On The Problem Of Entire Functions That Share a Small Function With Their Difference Operators

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Abstract. In this paper, we study uniqueness problems for an entire function that shares small functions of finite order with their difference operators. In particular, we give a generalization of results in [2, 3, 13].

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1 Introduction and Main Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. In what follows, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions ([9], [11], [17]). In addition, we will use $\rho(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros of f, we say that a meromorphic function $\varphi(z)$ is a small function of $f(z)$ if $T(r,\varphi)=S(r,f)$, where $S(r,f)=o(T(r,f))$, as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For a meromorphic function $f(z)$, we define its shift by $f_c(z)=f(z+c)$ and its difference operators by

$$\Delta_c f(z)=f(z+c)-f(z), \quad \Delta_c^n f(z)=\Delta_c^{n-1}(\Delta_c f(z)),\ n\in\mathbb{N},\ n\geq 2.$$

In particular, $\Delta_c^n f(z)=\Delta^n f(z)$ for the case $c=1$.

Let f and g be two meromorphic functions and let a be a finite nonzero value. We say that f and g share the value a CM provided that $f-a$ and $g-a$ have the same zeros counting multiplicities. Similarly, we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. In [15], Rubel and Yang proved that if an entire function f shares two distinct complex numbers CM with its derivative f', then $f\equiv f'$. In 1986, Jank et al. (see [10]) proved that for a nonconstant meromorphic function f, if f, f' and f'' share a finite nonzero value CM, then $f'\equiv f$. This result suggests the following question:

Question 1 [17] *Let f be a nonconstant meromorphic function, let a be a finite nonzero constant, and let n and m $(n<m)$ be positive integers. If f, $f^{(n)}$ and $f^{(m)}$ share a CM, then can we get the result $f^{(n)}\equiv f$?*

The following example (see [18]) shows that the answer to the above question is, in general, negative. Let n and m be positive integers satisfying $m>n+1$, and let b be a constant satisfying $b^n=b^m\neq 1$. Set $a=b^n$ and $f(z)=e^{bz}+a-1$. Then f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, and $f^{(n)}\not\equiv f$. However, when f is an entire function of finite order and $m=n+1$, the answer to Question 1 is still positive. In fact, P. Li and C. C. Yang proved the following:

Theorem A [14] *Let f be a nonconstant entire function, let a be a finite nonzero constant, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f\equiv f'$.*

Recently several papers have focussed on the Nevanlinna theory with respect to difference operators see, e.g. [1], [5], [7], [8]. Many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts or difference operators. In [2, 3], B. Chen et al. proved a difference analogue of result of Jank et al. and obtained the following results:

Theorem B [2] *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)\,(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Theorem C [3] *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)\,(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ $(n \geq 2)$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^n f$.*

Theorem D [3] *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ share 0 CM, then $\Delta_c^n f(z) = C\Delta_c f(z)$, where C is a nonzero constant.*

Recently in [13], Z. Latreuch et al. proved the following results:

Theorem E [13] *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)\,(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Theorem F [13] *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) = C\Delta_c^n f(z)$, where C is a nonzero constant.*

For the case $n = 1$, A. El Farissi and others gave the following improvement.

Theorem G [6] *Let $f(z)$ be a non-periodic entire function of finite order, and let $a(z)\,(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

Remark 1.1 The Theorem G is essentially known in [6]. For the convenience of readers, we give his proof in the Lemma 2.4.

It is naturally now to ask the following question: *Under the hypotheses of Theorem E, can we get the result* $\Delta_c f(z) \equiv f(z)$*?* The aim of this paper is to answer this question and to give a difference analogue of result of P. Li and C. C. Yang in [14]. In fact we obtain the following results:

Theorem 1.1 *Let* $f(z)$ *be a nonconstant entire function of finite order such that* $\Delta_c^n f(z) \not\equiv 0$*, and let* $a(z) (\not\equiv 0) \in S(f)$ *be a periodic entire function with period* c*. If* $f(z)$*,* $\Delta_c^n f(z)$ *and* $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ *share* $a(z)$ *CM, then* $\Delta_c f(z) \equiv f(z)$*.*

Remark 1.2 The condition $\Delta_c^n f(z) \not\equiv 0$ is necessary. Let's take for example the entire function $f(z) = 1 + e^{2\pi i z}$ and $c = a = 1$, then $f - a$ and $\Delta^n f - a = \Delta^{n+1} f - a = -1$ have the same zeros but $\Delta f \neq f$. On the other hand, under the conditions of Theorem 1.1 $\Delta_c^n f(z) \not\equiv 0$ can not be a periodic entire function because $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ (Theorem E, [13]).

Example 1.1 Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share a CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c f(z) \equiv f(z)$. This example satisfies Theorem 1.1.

Theorem 1.2 *Let* $f(z)$ *be a nonconstant entire function of finite order such that* $\Delta_c^n f(z) \not\equiv 0$*, and let* $a(z)$*,* $b(z) (\not\equiv 0) \in S(f)$ *such that* $b(z)$ *is a periodic entire function with period* c *and* $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$*. If* $f(z) - a(z)$*,* $\Delta_c^n f(z) - b(z)$ *and* $\Delta_c^{n+1} f(z) - b(z)$ *share* 0 *CM, then* $\Delta_c f(z) \equiv f(z) + b(z) + \Delta_c a(z) - a(z)$*.*

Remark 1.3 The condition $b(z) \not\equiv 0$ is necessary in the proof of Theorem 1.2, for the case $b(z) \equiv 0$, please see Theorem 1.4.

Remark 1.4 The condition $\Delta_c^m a(z) \equiv 0$ in Theorem 1.2 is more general than the condition "periodic entire function of period c".

For the case $m = 1$, we deduce the following result.

Corollary 1.1 *Let* $f(z)$ *be a nonconstant entire function of finite order such that* $\Delta_c^n f(z) \not\equiv 0$*, and let* $a(z)$*,* $b(z) (\not\equiv 0) \in S(f)$ *be periodic entire functions with period* c*. If* $f(z) - a(z)$*,* $\Delta_c^n f(z) - b(z)$ *and* $\Delta_c^{n+1} f(z) - b(z)$ *share* 0 *CM, then* $\Delta_c f(z) \equiv f(z) + b(z) - a(z)$*.*

Example 1.2 Let $f(z) = e^{z\ln 2} - 2$, $a = -1$ and $b = 1$. It is clear that $f(z) - a$, $\Delta^n f(z) - b$ and $\Delta^{n+1} f(z) - b$ share 0 CM. Here, we also get $\Delta f(z) = f(z) + b - a$.

Example 1.3 Let $f(z) = e^{z\ln 2} + z^3 - 1$, $a(z) = z^3$ and $b = 1$. It is clear that $f(z) - z^3$, $\Delta^4 f(z) - 1$ and $\Delta^5 f(z) - 1$ share 0 CM. On the other hand, we can verify that $\Delta f(z) = f(z) + 1 + \Delta z^3 - z^3$ which satisfies Theorem 1.2.

Theorem 1.3 *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c f(z) \equiv Cf(z)$, where C is a nonzero constant.*

Example 1.4 Let $f(z) = e^{az}$ and $c = 1$ where $a \neq 2k\pi i$ $(k \in \mathbb{Z})$, it is clear that $\Delta_c^n f(z) = (e^a - 1)^n e^{az}$ for any integer $n \geq 1$. So, $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c f(z) \equiv Cf(z)$ where $C = e^a - 1$. This example satisfies Theorem 1.3.

Corollary 1.2 *Let $f(z)$ be a nonconstant entire function of finite order such that $f(z)$, $\Delta_c^n f(z)$ $(\not\equiv 0)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share 0 CM. If there exists a point z_0 and an integer $m \geq 1$ such that $\Delta_c^m f(z_0) = f(z_0) \neq 0$, then $\Delta_c^m f(z) \equiv f(z)$.*

By combining Theorem 1.2 and Theorem 1.3 we can prove the following result.

Theorem 1.4 *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z) \in S(f)$ such that $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$. If $f(z) - a(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c f(z) \equiv Cf(z) + \Delta_c a(z) - a(z)$, where C is a nonzero constant.*

2 Some lemmas

Lemma 2.1 [5] *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of*

$f(z)$, then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

By combining Theorem 1.4 in [4] and Theorem 2.2 in [12], we can prove the following lemma.

Lemma 2.2 *Let $a_0(z), a_1(z), \cdots, a_n(z) (\not\equiv 0)$, $F(z) (\not\equiv 0)$ be finite order meromorphic functions, c_k $(k = 0, \cdots, n)$ be constants, unequal to each other. If f is a finite order meromorphic solution of the equation*

$$a_n(z) f(z+c_n) + \cdots + a_1(z) f(z+c_1) + a_0(z) f(z+c_0) = F(z) \quad (2.1)$$

with

$$\max\{\rho(a_i), (i = 0, \cdots, n), \rho(F)\} < \rho(f),$$

then $\lambda(f) = \rho(f)$.

Proof. By (2.1) we have

$$\frac{1}{f(z+c_0)} = \frac{1}{F}\left(a_n \frac{f(z+c_n)}{f(z+c_0)} + \cdots + a_1 \frac{f(z+c_1)}{f(z+c_0)} + a_0\right). \quad (2.2)$$

Set $\max\{\rho(a_j) \ \ (j = 0, \cdots, n), \rho(F)\} = \beta < \rho(f) = \rho$. Then, for any given $\varepsilon \left(0 < \varepsilon < \frac{\rho-\beta}{2}\right)$, we have

$$\sum_{j=0}^{n} T(r, a_j) + T(r, F) \leq (n+2) \exp\left\{r^{\beta+\varepsilon}\right\} = o(1) \exp\left\{r^{\rho-\varepsilon}\right\}. \quad (2.3)$$

By (2.2), (2.3) and Lemma 2.1, we obtain

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1)$$

$$\leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{n} m(r, a_j) + \sum_{j=1}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_0)}\right) + O(1)$$

$$\leq N\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{F}\right) + \sum_{j=0}^{n} T(r, a_j) + \sum_{j=1}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_0)}\right) + O(1)$$

$$\leq N\left(r,\frac{1}{f}\right)+O\left(r^{\rho-1+\varepsilon}\right)+o\left(1\right)\exp\left\{r^{\rho-\varepsilon}\right\}. \tag{2.4}$$

By (2.4), we obtain that $\rho\left(f\right)\leq\lambda\left(f\right)$ and since $\lambda\left(f\right)\leq\rho\left(f\right)$ for every meromorphic function, we deduce that $\lambda\left(f\right)=\rho\left(f\right).$

Remark 2.1 Recently, Shun-Zhou Wu and Xiu-Min Zheng (see [16]) obtained Lemma 2.2 by using a different proof.

Lemma 2.3 [17] *Suppose* $f_j\left(z\right)\left(j=1,2,\cdots,n+1\right)$ *and* $g_j\left(z\right)\left(j=1,2,\cdots,n\right)$ $\left(n\geq 1\right)$ *are entire functions satisfying the following conditions:*
(i) $\sum\limits_{j=1}^{n} f_j\left(z\right)e^{g_j\left(z\right)}\equiv f_{n+1}\left(z\right);$
(ii) *The order of* $f_j\left(z\right)$ *is less than the order of* $e^{g_k(z)}$ *for* $1\leq j\leq n+1$, $1\leq k\leq n$. *And furthermore, the order of* $f_j\left(z\right)$ *is less than the order of* $e^{g_h(z)-g_k(z)}$ *for* $n\geq 2$ *and* $1\leq j\leq n+1$, $1\leq h<k\leq n$.
Then $f_j\left(z\right)\equiv 0,\ \left(j=1,2,\cdots n+1\right).$

Lemma 2.4 [6] *Let* $f\left(z\right)$ *be a non-periodic entire function of finite order, and let* $a\left(z\right)\left(\not\equiv 0\right)\in S\left(f\right)$ *be a periodic entire function with period* c. *If* $f\left(z\right),\ \Delta_c f\left(z\right)$ *and* $\Delta_c^2 f\left(z\right)$ *share* $a\left(z\right)$ *CM, then* $\Delta_c f\left(z\right)\equiv f\left(z\right).$

Proof. Suppose that $\Delta_c f\left(z\right)\not\equiv f\left(z\right).$ Since f, $\Delta_c f$ and $\Delta_c^2 f$ share $a\left(z\right)$ CM, then we have

$$\frac{\Delta_c f\left(z\right)-a\left(z\right)}{f\left(z\right)-a\left(z\right)}=e^{P(z)}$$

and

$$\frac{\Delta_c^2 f\left(z\right)-a\left(z\right)}{f\left(z\right)-a\left(z\right)}=e^{Q(z)}$$

where $P\ \left(e^P\not\equiv 1\right)$ and Q are polynomials. By using Theorem B, we obtain that $\Delta_c^2 f\equiv\Delta_c f$, which means that

$$\alpha\left(z\right)=\Delta_c f\left(z\right)-f\left(z\right) \tag{2.5}$$

is entire periodic function of period c. By (2.5) we have

$$\Delta_c f\left(z\right)-a\left(z\right)=f\left(z\right)-a\left(z\right)+\alpha\left(z\right),$$

then

$$\frac{\Delta_c f\left(z\right)-a\left(z\right)}{f\left(z\right)-a\left(z\right)}=1+\frac{\alpha\left(z\right)}{f\left(z\right)-a\left(z\right)}=e^{P(z)},$$

which is equivalent to

$$f(z) - a(z) = \frac{\alpha(z)}{e^{P(z)} - 1}. \tag{2.6}$$

Since $\alpha(z)$ and $a(z)$ are periodic functions of period c, then we have

$$\Delta_c f(z) = \alpha(z) \Delta_c \left(\frac{1}{e^{P(z)} - 1} \right) \tag{2.7}$$

and

$$\Delta_c^2 f(z) = \alpha(z) \Delta_c^2 \left(\frac{1}{e^{P(z)} - 1} \right). \tag{2.8}$$

We have the two following subcases:
(i) If $P \equiv K$ ($K \neq 2k\pi i,\ K \in \mathbb{Z}$), then by (2.7) we have $\Delta_c f(z) = 0$. On the other hand, by using (2.5), (2.6) and $\Delta_c f(z) = 0$, we deduce that

$$f(z) - a(z) = \frac{-f(z)}{e^K - 1},\ K \in \mathbb{C} - \{2k\pi i, k \in \mathbb{Z}\}.$$

So,

$$f(z) = \frac{e^K - 1}{e^K} a(z).$$

Hence

$$T(r, f) = S(r, f),$$

which is a contradiction.
(ii) If P is nonconstant and since $\Delta_c^2 f(z) = \Delta_c f(z)$, then

$$e^{P_c(z)+P(z)} - 3e^{P_{2c}(z)+P(z)} + 2e^{P_{2c}(z)+P_c(z)} + e^{P_{2c}(z)} - 3e^{P_c(z)} + 2e^{P(z)} = 0$$

which is equivalent to

$$e^{P_c(z)} + \left(2e^{\Delta_c P(z)} - 3\right) e^{P_{2c}(z)} = -e^{\Delta_c P_c(z) + \Delta_c P(z)} + 3e^{\Delta_c P(z)} - 2. \tag{2.9}$$

Since $\deg \Delta_c P = \deg P - 1$, then we have

$$\rho\left(e^{P_c} + \left(2e^{\Delta_c P} - 3\right) e^{P_{2c}}\right) = \rho\left(-e^{\Delta_c P_c + \Delta_c P} + 3e^{\Delta_c P} - 2\right) \leq \deg P - 1. \tag{2.10}$$

On the other hand,

$$\rho\left(e^{P_c} + \left(2e^{\Delta_c P} - 3\right) e^{P_{2c}}\right) = \rho\left(e^{P_c}\right) = \deg P \tag{2.11}$$

because if we have the contrary

$$\rho\left(e^{P_c}+\left(2e^{\Delta_c P}-3\right)e^{P_{2c}}\right)<\rho\left(e^{P_c}\right),$$

we obtain the following contradiction

$$\deg P=\rho\left(\frac{e^{P_c}+\left(2e^{\Delta_c P}-3\right)e^{P_{2c}}}{e^{P_c}}\right)=\rho\left(1+\left(2e^{\Delta_c P}-3\right)e^{\Delta P_c}\right)\leq \deg P-1.$$

By using (2.10) and (2.11), we obtain

$$\deg P\leq \deg P-1$$

which is a contradiction. This leads to $\Delta_c f(z)=f(z)$. Thus, the proof of Lemma 2.4 is completed.

3 Proof of the Theorems and Corollary

Proof of the Theorem 1.1. Obviously, suppose that $\Delta_c f(z)\not\equiv f(z)$. By using Theorem E, we have

$$\frac{\Delta_c^n f(z)-a(z)}{f(z)-a(z)}=e^{P(z)} \tag{3.1}$$

and

$$\frac{\Delta_c^{n+1} f(z)-a(z)}{f(z)-a(z)}=e^{P(z)}, \tag{3.2}$$

where $P\left(e^P\not\equiv 1\right)$ is polynomial. Dividing the proof of Theorem 1.1 into two cases:

Case 1. P is a nonconstant polynomial. Setting now $g(z)=f(z)-a(z)$. Then, we have from (3.1) and (3.2)

$$\Delta_c^n g(z)=e^{P(z)}g(z)+a(z) \tag{3.3}$$

and

$$\Delta_c^{n+1} g(z)=e^{P(z)}g(z)+a(z). \tag{3.4}$$

By (3.3) and (3.4), we have

$$g_c(z)=2e^{P-P_c}g(z)+a(z)e^{-P_c}.$$

Using the principle of mathematical induction, we obtain

$$g_{ic}(z) = 2^i e^{P-P_{ic}} g(z) + a(z)\left(2^i - 1\right) e^{-P_{ic}},\ i \geq 1. \tag{3.5}$$

Now, we can rewrite (3.3) as

$$\Delta_c^n g(z) = \sum_{i=1}^{n} C_n^i (-1)^{n-i} \left(2^i e^{P-P_{ic}} g(z) + a(z)\left(2^i - 1\right) e^{-P_{ic}}\right)$$
$$+ (-1)^n g(z) = e^P g(z) + a(z),$$

which implies

$$\left(\sum_{i=0}^{n} C_n^i (-1)^{n-i} 2^i e^{P-P_{ic}} - e^P\right) g(z)$$
$$+ a(z)\left(\sum_{i=0}^{n} C_n^i (-1)^{n-i} \left(2^i - 1\right) e^{-P_{ic}} - 1\right) = 0.$$

Hence

$$A_n(z) g(z) + B_n(z) = 0, \tag{3.6}$$

where

$$A_n(z) = \sum_{i=0}^{n} C_n^i (-1)^{n-i} 2^i e^{P-P_{ic}} - e^P$$

and

$$B_n(z) = a(z)\left(\sum_{i=0}^{n} C_n^i (-1)^{n-i} \left(2^i - 1\right) e^{-P_{ic}} - 1\right).$$

By the same method, we can rewrite (3.4) as

$$A_{n+1}(z) g(z) + B_{n+1}(z) = 0, \tag{3.7}$$

where

$$A_{n+1}(z) = \sum_{i=0}^{n+1} C_{n+1}^i (-1)^{n+1-i} 2^i e^{P-P_{ic}} - e^P$$

and

$$B_{n+1}(z) = a(z)\left(\sum_{i=0}^{n+1} C_{n+1}^i (-1)^{n+1-i} \left(2^i - 1\right) e^{-P_{ic}} - 1\right).$$

We can see easily from the equations (3.6) and (3.7) that

$$h\left(z\right)=A_{n}\left(z\right)B_{n+1}\left(z\right)-A_{n+1}\left(z\right)B_{n}\left(z\right)\equiv 0. \tag{3.8}$$

On the other hand, we remark that

$$\begin{aligned}e^{P}B_{n}\left(z\right)=a\left(z\right)e^{P}\left(\sum_{i=0}^{n}C_{n}^{i}\left(-1\right)^{n-i}2^{i}e^{-P_{ic}}-\sum_{i=0}^{n}C_{n}^{i}\left(-1\right)^{n-i}e^{-P_{ic}}-1\right)\\ =a\left(z\right)e^{P}\left(\sum_{i=0}^{n}C_{n}^{i}\left(-1\right)^{n-i}2^{i}e^{-P_{ic}}-1-\Delta_{c}^{n}\left(e^{-P}\right)\right)\\ =a\left(z\right)\left(A_{n}\left(z\right)-e^{P}\Delta_{c}^{n}\left(e^{-P}\right)\right).\end{aligned}$$

Then

$$B_{n}\left(z\right)=a\left(z\right)\left(e^{-P}A_{n}\left(z\right)-\Delta_{c}^{n}\left(e^{-P}\right)\right). \tag{3.9}$$

By the same method, we obtain

$$B_{n+1}\left(z\right)=a\left(z\right)\left(e^{-P}A_{n+1}\left(z\right)-\Delta_{c}^{n+1}\left(e^{-P}\right)\right). \tag{3.10}$$

Return now to the equation (3.8), by using (3.9) and (3.10), we get

$$\begin{aligned}h\left(z\right)=A_{n}\left(z\right)B_{n+1}\left(z\right)-A_{n+1}\left(z\right)B_{n}\left(z\right)\\ =A_{n}\left(z\right)\left[a\left(z\right)\left(e^{-P}A_{n+1}\left(z\right)-\Delta_{c}^{n+1}\left(e^{-P}\right)\right)\right]\\ -A_{n+1}\left(z\right)\left[a\left(z\right)\left(e^{-P}A_{n}\left(z\right)-\Delta_{c}^{n}\left(e^{-P}\right)\right)\right]\\ =a\left(z\right)\left[A_{n+1}\left(z\right)\Delta_{c}^{n}\left(e^{-P}\right)-A_{n}\left(z\right)\Delta_{c}^{n+1}\left(e^{-P}\right)\right]\equiv 0.\end{aligned}$$

Hence

$$A_{n+1}\left(z\right)\Delta_{c}^{n}\left(e^{-P}\right)-A_{n}\left(z\right)\Delta_{c}^{n+1}\left(e^{-P}\right)\equiv 0.$$

Therefore

$$\begin{aligned}\Delta_{c}^{n}\left(e^{-P}\right)\left(\sum_{i=0}^{n+1}C_{n+1}^{i}\left(-1\right)^{n+1-i}2^{i}e^{-P_{ic}}-1\right)\\ -\Delta_{c}^{n+1}\left(e^{-P}\right)\left(\sum_{i=0}^{n}C_{n}^{i}\left(-1\right)^{n-i}2^{i}e^{-P_{ic}}-1\right)=0.\end{aligned}$$

Thus

$$\begin{aligned}
& \Delta_c^n\left(e^{-P}\right)\sum_{i=0}^{n+1}C_{n+1}^i\left(-1\right)^{n+1-i}2^ie^{-P_ic}-\Delta_c^{n+1}\left(e^{-P}\right)\sum_{i=0}^{n}C_n^i\left(-1\right)^{n-i}2^ie^{-P_{ic}} \\
= & \Delta_c^n\left(e^{-P}\right)-\Delta_c^{n+1}\left(e^{-P}\right)=\Delta_c^n\left(2e^{-P}-e^{-P_c}\right).
\end{aligned}$$

Then

$$\sum_{i=0}^{n}\left(\Delta_c^n\left(e^{-P}\right)C_{n+1}^i\left(-1\right)^{n+1-i}-\Delta_c^{n+1}\left(e^{-P}\right)C_n^i\left(-1\right)^{n-i}\right)2^ie^{-P_{ic}}$$

$$+\Delta_c^n\left(e^{-P}\right)2^{n+1}e^{-P_{(n+1)c}}=\Delta_c^n\left(2e^{-P}-e^{-P_c}\right),$$

which yields

$$\sum_{i=0}^{n}\left(\Delta_c^n\left(e^{-P}\right)C_{n+1}^i+\Delta_c^{n+1}\left(e^{-P}\right)C_n^i\right)\left(-1\right)^{n+1-i}2^ie^{P_{(n+1)c}-P_{ic}}$$

$$+\Delta_c^n\left(e^{-P}\right)2^{n+1}=e^{P_{(n+1)c}}\Delta_c^n\left(2e^{-P}-e^{-P_c}\right). \tag{3.11}$$

Let us denote

$$\alpha_i\left(z\right)=\left(-1\right)^{n+1-i}2^ie^{P_{(n+1)c}-P_{ic}},\ i=0,\cdots,n$$

and

$$\alpha_{n+1}\left(z\right)=e^{P_{(n+1)c}}\Delta_c^n\left(2e^{-P}-e^{-P_c}\right).$$

It is clear that $\rho\left(\alpha_i\right)\leq\deg P-1$ for all $i=0,2,\cdots,n+1$. The equation (3.11) will be

$$\sum_{i=0}^{n}\left(\Delta_c^n\left(e^{-P}\right)C_{n+1}^i+\Delta_c^{n+1}\left(e^{-P}\right)C_n^i\right)\alpha_i\left(z\right)+\Delta_c^n\left(e^{-P}\right)2^{n+1}$$

$$=\left(\sum_{i=0}^{n}C_{n+1}^i\alpha_i\left(z\right)+2^{n+1}\right)\Delta_c^n\left(e^{-P}\right)$$

$$+\left(\sum_{i=0}^{n}C_n^i\alpha_i\left(z\right)\right)\Delta_c^{n+1}\left(e^{-P}\right)=\alpha_{n+1}\left(z\right). \tag{3.12}$$

For convenience, we denote by $M(z)$ and $N(z)$ the following

$$M(z)=\sum_{i=0}^{n}C_{n+1}^{i}\alpha_{i}(z)+2^{n+1},\ N(z)=\sum_{i=0}^{n}C_{n}^{i}\alpha_{i}(z).$$

Equation (3.12) is equivalent to

$$M(z)\sum_{i=0}^{n}C_{n}^{i}(-1)^{n-i}e^{-P_{ic}}+N(z)\sum_{i=0}^{n+1}C_{n+1}^{i}(-1)^{n+1-i}e^{-P_{ic}}$$

$$=\sum_{i=0}^{n}\left(C_{n}^{i}M(z)-C_{n+1}^{i}N(z)\right)(-1)^{n-i}e^{-P_{ic}}+N(z)e^{-P_{(n+1)c}}=\alpha_{n+1}(z). \tag{3.13}$$

As conclusion, we can say that (3.13) can be written as follow

$$a_{n+1}(z)e^{-P(z+(n+1)c)}+a_{n}(z)e^{-P(z+nc)}+\cdots+a_{0}(z)e^{-P(z)}=\alpha_{n+1}(z), \tag{3.14}$$

where $a_0(z),\cdots,a_{n+1}(z)$ and $\alpha_{n+1}(z)$ are entire functions. We distingue the following two subcases.

(i) If $\deg P>1$, then we have

$$\max\{\rho(a_i)\ \ (i=0,\cdots,n+1),\rho(\alpha_{n+1})\}<\deg P. \tag{3.15}$$

In order to prove that $\alpha_{n+1}(z)\not\equiv 0$, it suffices to show that $\Delta_c^n\left(2e^{-P}-e^{-P_c}\right)\not\equiv 0$. Suppose the contrary. Thus

$$\sum_{i=0}^{n}C_{n}^{i}(-1)^{n-i}\left(2e^{-P_{ic}}-e^{-P_{(i+1)c}}\right)\equiv 0. \tag{3.16}$$

The equation (3.16) can be written as

$$\sum_{i=0}^{n+1}b_{i}e^{-P_{ic}}\equiv 0,$$

where

$$b_i=\begin{cases} 2(-1)^{n},\ \text{if } i=0\\ (2C_n^i+C_n^{i-1})(-1)^{n-i},\ \text{if } 1\leq i\leq n\\ -1,\ \text{if } i=n+1.\end{cases}$$

Since $\deg P = m > 1$, then for any two integers j and k such that $0 \leq j < k \leq n+1$, we have

$$\rho\left(e^{-P_{kc}+P_{jc}}\right) = \deg P - 1.$$

It's clear now that all the conditions of Lemma 2.3 are satisfied. So, by Lemma 2.3 we obtain $b_i \equiv 0$ for all $i = 0, ..., n+1$, which is impossible. Then, $\alpha_{n+1}(z) \not\equiv 0$. By Lemma 2.2, (3.14) and (3.15), we deduce that $\lambda\left(e^P\right) = \deg P > 1$, which is a contradiction.

(ii) $\deg P = 1$. Suppose now that $P(z) = \mu z + \eta$ $(\mu \neq 0)$. Assume that $\alpha_{n+1}(z) \equiv 0$. It easy to see that

$$\Delta_c^n\left(2e^{-P} - e^{-P_c}\right) = \left(2 - e^{-\mu c}\right)\Delta_c^n\left(e^{-P}\right).$$

In the following two subcases, we prove that both of $(2 - e^{-\mu c})$ and $\Delta_c^n\left(e^{-P}\right)$ are not vanishing.

(A) Suppose that $2 = e^{-\mu c}$. Then for any integer i, we have $e^{-i\mu c} = 2^i$ and $e^{-P_{ic}} = 2^i e^{-P}$, applying that on the equation (3.6), we get

$$A_n(z) = \sum_{i=0}^{n} C_n^i (-1)^{n-i} 2^i e^{-i\mu c} - e^P = 3^n - e^P,$$

and

$$B_n(z) = a(z)\left(\sum_{i=0}^{n} C_n^i (-1)^{n-i}\left(2^i - 1\right)e^{-P_{ic}} - 1\right)$$

$$= a(z)\left(\sum_{i=0}^{n} C_n^i (-1)^{n-i}\left(4^i - 2^i\right)e^{-P} - 1\right) = a(z)\left(\left(3^n - 1\right)e^{-P} - 1\right).$$

Then

$$\left(3^n - e^P\right)g(z) + a(z)\left(\left(3^n - 1\right)e^{-P} - 1\right) = 0,$$

which is equivalent to

$$g(z) = a(z)\frac{e^P - (3^n - 1)}{e^P\left(3^n - e^P\right)}. \tag{3.17}$$

By the same arguing as before and the equation (3.7), we obtain

$$g(z) = a(z)\frac{e^P - (3^{n+1} - 1)}{e^P\left(3^{n+1} - e^P\right)},$$

which contradicts (3.17).
(B) Suppose now that $\Delta_c^n\left(e^{-P}\right) \equiv 0$. Thus

$$\Delta_c^n\left(e^{-P}\right) = \sum_{i=0}^{n} C_n^i\left(-1\right)^{n-i} e^{-\mu(z+ic)-\eta} = e^{-P}\sum_{i=0}^{n} C_n^i\left(-1\right)^{n-i} e^{-\mu ic}$$
$$= e^{-P}\left(e^{-\mu c}-1\right)^n.$$

This together with $\Delta_c^n e^{-P} \equiv 0$ gives $\left(e^{-\mu c}-1\right)^n \equiv 0$, which yields $e^{\mu c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$

$$e^{P(z+jc)} = e^{\mu z+\mu jc+\eta} = \left(e^{\mu c}\right)^j e^{P(z)} = e^{P(z)}. \tag{3.18}$$

On the other hand, we have from (3.1)

$$\Delta_c^n f\left(z\right) = e^{P(z)}\left(f\left(z\right) - a\left(z\right)\right) + a\left(z\right). \tag{3.19}$$

By (3.18) and (3.19), we have

$$\Delta_c^{n+1} f\left(z\right) = e^{P(z)}\Delta_c f\left(z\right) \tag{3.20}$$

Combining (3.2) and (3.20), we obtain

$$\Delta_c f\left(z\right) = \left(f\left(z\right) - a\left(z\right)\right) + a\left(z\right) e^{-P(z)}$$

which means that $\Delta_c^{n+1} f\left(z\right) = \Delta_c^n f\left(z\right)$ for all $n \geq 1$. Therefore, $f\left(z\right)$, $\Delta_c f\left(z\right)$ and $\Delta_c^2 f\left(z\right)$ share $a\left(z\right)$ CM and by Lemma 2.4 we obtain $\Delta_c f\left(z\right) = f\left(z\right)$, which contradicts the hypothesis. Then $\Delta_c^n\left(e^{-P}\right) \not\equiv 0$. From the subcases (A) and (B), we can deduce that $\alpha_{n+1}\left(z\right) \not\equiv 0$. It is clear that

$$\max\left\{\rho\left(a_i\right), \rho\left(\alpha_{n+1}\right), i = 0, ..., n+1\right\} < \deg P = 1.$$

By using Lemma 2.2, we obtain $\lambda\left(e^P\right) = \deg P = 1$, which is a contradiction, and P must be a constant.
Case 2. $P\left(z\right) \equiv K$, $K \in \mathbb{C} - \left\{2k\pi i, k \in \mathbb{Z}\right\}$. We have from (3.1)

$$\Delta_c^n f\left(z\right) = e^K\left(f\left(z\right) - a\left(z\right)\right) + a\left(z\right).$$

Hence

$$\Delta_c^{n+1} f\left(z\right) = e^K \Delta_c f\left(z\right). \tag{3.21}$$

Combining (3.2) and (3.21), we obtain

$$\Delta_c f(z) = (f(z) - a(z)) + a(z) e^{-K}$$

which means that $\Delta_c^{n+1} f(z) = \Delta_c^n f(z)$ for all $n \geq 1$. Therefore, $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM and by Lemma 2.4 we obtain $\Delta_c f(z) = f(z)$, which contradicts the hypothesis. Then $e^P \equiv 1$ and the proof of Theorem 1.1 is completed.

Proof of the Theorem 1.2. Setting $g(z) = f(z) + b(z) - a(z)$. Since $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$, then we can remark that

$$g(z) - b(z) = f(z) - a(z),$$

$$\Delta_c^n g(z) - b(z) = \Delta_c^n f(z) - b(z)$$

and

$$\Delta_c^{n+1} g(z) - b(z) = \Delta_c^n f(z) - b(z),\ n \geq 2.$$

Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share $b(z)$ CM. By using Theorem 1.1, we deduce that $\Delta_c g(z) \equiv g(z)$, which leads to $\Delta_c f(z) \equiv f(z) + b(z) + \Delta_c a(z) - a(z)$ and the proof of Theorem 1.2 is completed.

Proof of the Theorem 1.3. Note that $f(z)$ is a nonconstant entire function of finite order. Since $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then it is known by Theorem F that $\Delta_c^{n+1} f(z) = C\Delta_c^n f(z)$, where C is a nonzero constant. Then we have

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{P(z)} \tag{3.22}$$

and

$$\frac{\Delta_c^{n+1} f(z)}{f(z)} = Ce^{P(z)}, \tag{3.23}$$

where P is a polynomial. By (3.22) and (3.23) we obtain

$$f_{ic}(z) = (C+1)^i e^{P-P_{ic}} f(z). \tag{3.24}$$

Then

$$\Delta_c^n f(z) = \left(\sum_{i=0}^{n} C_n^i (-1)^{n-i} (C+1)^i e^{P-P_{ic}} \right) f(z) = e^{P(z)} f(z). \tag{3.25}$$

The equality (3.25) leads to $\deg P = 0$. Hence $P(z) - P_{ic}(z) \equiv 0$ and (3.25) will be

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} (C+1)^i = C^n = e^{P(z)}. \tag{3.26}$$

By $(3.22), (3.23)$ and (3.26) we deduce

$$\Delta_c^n f(z) = C^n f(z)$$

and

$$\Delta_c^{n+1} f(z) = C^{n+1} f(z).$$

Then

$$\Delta_c^{n+1} f(z) = \Delta_c (\Delta_c^n f(z)) = \Delta_c (C^n f(z)) = C^n \Delta_c f(z) = C^{n+1} f(z),$$

which implies $\Delta_c f(z) = C f(z)$. Thus, the proof of Theorem 1.3 is completed.

Proof of Corollary 1.2. By Theorem 1.3 we have $\Delta_c f(z) = C f(z)$, where C is a nonzero constant. Then

$$\Delta_c^m f(z) = C \Delta_c^{m-1} f(z) = C^m f(z), \ m \geq 1. \tag{3.27}$$

On the other hand, for $z_0 \in \mathbb{C}$ we have

$$\Delta_c^m f(z_0) = f(z_0). \tag{3.28}$$

By (3.27) and (3.28) we deduce that $C^m = 1$. Hence $\Delta_c^m f(z) = f(z)$.

Proof of the Theorem 1.4. Setting $g(z) = f(z) - a(z)$, we can remark that

$$g(z) = f(z) - a(z),$$
$$\Delta_c^n g(z) = \Delta_c^n f(z) - b(z)$$

and

$$\Delta_c^{n+1} g(z) = \Delta_c^n f(z) - b(z), \ n \geq 2.$$

Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share 0 CM. By using Theorem 1.3, we deduce that $\Delta_c g(z) \equiv C g(z)$, where C is a nonzero constant, which leads to $\Delta_c f(z) \equiv C f(z) + \Delta_c a(z) - a(z)$ and the proof of Theorem 1.4 is completed.

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