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Orders of solutions of an n - th order linear differential equation with entire coefficients *

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Abstract

We study the solutions of the differential equation

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+\cdots+A_1(z)f'+A_0(z)f=0,$$

where the coefficients are entire functions . We find conditions on the coefficients so that every solution that is not identically zero has infinite order .

1 Introduction

For $n\geq 2$, we consider the linear differential equation

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_0(z),...,A_{n-1}(z)$ are entire functions with $A_0(z)\not\equiv 0$. Let $\rho(f)$ denote the order of the growth of an entire function f as defined in [4] :

$$\rho(f)=\lim_{r\to\infty}\sup \log (\log (\max_{|} \log_r^{z|=r} \mid f(z) \mid)).$$

The value $T(r,f)=\log(\max_{|z|=r} \mid f(z) \mid)$ is known as the Nevanlinna character - istic of f[4]. It is well known that all solutions of (1 . 1) are entire functions and when some of the coefficients of (1 . 1) are transcendental , (1 . 1) has at least one solution with order $\rho(f)=\infty$. The question which arises is :

What conditions on $A_0(z),...,A_{n-1}(z)$ will guarantee that every solution $f\not\equiv 0$ has infinite order ?

In this paper we prove two results concerning this question .

When $A_0(z),...,A_{n-1}(z)$ are polynomials with $A_0(z)\not\equiv 0$, every solution of (1 . 1) is an entire function with finite rational order ; see for example [3] , [5 , pp . 1 99 - 209] , [6 , pp . 1 6 - 1 8] , and [7 , pp . 65 - 67] .

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 In the study of the differential equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.2}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions , Gundersen proved the following results .
Theorem 1 . 1 ([1 , p . 4 18]) *Let* $A(z)$ *and* $B(z) \not\equiv 0$ *be entire functions such that for real constants* $\alpha, \beta, \theta_1, \theta_2$ *with* $\alpha > 0, \beta > 0,$ *and* $\theta_1 < \theta_2$, *we have*

$$| B(z) | \geq \exp\{(1 + o(1))\alpha \mid z \mid^{\beta}\} \tag{1.3}$$

and

$$| A(z) | \leq \exp\{o(1) \mid z \mid^{\beta}\} \tag{1.4}$$

as $z \to \infty$ *with* $\theta_1 \leq \arg z \leq \theta_2$. *Then every s o lution* $f equivalence - negationslash 0$ *of (1 . 2) has infinite o rder .*
Theorem 1 . 2 ([1 , p . 4 19]) *Let* $\{\Phi_k\}$ *and* $\{\theta_k\}$ *be two finite collections of real numbers satisfying* $\Phi_1 < \theta_1 < \Phi_2 < \theta_2 < \cdots < \Phi_n < \theta_n < \Phi_{n+1}$, *where* $\Phi_{n+1} = \Phi_1 + 2\pi$, *and s e t*

$$\mu = 1^{\max} \leq k \leq n^{(\Phi_{k+1}} - \theta_k). \tag{1.5}$$

Suppose that $A(z)$ *and* $B(z)$ *are entire functions such that for s ome constant*

$$\alpha \geq 0,$$
$$| A(z) | = O(| z |^{\alpha}) \quad (1.6)$$

as $z \to \infty$ *with* $\Phi_k \leq \arg z \leq \theta_k$ *for* $k = 1, ..., n$ *and where* $B(z)$ *is tran -* $s_o^{cendental}{}_{rder.}$ *with* $\rho(B) < \pi\mu$. *Then e very s o lution* $f equivalence - negationslash\ 0$ *of (1 . 2) has infinite*

2 Statement and proof of results

In this paper we prove the following two theorems :
Theorem 2 . 1 *Let* $A_0(z), ..., A_{n-1}(z), A_0(z) \not\equiv 0$ *be entire functions such that for real constants* $\alpha, \beta, \mu, \theta_1, \theta_2$, *where* $0 \leq \beta < \alpha, \mu > 0$ *and* $\theta_1 < \theta_2$ *we have*

$$| A_0(z) | \geq e^{\alpha |z|^{\mu}} \tag{2.1}$$

and

$$| A_k(z) | \leq e^{\beta |z|^{\mu}}, \quad k = 1, ..., n-1 \tag{2.2}$$

as $z \to \infty$ *with* $\theta_1 \leq \arg z \leq \theta_2$. *Then every s o lution* $f equivalence - negationslash 0$ *of (1 . 1) has infinite o rder .*

Theorem 2 . 2 *Let $\{\Phi_k\}$ and $\{\theta_k\}$ be two finite co l lections of real numbers sat -*

$$isfying \Phi_1 < \theta_1 < \Phi_2 < \theta_2 < \cdots < \Phi_m < \theta_m < \Phi_{m+1} where \Phi_{m+1} = \Phi_1 + 2\pi,$$

and s e t

$$\mu = 1 \max \leq km_{\leq}^{(\Phi_{k+1}} - \theta_k). \tag{2.3}$$

Suppose that $A_0(z), ..., A_{n-1}(z)$ are entire functions such that for s ome constant

$$\alpha \geq 0,$$

$$| A_j(z) | = O(| z |^{\alpha}), \quad j = 1, ..., n-1 \quad (2.4)$$

as $z \to \infty$ with $\Phi_k \leq \arg z \leq \theta_k$ for $k = 1, ..., m$ and where $A_0(z)$ is tran - s cendental with $\rho(A_0) < \pi/\mu$. Then every s o lution $f \not\equiv 0$ of (1 . 1) has infinite o rder .

Next , we provide a lemma that is used in the proofs of our theorems .

Lemma 2 . 3 ([2 , p . 89]) *Let w be a transcendental entire function of finite o rder ρ. Let $\Gamma = \{(k_1, j1), (k_2, j2), ..., (k_m, j_m)\}$ denote a finite s e t of distinct pairs of integers satisfying $k_i > ji \geq 0$ for $i = 1, ..., m$, and le t $\varepsilon > 0$ be a given constant . Then there exists a s e t $E \subset [0, 2\pi)$ that has lin ear measure zero , such that if $\psi 0 \in [0, 2\pi) - E$, then th ere is a constant $R_0 = R_0(\psi 0) > 0$ such that for all z satisfying $\arg z = \psi 0$ and $| z | \geq R_0$ and for al l $(k, j) \in \Gamma$, we have*

$$|w^{(k)}_{w^{(j)}(z)}{}^{(z)}| \leq | z |^{(k-j)(\rho-1+\varepsilon)} .$$

Proof of Theorem 2 . 1

Suppose that $f equivalence - negationslash 0$ is a solution of (1 . 1) with $\rho(f) < \infty$. Set $\delta = \rho(f)$. Then from Lemma 1 , there exists a real constant $\psi 0$ where $\theta_1 \leq \psi 0 \leq \theta_2$, such that

$$|f^{(k)(z)}_{f(z)}| = o(1) \mid z \mid^k \delta, \quad k = 1, ..., n \tag{2.5}$$

as $z \to \infty$ with $\arg z = \psi 0$. Then from (2 . 5) and (1 . 1) , we obtain that

$$| A_0(z) | \leq o(1) \mid z \mid^{\delta} | A_1(z) | + \cdots + o(1) \mid z \mid^{(n-1)} \delta_{|A_{n-1}(z)|} + o(1) \mid z \mid^{n\delta} \tag{2.6}$$

as $z \to \infty$ with $\arg z = \psi 0$. However this contradicts (2 . 1) and (2 . 2) . Therefore , every solution $f equivalence - negationslash 0$ of (1 . 1) has infinite order .

Next we give an example that illustrates Theorem 2 . 1 .

Example 1 . Consider the differential equation

$$f'' - (3 + 6e^z)f'' + (2 + 6e^z + 11e^{2z})f' - 6e^{3z}f = 0 \tag{2.7}$$

In this equation , for $z = re^{i\theta}, r \to +\infty, \pi 6 \leq \theta \leq \pi 4$ we have

$$| A_0(z) | = \quad | -6e^{3z} | = 6e^{3r} \cos\theta > e^3 \sqrt{2}_2 r,$$
$$| A_1(z) | = 2 + 6e^z + 11e^{2z} | \leq 19e^{2r\cos\theta} \leq 19e\sqrt{3}r < e^{2r}$$
$$| A_2(z) | = | -(3 + 6e^z) | \leq 9e^r \cos\theta \leq 9e\sqrt{3}_2 r < e^{2r}.$$

As we see , conditions (2 . 1) and (2 . 2) of Theorem 2 . 1 are verified . The three linearly independent functions $f1(z) = e^{e^z}$, $f2(z) = e^{2e^z}$, $f3(z) = e^{3e^z}$ are solutions of (2 . 7) with $\rho(f1) = \rho(f2) = \rho(f3) = \infty$.

Next we give a generalization of Example 1 .

Example 2 . Consider the differential equation

$$f^{(n)} + P_{n-1}(e^z)f^{(n-1)} + \cdots + P_1(e^z)f' + \beta e^{\alpha z} f = 0, \tag{2.8}$$

where $\alpha \in \mathbb{R}, \alpha > 0, \beta \in \mathbb{C}$, $\mid \beta \mid\geq 1$, and $P_1, ..., P_{n-1}$ are polynomials . If we t ake the sector $\theta_1 \leq \arg z \leq \theta_2, \theta_1, \theta_2 \in]0, \pi 2[$ with θ_1 near enough to θ_2 such that $\max_{1\leq k\leq n-1} \deg\ (P_k) < \alpha^{\cos\theta_2}_{\cos\theta_1}$, then conditions (2 . 1) and (2 . 2) of Theorem 2 . 1 are satisfied as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. From Theorem 2 . 1 , it follows that every solution $f \not\equiv 0$ of (2 . 8) has infinite order .

Proof of Theorem 2 . 2

Suppose that $f \not\equiv 0$ is a solution of (1 . 1) where $\rho(f) < \infty$ and we set $\beta = \rho(f)$. From Lemma 1 , there exists a set $E \subset [0, 2\pi)$ that has linear measure zero , such that if $\psi 0 \in [\Phi_k, \theta_k) - E$ for some k, then

$$\mid f^{(l)\ (z)}_{f(z)} \mid= O(\mid z \mid^{l\beta}), \quad l = 1, ..., n \tag{2.9}$$

as $z \to \infty$ with $\arg z = \psi 0$. From (2 . 9) , (2 . 4) and (1 . 1) , we obtain that

$$\mid A_0(z) \mid\leq\mid f^{(n)}_f \mid + \mid A_{n-1}(z) \mid\mid f^{(n-1)}_f \mid + \cdots + \mid A_1(z) \mid\mid f^{/}_f \mid= O(\mid z \mid^\sigma) \tag{2.10}$$

as $z \to \infty$ with $\arg z = \psi 0$, where $\sigma = \alpha + n\beta$. Let $\varepsilon > 0$ be a small constant that satisfies $\rho(A_0) < \mu\pi{+}2\varepsilon$ (this is possible since $\rho(A_0) < \pi\mu)$. By using the Phragm é n - Lindel ö f theorem on (2 . 1 0) , it can be deduced that for some integer

$s > 0$

$$\mid A_0(z) \mid= O(\mid z \mid^s) \quad (2.11)$$

as $z \to \infty$ with $\Phi_k + \varepsilon \leq \arg z \leq \theta_k - \varepsilon$ for $k = 1, ..., m$.

Now for each k, we have from (2 . 3) that $\Phi_{k+1} + \varepsilon - (\theta_k - \varepsilon) \leq \mu + 2\varepsilon$, and so $\rho(A_0) < \Phi_{k+1}\pi_{-\theta_k} + 2\varepsilon$. Hence using the Phragm é n - Lindel ö f theorem on (2 . 1 1) we can deduce that $\mid A_0(z) \mid= O(\mid z \mid^s)$ as $z \to \infty$ in the whole complex plane . This means that $A_0(z)$ is a polynomial which contradicts our hypothesis and completes the proof of Theorem 2 . 2 .

Next we give an example that illustrates Theorem 2 . 2 .

Example 3 . If $A_0(z)$ is transcendental with $\rho(A_0) < 2$, then from Theorem 2 . 2 , every solution $f equivalence - negationslash 0$ of the equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_2(z)f'' + (e^{z^3} + e^i z^3_) f\prime + A_0(z)f = 0,$$

where $P_{n-1}, ..., P_2$ are polynomials , is of infinite order .

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