range $(0,\frac{1}{N}S_\mu^{\frac{N}{2}})$. Relevant papers on this matter see [6,9–11].

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doi:10.1016/j.aml.2008.03.024

In the case $\mu > 0$ and $s = 0$, Chen [5] studied the asymptotic behavior of solutions by using Moser's iteration method, and he gave the following existence results:

- Existence of a local minimizer of associated energy functional to $(\mathcal{P}_{\mu,0})$, and
- Existence of a second positive solution by variational methods.

The case $\lambda = 0$ and $0 < s < 2$, Kang et al. [12] proved that, for $\varepsilon > 0$ and $\beta = \sqrt{\overline{\mu} - \mu}$, the functions

$$U_\varepsilon(x) = \frac{C_\varepsilon}{|x|^{\sqrt{\overline{\mu}}-\beta}\left(\varepsilon + |x|^{(2-s)\beta/\sqrt{\overline{\mu}}}\right)^{(N-2)/(2-s)}} \quad \text{with } C_\varepsilon = \left(\frac{2\varepsilon(\overline{\mu}-\mu)(N-s)}{\sqrt{\overline{\mu}}}\right)^{\sqrt{\overline{\mu}}/(2-s)}, \tag{1.1}$$

solve the equation

$$-\Delta u - \frac{\mu}{|x|^2}u = \frac{|u|^{2^*(s)-2}}{|x|^s}u \quad \text{in } \mathbb{R}^N\backslash\{0\},$$

and satisfy

$$\int_{\mathbb{R}^N}\left(|\nabla U_\varepsilon|^2 - \mu\frac{|U_\varepsilon(x)|^2}{|x|^2}\right)\mathrm{d}x = \int_{\mathbb{R}^N}\frac{|U_\varepsilon(x)|^{2^*(s)}}{|x|^s}\mathrm{d}x = S_{\mu,s}^{(N-s)/(2-s)}, \tag{1.2}$$

with $S_{\mu,s}$ is the best constant defined as

$$S_{\mu,s} := \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)\mathrm{d}x}{\left(\int_\Omega\frac{|u(x)|^{2^*(s)}}{|x|^s}\mathrm{d}x\right)^{2\backslash 2^*(s)}} \tag{1.3}$$

which is independent of Ω.

A natural interesting question is whether the results concerning the solutions of $(\mathcal{P}_{\mu,0})$ remain true for $(\mathcal{P}_{\mu,s})$. Borrowing ideas from [5], we give a positive answer.

The main result of this paper is

Theorem 1. *Suppose that $0 \le \mu < \overline{\mu} - 1$ and $0 \le s < 2$, then there exists $\Lambda > 0$ such that $(\mathcal{P}_{\mu,s})$ has at least two positive solutions in $H_0^1(\Omega)$ for any $\lambda \in (0, \Lambda)$.*

This paper is organized as follows. In Section 2 we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

$L^p\left(\Omega, |x|^t\mathrm{d}x\right)$, $1 \le p < +\infty$ and $-2 \le t < 0$, denote weighted Lebesgue Sobolev spaces with norm the $|.|_{p,t}$; $H_0^1(\Omega)$ endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^2\,\mathrm{d}x)^{\frac{1}{2}}$, B_r denotes the ball of radius r centred at the origin and C denotes various generic positive constants.

The following lemma is essentially due to Caffarelli et al. [2].

Lemma 1. *Suppose that $0 \le s < 2$ and $0 \le \mu < \overline{\mu}$. Then for all $u \in H_0^1(\Omega)$,*

(i) $\int_\Omega \frac{u^2}{|x|^2}\mathrm{d}x \le \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^2\,\mathrm{d}x$
(ii) *there exists a constant $C > 0$ such that*

$$\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\mathrm{d}x \le C\int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

For any $\mu \in [0, \overline{\mu})$ fixed, we consider $H_\mu(\Omega)$ be the space $H_0^1(\Omega)$ endowed with the following scalar product

$$\langle u, v\rangle = \int_\Omega\left(\nabla u.\nabla v - \mu\frac{uv}{|x|^2}\right)\mathrm{d}x, \quad \forall u, v \in H_\mu(\Omega).$$

In view of Lemma 1 (i), the induced norm

$$\|u\|_\mu := \left(\int_\Omega\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)\mathrm{d}x\right)^{\frac{1}{2}}$$

is equivalent to the standard norm $\|u\|$ of $H_0^1(\Omega)$.

The corresponding energy functional to problem $(\mathcal{P}_{\mu,s})$ is defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{\lambda}{q}\int_\Omega \left(u^+\right)^q \mathrm{d}x - \frac{1}{2^*(s)}\int_\Omega \frac{\left(u^+\right)^{2^*(s)}}{|x|^s}\mathrm{d}x, \quad u \in H_\mu(\Omega)$$

where $u^+ = \max(u, 0)$.

From Lemma 1(ii), $I_\lambda(u)$ is well defined and of class C^1 on $H_\mu(\Omega)$.

$u \in H_\mu(\Omega)$ is said to be a weak solution of problem $(\mathcal{P}_{\mu,s})$ if

$$\int_\Omega \left(\nabla u \nabla v - \frac{\mu}{|x|^2}uv - \lambda\left(u^+\right)^{q-1}v - \frac{\left(u^+\right)^{2^*(s)-1}}{|x|^s}v\right)\mathrm{d}x = 0, \quad \forall v \in H_\mu(\Omega),$$

and by the standard elliptic regularity argument, we have that $u \in C^2(\Omega\backslash\{0\})$.

3. Proof of Theorem 1

The proof of Theorem 1 is given in two parts, we start by proving the existence of a first positive solution by using the concentration-compactness method [13]. Moreover the second positive solution is given by applying the Mountain Pass Theorem.

3.1. Existence of a first positive solution

In this subsection, we prove that there is $\Lambda > 0$ such that I_λ can achieve a local minimizer for any $\lambda \in (0, \Lambda)$. In order to check a local Palais–Smale condition, we use the concentration-compactness method. More precisely we have the following result.

Lemma 2. *There exists a constant $C = C(N, \Omega, q, s)$ such that, for all sequence $(u_n) \subset H_\mu(\Omega)$ satisfying*

$$I_\lambda(u_n) \leq c < \frac{2-s}{2(N-s)}S_{\mu,s}^{(N-s)/(2-s)} - C(N, \Omega, q, s) \tag{3.1}$$

and

$$I_\lambda^{'}(u_n) \to 0 \quad \textit{in } \left(H_\mu(\Omega)\right)' \textit{ (dual of } H_\mu(\Omega)). \tag{3.2}$$

Then there exists a subsequence strongly convergent in $H_\mu(\Omega)$.

Proof. From (3.1) and (3.2) we deduce that (u_n) is bounded. Up to a subsequence if necessary, we have that

(1) $u_n \rightharpoonup u_\lambda$ in $H_\mu(\Omega)$,
(2) $u_n \to u_\lambda$ in $L^t(\Omega)$, for $1 \leq t < 2^*$ and a.e in Ω,
(3) $u_n \rightharpoonup u_\lambda$ in $L^2(\Omega, |x|^{-2}\mathrm{d}x)$,
(4) $u_n \rightharpoonup u_\lambda$ in $L^{2^*(s)}(\Omega, |x|^{-s}\mathrm{d}x)$.

Using the concentration-compactness lemma of Lions and Sobolev–Hardy inequality we get a subsequence, still denoted by (u_n) such that

(a) $|\nabla u_n|^2 - \mu\frac{|u_n|^2}{|x|^2} \rightharpoonup \mathrm{d}\mu \geq |\nabla u_\lambda|^2 - \mu\frac{|u_\lambda|^2}{|x|^2} + \sum_{j\in J}\mu_j\delta_{x_j}$,
(b) $frac|u_n|^{2^*(s)}\,|x|^s \rightharpoonup \mathrm{d}\nu = \frac{|u_\lambda|^{2^*(s)}}{|x|^s} + \sum_{j\in J}\nu_j\delta_{x_j}$,
(c) $\nu_j^{\frac{2}{2^*(s)}} \leq S_{\mu,s}^{-1}\mu_j$ for all $j \in J$, where J is at most countable.

Thus we have the following consequence.

Claim 1. *Either $\mu_j = 0$ or $\mu_j \geq S_{\mu,s}^{(N-s)/(2-s)}$ for all $j \in J$.*

Proof of claim. We assume that there exist some $j \in J$ such that $\mu_j \neq 0$. Let $\varepsilon > 0$ and Φ a cut-off function centred at x_j with

$$\Phi(x) = \begin{cases} 1 & \text{if} \quad |x - x_j| \leq \frac{1}{2}\varepsilon, \\ 0 & \text{if} \quad |x - x_j| \geq \varepsilon, \end{cases} \quad \text{and} \quad |\nabla\Phi| \leq \frac{4}{\varepsilon}.$$

Then we get

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\langle I_\lambda^{'}(u_n), \Phi u_n\right\rangle \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^2\,\Phi \mathrm{d}x + \int_\Omega u_n\nabla u_n \nabla\Phi \mathrm{d}x - \mu\int_\Omega \frac{u_n^2}{|x|^2}\Phi \mathrm{d}x - \int_\Omega \frac{u_n^{2^*(s)}}{|x|^s}\Phi \mathrm{d}x - \lambda\int_\Omega u_n^q\Phi\, \mathrm{d}x\right) \\
&\geq \mu_j - \nu_j.
\end{aligned}
$$

and by (c) we deduce that $\mu_j \geq S_{\mu,s}^{(N-s)/(2-s)}$.

Then as a conclusion $\mu_j = 0$ or $\mu_j \geq S_{\mu,s}^{(N-s)/(2-s)}$.

From (3.1) and (3.2) we have

$$
I_\lambda(u_n) - \frac{1}{2^*(s)}(I_\lambda^{'}(u_n), u_n) = \frac{2-s}{2(N-s)}\,\|u_n\|_\mu^2 - \lambda\left(\frac{1}{q} - \frac{1}{2^*(s)}\right)\int_\Omega (u_n^+)^q \mathrm{d}x.
$$

Then, by Sobolev inequality, we find that

$$
I_\lambda(u_n) - \frac{1}{2^*(s)}(I_\lambda^{'}(u_n), u_n) \geq \frac{2-s}{2(N-s)}\,\|u_n\|_\mu^2 - \lambda\left(\frac{1}{q} - \frac{1}{2^*(s)}\right) C\,\|u_n\|_\mu^q\,.
$$

Thus there exists $C := C(N, \Omega, q, s)$ such that

$$
\frac{2-s}{2(N-s)}t^2 - \lambda\left(\frac{1}{q} - \frac{1}{2^*(s)}\right) Ct^q \geq -C(N, \Omega, q, s)\lambda^{2/(2-q)} \quad \forall t \geq 0.
$$

If we assume that $\mu_j \neq 0$ for some $j \in J$, then

$$
\begin{aligned}
c &\geq \frac{2-s}{2(N-s)}S_{\mu,s}^{(N-s)/(2-s)} + \frac{2-s}{2(N-s)}\,\|u_\lambda\|_\mu^2 - \lambda\left(\frac{1}{q} - \frac{1}{2^*(s)}\right)\int_\Omega (u_\lambda^+)^q \mathrm{d}x \\
&\geq \frac{2-s}{2(N-s)}S_{\mu,s}^{(N-s)/(2-s)} - C(N, \Omega, q, s)\lambda^{2/(2-q)},
\end{aligned}
$$

which contradicts our assumption. Hence $u_n \to u_\lambda$, as n goes to $+\infty$, strongly in $H_\mu(\Omega)$. □

The geometry conditions of I_λ will be obtained after the following computations. Using the Sobolev and Sobolev–Hardy inequalities, we obtain

$$
I_\lambda\,(u) \geq \frac{1}{2}\,\|u\|_\mu^2 - \frac{\lambda C}{q}\,\|u\|_\mu^q - \frac{C}{2^*(s)}\,\|u\|_\mu^{2^*(s)}\,,
$$

Let $\|u\|_\mu = \rho$, then we have

$$
I_\lambda\,(u) \geq \frac{1}{2}\rho^2 - \frac{\lambda C}{q}\rho^q - \frac{C}{2^*(s)}\rho^{2^*(s)}.
$$

Hence we can choose ρ_0 and Λ such that, for $\lambda \in (0, \Lambda)$, $I_\lambda\,(u)$ is bounded from below in B_{ρ_0} and $I_\lambda(u) \geq r > 0$ for $\|u\|_\mu = \rho_0$.

Let $\Phi \in H_\mu(\Omega)$ such that $\|\Phi\|_\mu = 1$. Then, for $t > 0$, we have

$$
I_\lambda\,(t\Phi) = \frac{t^2}{2} - \frac{\lambda t^q}{q}\int_\Omega (\Phi^+)^q \mathrm{d}x - \frac{t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{(\Phi^+)^{2^*(s)}}{|x|^s}\mathrm{d}x.
$$

So there is t_0 such that for $0 < t < t_0$, $I_\lambda\,(t\Phi) < 0$. Then

$$
I := \inf_{u\in B_{\rho_0}} I_\lambda\,(u) < 0.
$$

Lemma 2 implies that I_λ can achieves its minimum I at u_λ, i.e. $I = I_\lambda\,(u_\lambda)$. Moreover u_λ satisfies $(\mathcal{P}_{\mu,s})$.

3.2. Existence of a second positive solution

To prove the existence of a second positive solution we need the following proposition.

Proposition 1. *For any solutions $u \in C^2\,(\Omega \setminus \{0\})$ of $(\mathcal{P}_{\mu,s})$ there exists a positive constant M such that*

$$
u(x) \geq M|x|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})},
$$

hold for any $x \in B_\rho\,(0) \setminus \{0\}$ with ρ is sufficiently small.

Proof. The proof is similar to ([5], Proposition 3.1). □

For fixed $\lambda \in (0, \Lambda)$ we look for a second solution of $(\mathcal{P}_{\mu,s})$ in the form $u = u_\lambda + v$ where u_λ is found in the previous subsection and $v > 0$ in $\Omega \setminus \{0\}$. The corresponding equation for v becomes

$$-\Delta v - \frac{\mu}{|x|^2} v = \lambda (u_\lambda + v)^{q-1} - \lambda u_\lambda^{q-1} + \frac{1}{|x|^s}\left((u_\lambda + v)^{2^*(s)-1} - u_\lambda^{2^*(s)-1}\right). \tag{3.3}$$

Let us define

$$g_\lambda(x,t) = \begin{cases} \lambda (u_\lambda + t)^{q-1} - \lambda u_\lambda^{q-1} + \dfrac{1}{|x|^s}\left((u_\lambda + t)^{2^*(s)-1} - u_\lambda^{2^*(s)-1}\right) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

$$G_\lambda(x, v) = \int_0^v g_\lambda(x,t)\mathrm{d}t,$$

and

$$J_\lambda(v) = \frac{1}{2}\|v\|_\mu^2 - \int_\Omega G_\lambda(x, v^+(x))\mathrm{d}x.$$

Lemma 3. *$v = 0$ is a local minimum of J_λ in $H_\mu(\Omega)$.*

Proof. The proof is similar to the lemma 5.1 of the paper [5]. □

We recall that J_λ satisfies the $(PS)_c$ condition if any sequence (v_n) in $H_\mu(\Omega)$ such that $J_\lambda(v_n) \longrightarrow c$ and $J_\lambda'(v_n) \longrightarrow 0$ in $\left(H_\mu(\Omega)\right)'$ as $n \to \infty$ has a convergent subsequence.

Lemma 4. *If $v = 0$ is the only critical point of J_λ, then J_λ satisfies the $(PS)_c$ condition for any $c < c^* = \frac{2-s}{2(N-s)} S_{\mu,s}^{(N-s)/(2-s)}$.*

Proof. Let $(v_n) \subset H_\mu(\Omega)$ be such that

$$J_\lambda(v_n) \longrightarrow c < \frac{2-s}{2(N-s)} S_{\mu,s}^{(N-s)/(2-s)} \quad \text{and} \quad J_\lambda'(v_n) \to 0 \text{ in } \left(H_\mu(\Omega)\right)' \text{ as } n \to \infty,$$

then (v_n) is bounded in $H_\mu(\Omega)$.

Going if necessary to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v && \text{in } H_\mu(\Omega), \\ &v_n \to v && \text{in } L^t(\Omega), \text{ for } 1 < t < 2^* \text{ and a.e in } \Omega, \\ &v_n \longrightarrow v && \text{in } L^p(\Omega, |x|^{-s}\mathrm{d}x) \text{ for } 2 \leq p < 2^*(s). \end{aligned} \tag{3.4}$$

Moreover v is a critical point of J_λ in $H_\mu(\Omega)$. From our hypothesis, we know that $v = 0$. Now we want to prove $v_n \longrightarrow 0$ strongly in $H_\mu(\Omega)$. From (3.4) and Ghoussoub–Yuan's relation [8]:

$$\int_\Omega \frac{\left(u_\lambda + v_n^+\right)^{2^*(s)}}{|x|^s}\mathrm{d}x - \int_\Omega \frac{u_\lambda^{2^*(s)}}{|x|^s}\mathrm{d}x = \int_\Omega \frac{(v_n^+)^{2^*(s)}}{|x|^s}\mathrm{d}x + \circ(1).$$

We have

$$\begin{aligned} \left\langle J_\lambda'(v_n), u_\lambda + v_n \right\rangle &= \int_\Omega \left(\nabla v_n \nabla(u_\lambda + v_n) - \frac{\mu}{|x|^2} v_n (u_\lambda + v_n)\right)\mathrm{d}x + \circ(1) \\ &\quad - \int_\Omega \left[\frac{\left(u_\lambda + v_n^+\right)^{2^*(s)-1}}{|x|^s}(u_\lambda + v_n) - \frac{u_\lambda^{\,2^*(s)-1}}{|x|^s}(u_\lambda + v_n)\right]\mathrm{d}x \\ &\quad - \lambda \int_\Omega \left[(u_\lambda + v_n^+)^{q-1}(u_\lambda + v_n) - u_\lambda^{q-1}(u_\lambda + v_n)\right]\mathrm{d}x \\ &= \int_\Omega \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} v_n^2\right)\mathrm{d}x - \int_\Omega \frac{\left(v_n^+\right)^{2^*(s)}}{|x|^s}\mathrm{d}x + \circ(1) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus we can assume that

$$\|v_n\|_\mu^2 \to b \quad \text{and} \quad \int_\Omega \frac{\left(v_n^+\right)^{2^*(s)}}{|x|^s}\mathrm{d}x \to b \geq 0 \quad \text{when } n \to \infty.$$

If $b = 0$, the proof is complete. If $b \neq 0$, we have from the definition of $S_{\mu,s}$ that

$$\int_\Omega \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} v_n^2 \right) \mathrm{d}x \geq S_{\mu,s} \left(\int_\Omega \frac{\left(v_n^+\right)^{2^*(s)}}{|x|^s} \mathrm{d}x \right)^{\frac{2}{2^*(s)}},$$

and so that $b \geq S_{\mu,s}^{(N-s)/(2-s)}$. Then we get that

$$\begin{aligned} c &= J_\lambda(v_n) + \circ(1) = \frac{1}{2} \|v_n\|_\mu^2 - \frac{1}{2^*(s)} \int_\Omega \frac{\left(v_n^+\right)^{2^*(s)}}{|x|^s} \mathrm{d}x + \circ(1) \quad \text{as } n \to \infty \\ &= \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) b \geq \frac{2-s}{2(N-s)} S_{\mu,s}^{(N-s)/(2-s)}, \end{aligned}$$

which gives a contradiction. $\square$

In the following, we shall give some estimates for the extremal functions defined in (1.1). Let

$$V_\varepsilon = U_\varepsilon / C_\varepsilon \text{ and } \Psi(x) \in C_0^\infty(\Omega) \quad \text{such that}$$

$0 \leq \Psi(x) \leq 1$, $\Psi(x) = 1$ for $|x| \leq \rho$, $\Psi(x) = 0$ for $|x| \geq 2\rho$, ρ is chosen as in Proposition 1.

Set

$$v_\varepsilon(x) = (\Psi(x) V_\varepsilon) \Big/ \left(\int_\Omega \frac{|\Psi(x) V_\varepsilon|^{2^*(s)}}{|x|^s} \mathrm{d}x \right)^{1/2^*(s)}.$$

By a straightforward computation one finds

$$\int_\Omega \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} = 1, \ \|v_\varepsilon\|_\mu^2 = S_{\mu,s} + 0(\varepsilon^{\frac{N-2}{2-s}}),$$

and

$$\int_\Omega |v_\varepsilon|^r \,\mathrm{d}x = \begin{cases} 0(\varepsilon^{\frac{r-\sqrt{\mu}}{2-s}}) & \text{if } 1 < r < \dfrac{N}{\sqrt{\mu} + \sqrt{\bar{\mu} - \mu}} \\ 0(\varepsilon^{\frac{r-\sqrt{\mu}}{2-s}} |\ln \varepsilon|) & \text{if } r = \dfrac{N}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} \\ 0(\varepsilon^{\frac{\sqrt{\bar{\mu}}(N - r\sqrt{\bar{\mu}})}{(2-s)\sqrt{\bar{\mu}-\mu}}}) & \text{if } \dfrac{N}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} < r < 2^*. \end{cases}$$

Lemma 5. *Let c^* be defined in Lemma 4, then we have that*

$$\sup_{t \geq 0} J_\lambda(t v_\varepsilon) < c^*.$$

Proof. From the elementary inequality [1]

$$(a+b)^p \geq a^p + b^p + p\, a^{p-1} b, \quad p > 1, a, b \geq 0,$$

we get

$$g(x, v_\varepsilon) \geq \frac{\left(v_\varepsilon^+\right)^{2^*(s)-1}}{|x|^s} + (2^*(s) - 1) \frac{u_\lambda^{(2^*(s)-2)}}{|x|^s} v_\varepsilon^+,$$

and

$$G(x, t v_\varepsilon) \geq \frac{t^{2^*(s)}}{2^*(s)} \frac{(v_\varepsilon^+)^{2^*(s)}}{|x|^s} + \frac{(2^*(s)-1)t^2}{2} \frac{u_\lambda^{(2^*(s)-2)}}{|x|^s} (v_\varepsilon^+)^2.$$

Since $u_\lambda(x) \geq M|x|^{-(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu})} > 0$, on $B_{\rho_0}(0) \setminus \{0\}$ (result from Proposition 1)

thus

$$(2^*(s) - 1) \frac{u_\lambda^{(2^*(s)-2)}}{|x|^s} \geq (2^*(s) - 1) M^{2^*(s)-2} |x|^{-\left((2^*(s)-2)(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu}) + s\right)} \geq M_0 > 0 \quad \text{on } B_{\rho_0}(0) \setminus \{0\}.$$

The function

$$J_\lambda(tv_\varepsilon) = \frac{t^2}{2}\,\|v_\varepsilon\|_\mu^2 - \int_\Omega G_\lambda(tv_\varepsilon)\mathrm{d}x$$

becomes

$$J_\lambda(tv_\varepsilon) \le h_\varepsilon(t) = \frac{t^2}{2}\,\|v_\varepsilon\|_\mu^2 - \frac{t^{2^*(s)}}{2^*(s)} - \frac{t^2}{2}M_0\int_\Omega v_\varepsilon^2\mathrm{d}x.$$

From

$$h_\varepsilon'(t) = t\left(\|v_\varepsilon\|_\mu^2 - t^{2^*(s)-2} - M_0\int_\Omega v_\varepsilon^2\mathrm{d}x\right).$$

We get

$$\max_{t\ge 0} h(t) = h(t_\varepsilon),\text{ where } t_\varepsilon = \left(\|v_\varepsilon\|_\mu^2 - M_0\int_\Omega v_\varepsilon^2\mathrm{d}x\right)^{1/(2^*(s)-2)}.$$

Thus

$$\begin{aligned} J_\lambda(tv_\varepsilon) &\le h(t_\varepsilon) \\ &= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\left(\|v_\varepsilon\|_\mu^2 - M_0\int_\Omega v_\varepsilon^2\mathrm{d}x\right)^{2^*(s)/(2^*(s)-2)} \\ &= \begin{cases} \dfrac{2-s}{2(N-s)}S_{\mu,s}^{(N-s)/(2-s)} + (\varepsilon^{\frac{N-2}{2-s}}) - 0(\varepsilon^{\frac{N-2}{(2-s)\sqrt{\overline{\mu}-\mu}}}) & \text{if } 0<\mu<\overline{\mu}-1 \\ \dfrac{2-s}{2(N-s)}S_{\mu,s}^{(N-s)/(2-s)} + (\varepsilon^{\frac{N-2}{2-s}}) - 0(\varepsilon^{\frac{N-2}{(2-s)}}\,|\ln\varepsilon|) & \text{if } \mu=\overline{\mu}-1. \end{cases} \end{aligned}$$

Thus we get

$$\sup_{t\ge 0} J_\lambda(tv_\varepsilon) < c^*. \quad \square$$

Proof (*Proof of* Theorem 1 *Completed*). From Lemma 3, $v=0$ is a local minimizer of J_λ then there exists a sufficiently small positive number $\overline{\rho}$ such that $J_\lambda(v)>0$ for $\|v\|_\mu = \overline{\rho}$. Since $J_\lambda(tv_\varepsilon)\to -\infty$ as $t\to\infty$, then there exists $T>0$ such that $\|Tv_\varepsilon\|_\mu > \overline{\rho} > 0$ and $J_\lambda(Tv_\varepsilon)<0$. We defined

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) \quad \text{where } \Gamma = \left\{\gamma\in C([0,1], H_\mu(\Omega)), \gamma(0)=0, \gamma(1)=Tu_\varepsilon\right\}.$$

For $c<c^*$, $(PS)_c$ is satisfied by Lemma 4, then we conclude by Lemma 5 that

$$c \le \sup_{t\ge 0} J_\lambda(tTv_\varepsilon) \le \sup_{t\ge 0} J_\lambda(tv_\varepsilon) < c^*.$$

Then by applying the Mountain Pass theorem whenever $c>0$ and the Ghoussoub–Preiss version whenever $c=0$ see [7]. We obtain a nontrivial critical point v of J_λ. $\square$

Acknowledgement

The authors thank the anonymous referee for carefully reading this paper and suggesting many useful comments.

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