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**NEW ESTIMATIONS ABOUT THE ORDER OF GROWTH AND
THE TYPE OF MEROMORPHIC FUNCTIONS IN THE COMPLEX
PLANE**

ZINELAÂBIDINE LATREUCH¹ AND BENHARRAT BELAÏDI²

ABSTRACT. In this paper, we will give sufficient conditions to obtain new estimates about the order of growth and the type of meromorphic functions in the complex plane. We give also some examples to explain the sharpness of these estimations.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, we assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [1], [2], [3], [7]).

Definition 1.1. [2], [3], [7] The order of a meromorphic function f is defined as

$$\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f . If f is an entire function, then

$$\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log \log M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Definition 1.2. [1] A function $\rho(r)$ defined on $[0, +\infty)$ is called a proximate order if it satisfies the conditions:

- i) $\rho(r) \geq 0$;
- ii) $\lim_{r \rightarrow +\infty} \rho(r) = \rho$, $0 \leq \rho < +\infty$;
- iii) $\rho(r)$ is continuously differentiable on $[0, +\infty)$;
- iv) $\lim_{r \rightarrow +\infty} r\rho'(r) \log r = 0$.

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Definition 1.3. [2], [3], [7] Let f be a meromorphic function. Then the exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda(f) = \limsup_{r \rightarrow +\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r},$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$.

Definition 1.4. [2], [4] Let f be a meromorphic function of order ρ ($0 < \rho < \infty$), the type of f is defined as

$$\tau(f) = \limsup_{r \rightarrow +\infty} \frac{T(r, f)}{r^\rho}.$$

If f is an entire function, we can define the type by

$$\tau_M(f) = \limsup_{r \rightarrow +\infty} \frac{\log M(r, f)}{r^\rho}.$$

Remark 1.1. We have not always the equality $\tau_M(f) = \tau(f)$, for example $\tau(e^z) = \frac{1}{\pi} < 1 = \tau_M(e^z)$. By $T(r, f) \leq \log^+ M(r, f)$, it's easy to obtain the following $\tau(f) \leq \tau_M(f)$.

In [1], Goldberg and Ostrovskii proved the following inequalities

$$\begin{cases} \tau_M(f) \leq \frac{\pi\rho}{\sin \pi\rho} \tau(f), & \text{for } 0 < \rho \leq \frac{1}{2}, \\ \tau_M(f) \leq \pi\rho\tau(f), & \text{for } \frac{1}{2} \leq \rho < \infty. \end{cases}$$

The determination of the order of growth is very important, in this regard, many of the research task dealt with this problem, was reached following:

Theorem 1.1. [4] *Let f and g be entire functions. Then*

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\}, \quad (1.1)$$

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\} \quad (1.2)$$

and

$$\tau_M(f + g) \leq \max\{\tau_M(f), \tau_M(g)\}, \quad (1.3)$$

$$\tau_M(fg) \leq \tau_M(f) + \tau_M(g). \quad (1.4)$$

Theorem 1.2. ([1], Theorem 1.7) *Let f and g be meromorphic functions. If $\rho(f) > \rho(g)$, then*

$$\rho(f + g) = \rho(fg) = \rho(f). \quad (1.5)$$

The following theorem due to Valiron [6] and Mohon'ko [5], is of essential importance in the theory of complex differential equations.

Theorem 1.3. [5], [6] *Let f be a meromorphic function. Then for all irreducible rational functions in f ,*

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^p a_i(z) f^i}{\sum_{j=0}^q b_j(z) f^j}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that

$$\begin{cases} T(r, a_i) = S(r, f), & i = 0, \dots, p, \\ T(r, b_j) = S(r, f), & j = 0, \dots, q, \end{cases}$$

where $S(r, f)$ is a quantity satisfying

$$S(r, f) = O(\log r + \log T(r, f)) = o(T(r, f)),$$

as $r \rightarrow +\infty$ outside of a possible exceptional set $E \subset (0, +\infty)$ of finite linear measure, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f)) = dT(r, f) + S(r, f),$$

where $d = \max\{p, q\}$.

It is natural to ask: What can be said about $\rho(f + g)$ and $\rho(fg)$ if $\rho(f) = \rho(g)$? The main purpose of this paper is to answer to this question and we give also sufficient conditions to obtain similar results as in the Theorem 1.1 and Theorem 1.2 for the type of meromorphic functions.

Theorem 1.4. *Let f and g be meromorphic functions.*

(i) *If $\rho(f) > \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$), then we have*

$$\tau(f + g) = \tau(fg) = \tau(f). \tag{1.6}$$

(ii) *If $0 < \rho(f) = \rho(g) = \rho(f + g) = \rho(fg) < \infty$, then we obtain*

$$|\tau(f) - \tau(g)| \leq \tau(f + g) \leq \tau(f) + \tau(g) \tag{1.7}$$

and

$$|\tau(f) - \tau(g)| \leq \tau(fg) \leq \tau(f) + \tau(g). \tag{1.8}$$

Remark 1.2. To explain the sharpness of inequalities (1.7) and (1.8), we take $f(z) = \frac{1}{1+e^z}$ and $g(z) = \frac{1}{1-e^z}$. We can see that

$$\rho(f) = \rho(g) = \rho(f + g) = \rho(fg) = 1, \quad \tau(f) = \tau(g) = \frac{1}{\pi}$$

and

$$\tau(f + g) = \tau\left(\frac{2}{1 - e^{2z}}\right) = \frac{2}{\pi} = \tau(f) + \tau(g),$$

$$\tau(fg) = \tau\left(\frac{1}{1-e^{2z}}\right) = \frac{2}{\pi} = \tau(f) + \tau(g).$$

Theorem 1.5. *Let f and g be meromorphic functions such that $\rho(f) = \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$) and $\tau(f) \neq \tau(g)$. Then*

$$\rho(f+g) = \rho(fg) = \rho(f) = \rho(g). \quad (1.9)$$

Remark 1.3. If $\rho(f) = \rho(g)$ and $\tau(f) = \tau(g)$, then we cannot say any thing about $\rho(f+g)$ and $\rho(fg)$. If we have for example $f(z) = e^z$, $g(z) = -e^z$ and $h(z) = e^{-z}$, then $\rho(f+g) = 0 \neq \rho(f) = \rho(g) = 1$ and $\rho(fg) = \rho(f) = \rho(g) = 1$. On the other hand, $\rho(f+h) = \rho(f) = \rho(h) = 1$, $\rho(fh) = 0 \neq \rho(f) = \rho(h) = 1$.

Theorem 1.6. *Let f and g be entire functions.*

(i) *If $\rho(f) > \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$), then we have*

$$\tau_M(f+g) = \tau_M(f) \quad (1.10)$$

and

$$\tau_M(fg) \leq \tau_M(f). \quad (1.11)$$

(ii) *If $0 < \rho(f) = \rho(g) = \rho(f+g) = \rho(fg) < \infty$, then we obtain*

$$\tau_M(f+g) \leq \max\{\tau_M(f), \tau_M(g)\} \quad (1.12)$$

and

$$\tau_M(fg) \leq \tau_M(f) + \tau_M(g). \quad (1.13)$$

Furthermore, if $\tau_M(f) \neq \tau_M(g)$, then we get

$$\tau_M(f+g) = \max\{\tau_M(f), \tau_M(g)\}. \quad (1.14)$$

Theorem 1.7. *Let f and g be entire functions such that $\rho(f) = \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$) and $\tau_M(f) \neq \tau_M(g)$. Then*

$$\rho(f+g) = \rho(f) = \rho(g). \quad (1.15)$$

Theorem 1.8. *Let f be meromorphic function of non-integer order $0 < \rho < \infty$, and let g be meromorphic function such that $\rho(f) > \rho(g)$ ($0 < \rho(g) < \infty$) or $\tau(f) \neq \tau(g)$ if $\rho(f) = \rho(g)$. Then*

$$\lambda(f+g) = \lambda(fg) = \lambda(f) = \rho(f). \quad (1.16)$$

Lemma 1.9. ([1], Theorem 4.2) *If the order ρ of a meromorphic function $f(z)$ is non-integer, then the order of $N(r; 0, \infty)$ is equal to ρ . The types of $T(r, f)$ and $N(r; 0, \infty)$ with respect to an arbitrary proximate order $\rho(r)$, $\rho(r) \rightarrow \rho$, coincide.*

2. PROOF OF THEOREM 1.4

(i) By the definition of the type, we have

$$\tau(f+g) = \limsup_{r \rightarrow +\infty} \frac{T(r, f+g)}{r^{\rho(f+g)}} \leq \limsup_{r \rightarrow +\infty} \frac{T(r, f) + T(r, g) + O(1)}{r^{\rho(f+g)}}. \quad (2.1)$$

Since $\rho(g) < \rho(f)$, then $\rho(f+g) = \rho(f)$. Thus, from (2.1), we obtain

$$\tau(f+g) \leq \limsup_{r \rightarrow +\infty} \frac{T(r, f)}{r^{\rho(f)}} + \limsup_{r \rightarrow +\infty} \frac{T(r, g) + O(1)}{r^{\rho(f)}} = \tau(f). \quad (2.2)$$

On the other hand, since

$$\rho(f+g) = \rho(f) > \rho(g), \quad (2.3)$$

then by (2.2), we get

$$\tau(f) = \tau(f+g-g) \leq \tau(f+g). \quad (2.4)$$

Hence by (2.2) and (2.4), we obtain $\tau(f+g) = \tau(f)$. Now we prove $\tau(fg) = \tau(f)$. Since $\rho(g) < \rho(f)$, then $\rho(fg) = \rho(f)$. By the definition of the type, we have

$$\begin{aligned} \tau(fg) &= \limsup_{r \rightarrow +\infty} \frac{T(r, fg)}{r^{\rho(fg)}} \leq \limsup_{r \rightarrow +\infty} \frac{T(r, f) + T(r, g)}{r^{\rho(f)}} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{T(r, f)}{r^{\rho(f)}} + \limsup_{r \rightarrow +\infty} \frac{T(r, g)}{r^{\rho(f)}} = \tau(f). \end{aligned} \quad (2.5)$$

Since

$$\rho(fg) = \rho(f) > \rho(g) = \rho\left(\frac{1}{g}\right), \quad (2.6)$$

then by (2.5), we obtain

$$\tau(f) = \tau\left(fg\frac{1}{g}\right) \leq \tau(fg). \quad (2.7)$$

Thus, by (2.5) and (2.7), we have $\tau(fg) = \tau(f)$. (ii) Suppose that $\rho(f) = \rho(g) = \rho(fg) = \rho(f+g) = \rho$ ($0 < \rho < \infty$). Given $\varepsilon > 0$, we have

$$\begin{aligned} T(r, f+g) &\leq T(r, f) + T(r, g) + O(1) \leq (\tau(f) + \varepsilon)r^\rho + (\tau(g) + \varepsilon)r^\rho + O(1) \\ &\leq (\tau(f) + \tau(g) + 2\varepsilon)r^\rho + O(1) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} T(r, fg) &\leq T(r, f) + T(r, g) \leq (\tau(f) + \varepsilon)r^\rho + (\tau(g) + \varepsilon)r^\rho \\ &\leq (\tau(f) + \tau(g) + 2\varepsilon)r^\rho. \end{aligned} \quad (2.9)$$

Since $\varepsilon > 0$ is arbitrary by (2.8), (2.9) and the definition of the type, we obtain

$$\tau(f+g) \leq \tau(f) + \tau(g) \quad (2.10)$$

and

$$\tau(fg) \leq \tau(f) + \tau(g). \quad (2.11)$$

On the other hand, since $\rho(g) = \rho(f+g)$, then by using (2.10)

$$\tau(f) = \tau(f+g-g) \leq \tau(f+g) + \tau(g) \quad (2.12)$$

and by $\rho(f) = \rho(f + g)$, we obtain

$$\tau(g) = \tau(f + g - f) \leq \tau(f + g) + \tau(f). \quad (2.13)$$

From (2.12) and (2.13), we have

$$|\tau(f) - \tau(g)| \leq \tau(f + g). \quad (2.14)$$

Also, since $\rho(g) = \rho\left(\frac{1}{g}\right) = \rho(fg)$, then by using (2.11)

$$\tau(f) = \tau\left(fg\frac{1}{g}\right) \leq \tau(fg) + \tau(g). \quad (2.15)$$

and by $\rho(f) = \rho\left(\frac{1}{f}\right) = \rho(fg)$, we obtain

$$\tau(g) = \tau\left(fg\frac{1}{f}\right) \leq \tau(fg) + \tau(f). \quad (2.16)$$

By (2.15) and (2.16), we get

$$|\tau(f) - \tau(g)| \leq \tau(fg). \quad (2.17)$$

From (2.10) and (2.14) we get (1.7) and by (2.11), (2.17) we obtain (1.8).

3. PROOF OF THEOREM 1.5

Without lost of generality, we suppose that $\rho(f) = \rho(g)$ and $\tau(g) < \tau(f)$. Then, by Theorem 1.1 we have

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\} = \rho(f) = \rho(g). \quad (3.1)$$

If we suppose that $\rho(f + g) < \rho(f) = \rho(g)$, then by (1.6), we get $\tau(g) = \tau(f + g - f) = \tau(f)$ and this is a contradiction. Hence $\rho(f + g) = \rho(f) = \rho(g)$. Now, we prove that $\rho(fg) = \rho(f) = \rho(g)$. Also we have by Theorem 1.1

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\} = \rho(f) = \rho(g). \quad (3.2)$$

If we suppose that $\rho(fg) < \rho(f) = \rho(g) = \rho\left(\frac{1}{f}\right)$, then by (1.6), we can write $\tau(g) = \tau\left(fg\frac{1}{f}\right) = \tau\left(\frac{1}{f}\right) = \tau(f)$ and this is a contradiction. Hence $\rho(fg) = \rho(f) = \rho(g)$.

4. PROOF OF THEOREM 1.6

(i) Suppose that $\rho(f) > \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$). By the definition of the type, we have

$$\begin{aligned} \tau_M(f + g) &= \limsup_{r \rightarrow +\infty} \frac{\log M(r, f + g)}{r^{\rho(f+g)}} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\log M(r, f) + \log M(r, g) + O(1)}{r^{\rho(f+g)}} \end{aligned} \quad (4.1)$$

Since $\rho(g) < \rho(f)$, then $\rho(f + g) = \rho(f)$. Thus, from (4.1), we obtain

$$\tau_M(f + g) \leq \limsup_{r \rightarrow +\infty} \frac{\log M(r, f)}{r^{\rho(f)}} + \limsup_{r \rightarrow +\infty} \frac{\log M(r, g)}{r^{\rho(f)}} = \tau_M(f). \quad (4.2)$$

On the other hand, since

$$\rho(f+g) = \rho(f) > \rho(g), \quad (4.3)$$

then by (4.2), we get

$$\tau_M(f) = \tau_M(f+g-g) \leq \tau_M(f+g). \quad (4.4)$$

Hence by (4.2) and (4.4), we obtain $\tau_M(f+g) = \tau_M(f)$. Now we prove $\tau_M(fg) \leq \tau_M(f)$. Since $\rho(g) < \rho(f)$, then $\rho(fg) = \rho(f)$. By the definition of the type, we have

$$\begin{aligned} \tau_M(fg) &= \limsup_{r \rightarrow +\infty} \frac{\log M(r, fg)}{r^{\rho(fg)}} \leq \limsup_{r \rightarrow +\infty} \frac{\log M(r, f) + \log M(r, g)}{r^{\rho(f)}} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\log M(r, f)}{r^{\rho(f)}} + \limsup_{r \rightarrow +\infty} \frac{\log M(r, g)}{r^{\rho(f)}} = \tau_M(f). \end{aligned} \quad (4.5)$$

(ii) Suppose that $\rho(f) = \rho(g) = \rho(fg) = \rho(f+g) = \rho$ ($0 < \rho < +\infty$). Given $\varepsilon > 0$, we have

$$\begin{aligned} M(r, f+g) &\leq M(r, f) + M(r, g) \leq \exp((\tau_M(f) + \varepsilon)r^\rho) \\ &+ \exp((\tau_M(g) + \varepsilon)r^\rho) \leq 2 \exp((\max\{\tau_M(f), \tau_M(g)\} + \varepsilon)r^\rho) \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} M(r, fg) &\leq M(r, f)M(r, g) \leq \exp((\tau_M(f) + \varepsilon)r^\rho + (\tau_M(g) + \varepsilon)r^\rho) \\ &= \exp((\tau(f) + \tau(g) + 2\varepsilon)r^\rho). \end{aligned} \quad (4.7)$$

Since $\varepsilon > 0$ is arbitrary by (4.6), (4.7) and the definition of the type, we have

$$\tau_M(f+g) \leq \max\{\tau_M(f), \tau_M(g)\} \quad (4.8)$$

and

$$\tau_M(fg) \leq \tau_M(f) + \tau_M(g). \quad (4.9)$$

On the other hand and without lost of generality we suppose that $\tau_M(f) > \tau_M(g)$. Since $\rho(g) = \rho(f+g)$, then by using (4.8) we have

$$\tau_M(f) = \tau_M(f+g-g) \leq \max\{\tau_M(f+g), \tau_M(g)\} = \tau_M(f+g). \quad (4.10)$$

From (4.8) and (4.10) we obtain $\tau_M(f+g) = \tau_M(f) = \max\{\tau_M(f), \tau_M(g)\}$.

5. PROOF OF THEOREM 1.7

Suppose that $\rho(f) = \rho(g)$ ($0 < \rho(f), \rho(g) < \infty$) and $\tau_M(f) \neq \tau_M(g)$. It's easy to see that

$$\rho(f+g) \leq \rho(f) = \rho(g).$$

If we suppose that $\rho(f+g) < \rho(f) = \rho(g)$, then by (1.10)

$$\tau_M(g) = \tau_M(f+g-f) = \tau_M(f)$$

which is a contradiction. Hence $\rho(f+g) = \rho(f) = \rho(g)$.

6. PROOF OF THEOREM 1.8

It's known that if f is of non-integer order then by Lemma 1.9, we have $\rho(f) = \lambda(f)$. Since $\rho(f) > \rho(g)$ and $\tau(f) \neq \tau(g)$ if $\rho(f) = \rho(g)$, then by Theorem 1.2 and Theorem 1.5, we have

$$\rho(f + g) = \rho(fg) = \rho(f) = \lambda(f).$$

On the other hand, $f + g$ and fg are of non-integer orders. Then

$$\rho(f + g) = \lambda(f + g) = \rho(fg) = \lambda(fg) = \rho(f) = \lambda(f).$$

7. OPEN PROBLEM

It is interesting to study the growth ratios of Nevanlinna's characteristic of composite meromorphic function with that of left or right factor of the composition by using the results obtained in this paper.

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