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COURSE HANDOUT

Mechanical Waves and Vibrations Course and Exercises

Stream: Electrical Engineering, Civil Engineering and Process Engineering

Degree: 2nd year engineer

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Preface

This handout is intended for 2nd-year engineer students specializing in the following fields: Technological Sciences (ST): Electrotechnics (ETT), Electronics (EN), Process Engineering and Civil Engineering (GC).

It is a core module that deals with the oscillations of mechanical and electrical systems, a subject that has seen significant growth in recent years. This field has greatly contributed to the development of techniques capable of solving physical problems across various disciplines.

This document is a detailed course with exercises. It is divided into two main parts: "Vibrations and Mechanical Waves," structured into five chapters listed below:

Chapter I: Linear Systems with One Degree of Freedom.

Chapter II: Free Damped Linear Systems with One Degree of Freedom.

Chapter III: Damped Forced Oscillations: Systems with One Degree of Freedom.

Chapter IV: Oscillations of multi-degree-of-freedom systems.

Chapter V: Wave propagation

We were compelled to condense the course slightly, as the subject is covered in a single 12-week semester, with one lecture and one tutorial per week. However, we have ensured that the essential content remains intact.

Additional knowledge will be acquired in a separate subject, the "Vibrations and Mechanical Waves" practical work, with a 3-hour lab session every 15 days.

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Linear Systems with One Degree of Freedom

Chapter I

Linear Systems with One Degree of Freedom

I.1. Overview of Vibrations

I.1.1. Definition of an Oscillation

Oscillation is the back and forth repetitions, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. Familiar examples of oscillation include a swinging pendulum and alternating current. Oscillations can be used in physics to approximate complex interactions, such as those between atoms.

Oscillations occur not only in mechanical systems but also in dynamic systems in virtually every area of science: for example the beating of the human heart (for circulation), business cycles in economics, predator–prey population cycles in ecology, geothermal geysers in geology, vibration of strings in guitar and other string instruments, periodic firing of nerve cells in the brain, and the periodic swelling of Cepheid variable stars in astronomy. The term *vibration* is precisely used to describe a mechanical oscillation.

Oscillation, especially rapid oscillation, may be an undesirable phenomenon in process control and control theory (e.g. in sliding mode control), where the aim is convergence to stable state. In these cases it is called chattering or flapping, as in valve chatter, and route flapping. A very fast oscillation is called vibration.

Examples: the motion of a pendulum, a weight suspended from a spring, or a floating cylinder in a liquid, etc.

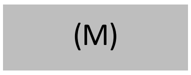
Oscillations can be subdivided into:

- **Free oscillations**: An oscillator is considered free if it oscillates without external intervention (i.e., without friction) as it returns to equilibrium.
- **Damped oscillations**: The oscillator is subject to frictional forces that dissipate energy, causing the oscillations to gradually diminish and eventually stop.
- **Forced oscillations**: An oscillator is forced when an external action supplies energy to it.
- **Damped forced oscillations**: Here, a periodic external force (excitation) compensates for energy losses due to friction, allowing the oscillations to be sustained without damping.

I.2. Vibration

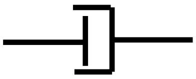
A motion that repeats itself over time is called vibratory or oscillatory motion. Any mechanical system with mass and a flexible element (such as a spring) or its equivalent in an electrical system (like inductance or capacitance) can undergo vibratory motion. In most cases, these systems include a damper in mechanical systems or a resistor in electrical systems. In general, a vibratory system can be represented by the following schematics:

- **Mechanical System:**



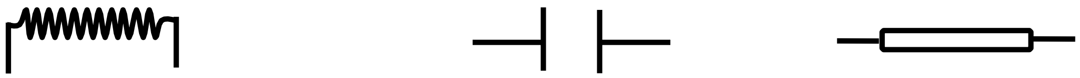
Mass

spring



damper

- **Electrical System:**



Inductor Capacitor Resistor

Figure I.1 Graphical Representation of the Vibratory System Elements

In another way, it can be said that every vibratory system consists of three components:

- **Mechanical System**

- A means of storing kinetic energy, which is the **mass**.
- A means of storing potential energy, which is the **spring**.
- A means of dissipating energy, which is the **damper**.

- **Electrical System**

- A means of storing electrical energy, which is the **capacitor**.
- A means of storing magnetic energy, which is the **inductor**.
- A means of dissipating energy, which is the **electrical resistor**.

During vibrations (oscillations), energy transforms from one form to another: from kinetic energy to potential energy and vice versa in a mechanical system, or from electrical energy to magnetic energy and vice versa in an electrical system. The simplest example of a mechanical vibratory system is the simple pendulum. In each oscillation, energy is converted from potential energy to kinetic energy and vice versa.

Assuming that, at the initial moment, the mass is at position 1 (Fig. I.2), the total energy is in the form of potential energy.

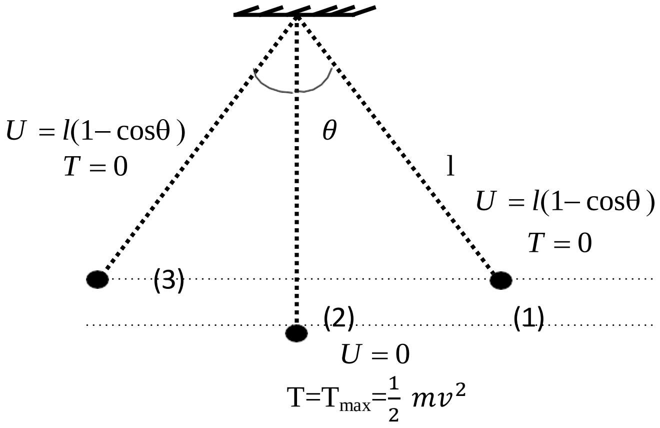


Figure I.2 Transformation of potential energy to kinetic energy and vice versa in the vibrations of a simple pendulum.

When the mass is released, the kinetic energy increases while the potential energy decreases until the complete transformation of potential energy into kinetic energy at position 2 (fig. I.2), where the mass has completed a quarter of the cycle. From position 2, potential energy increases and kinetic energy decreases until the complete transformation into potential energy at position 3 (the mass has completed half a cycle). The return of the mass to position 1 follows the same process, constituting a complete cycle. In a real scenario, during the movement of the mass, some energy is exchanged with the external environment. This portion is irrecoverable, leading to energy loss in the system with each cycle.

I.3. Classification of Vibrations

A vibration is a motion around the equilibrium position. It is characterized by a motion equation of the type of a second-order differential equation of the form:

$$\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = A(t) \qquad \text{(I.1)}$$

with:

y: The displacement (m)

$\dot{y}$: The velocity (m/s)

$\ddot{y}$: The acceleration (m/s^2)

δ: The damping coefficient

ω_0: The natural frequency (rad/s)

A(t): The forcing function.

I.3.1 Free Vibrations and Forced Vibrations

After an initial disturbance, a vibrating system is left without any action from an external force; in this case, the vibrations are known as free vibrations. The oscillation of a simple pendulum is an example of this type of vibration. If the system is subjected to an external force throughout these vibrations, the resulting vibrations are called forced vibrations. The simple pendulum can also serve as an example if the mass is subjected to an external periodic force.

I.3.2 Damped Vibrations and Undamped Vibrations

If the total energy of the system is conserved during the vibrations (meaning there is no energy dissipation), these vibrations are referred to as undamped vibrations. In contrast, if the system loses energy during these vibrations (considering the air resistance on the mass of the simple pendulum in the previous example), after a certain time, the mass stops due to energy dissipation. These vibrations are called damped vibrations.

I.3.3 Regular Vibrations and Irregular Vibrations

If the amplitude of the vibratory motion is known at all times, the resulting vibrations are called regular or deterministic. In other words, it is the vibratory motion for which the amplitude can be predicted at any moment. Conversely, if the amplitude cannot be predicted, the vibrations are referred to as irregular or non-deterministic.

I.4 Definition of a Periodic Motion

A periodic motion is one that repeats itself, with each cycle being identical. The duration of one cycle is called the period T, which is expressed in seconds (s).

- The number of repetitions per second is called frequency (denoted as f, measured in Hertz (Hz) or (s^{-1}) is related to the period by:

$$f = \frac{1}{T} \tag{I.2}$$

The number of rotations per second is called angular frequency (denoted as ω\omegaω, measured in radians per second (rad/s)).

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{I.3}$$

Mathematically, the periodic motion with period T is defined by: $x(t+T)=x(t)$.

I.4.1 Vibratory Motion

A vibratory motion is a periodic sinusoidal motion, also referred to as harmonic motion, when the material body in motion reaches the same position and has the same velocity after regular time intervals of duration T. A vibratory motion can also be defined by its frequency f. The frequency indicates the number of complete oscillations (in the back-and-forth sense) occurring per second.

I.4.2 Sinusoidal Vibratory Motion

If the displacement x (y or z) of a vibrating point is a simple sinusoidal function of time of the form:

$x(t) = A\sin(\omega t + \varphi)$ or $x(t) = A\cos(\omega t + \varphi)$

x(t) is called the displacement (or position) at time t.

A: the amplitude of a motion or the maximum displacement.

ω: the angular frequency of the motion, expressed in (rad/s).

φ: the initial phase, corresponding to the phase at time t=0s.

Example: In Figure I.3, consider a sinusoidal vibratory motion of the form:

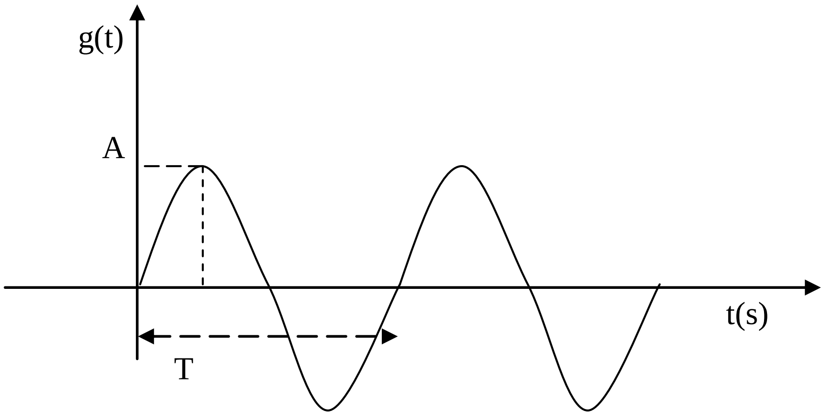


Figure I.3 Example of a sinusoidal periodic motion.

x(t)= Acos(ωt + φ), with a period T=1s and an amplitude of 1cm.

$$\omega = \frac{2\pi}{T} = \frac{6.28}{1} = 6.28 rad/s$$

A = 1 ⇒ x(t) = cos (2πt + φ).

À t = 0 ⇒ x(0) = 0 ⇒ φ $=\frac{\pi}{2}$

⇒X(t)=cos (2πt + $\frac{\pi}{2}$) (cm)

I.4.3 Oscillatory Motion

An oscillating system is characterized by periodic movements around an equilibrium position due to an external disturbance. When the motion is sinusoidal, the oscillator is referred to as harmonic.

Note: The oscillation is said to be anharmonic if the system evolves according to a periodic law of any non-sinusoidal form (Fig I.4).

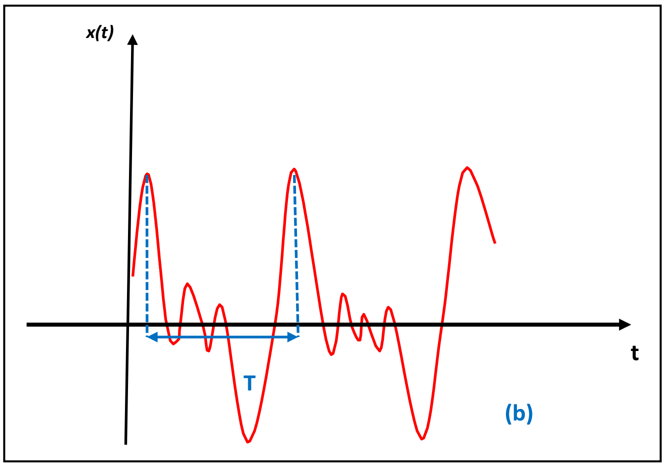


Figure 1.4 Representation of an anharmonic oscillation.

Two types of vibrations (oscillations) are distinguished:

- Mechanical oscillations (simple pendulum, vibrating string, etc.)
- Electromagnetic oscillations (light, radio waves, etc.)

I.5 Generalized Coordinates of a Physical System

I.5.1 Generalized Coordinates

Generalized coordinates are any set of variables that can specify the state of a physical system. These coordinates are not always assumed to be independent, and their advantage over Cartesian coordinates is the ability to choose the most suitable coordinates to represent the system, taking its constraints into account.

Example: In the case of a pendulum, it is advantageous to use the angle of the pendulum as one of the generalized coordinates. Generalized coordinates number n $\leq 3N$, where N is the number of points needed to describe the system, and are often denoted as $q_1(t)$, $q_2(t)$ … … … $q_N(t)$.

I.5.2 Degrees of Freedom

The degree of freedom (DOF) of a system refers to the system's ability to perform translational and rotational motion relative to the axes. It is calculated as the number of related generalized coordinates needed to configure all elements of the system at any given moment minus the number of relationships that connect these coordinates:

$$d = N - r \qquad \text{(I.4)}$$

d: Degree of freedom.

N: Number of generalized coordinates.

r: Number of relationships connecting these coordinates to each other.

- **Example:**

A homogeneous cylinder of mass M and radius R rolls without slipping on a horizontal platform.

We have two generalized coordinates x and θ, so N=2.

x and θ are related by the equation: x =rθ, therefore r=1.

Thus, the number of degrees of freedom d=N−r=1.

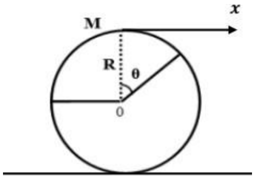


Figure I.5.a Cylinder

It is also possible to determine the degree of freedom (DOF) using the following relation:

DOF = [number of generalized coordinates] - [coordinates = 0 or constant].

We can list a few examples:

- **Example:** (Particle in free fall)

Out of the three coordinates x,y,z, the number of generalized coordinates is 3.

We have two constant coordinates: y and z.

DOF= 3-2 =1.

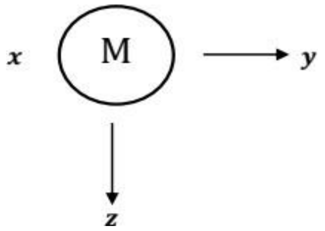


Figure I.5.b Particle in Free Fall

- **Example:**

Consider a mechanical system consisting of two points connected by a rod of length L. Find the number of degrees of freedom.

$M_1(x_1, y_1, z_1)$ so N=6

$M_2(x_2, y_2, z_2)$

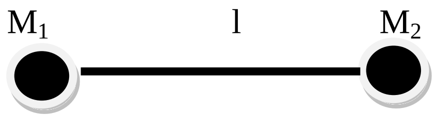


Figure I.5.c Particle in Free Fall

Constraint equation, $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = cst \Rightarrow r = 1$

So d=5 (d=N-r=6-1=5)

I.6 Choice of Method

When choosing the calculation method to derive the equation of motion (EOM), we distinguish between:

- Newton's equation.
- Lagrange's equation.

I.6.1 Newton's Equation

This formalism is based on the fundamental principle of dynamics and is applied according to the type of motion, whether it be translation or rotation.

I.6.1.1 Translational Motion

If a mass m system is subjected to external forces, the fundamental law of dynamics (F.L.D.) gives us:

$$\sum_i \overrightarrow{F_i} = m\overrightarrow{\gamma_i} = m\overrightarrow{a_i} \qquad \text{(I.5)}$$

I.6.1.2 Rotational Motion

The fundamental law of dynamics for rotational motion is expressed as:

$$\sum_i M_\Delta\left(\overrightarrow{F_i}\right) = J\ddot{\theta} = J\frac{d^2\theta}{dt^2} \qquad \text{(1.6)}$$

The kinetic energy of rotation T of a body with moment of inertia $J_{/\Delta}$ is:

$$T = \frac{1}{2}J_{/\Delta}\dot{\theta}^2 \qquad \text{(I.7)}$$

(J is the moment of inertia with respect to the point on the axis of rotation.

Table I.1 Moments of inertia of solids

Shape	Moment of inertia with respect to the center of gravity G (J/s)
Rod (Length **L**, Mass **M**)	$\frac{1}{12}ML^2$
Disk (Radius ***R***, Mass ***M***)	$\frac{1}{2}MR^2$
Ring (Radius ***R***, Mass ***M***)	MR^2
Cylinder (Radius ***R***, Mass ***M***)	$\frac{1}{2}MR^2$
Solid Sphere (Radius ***R***, Mass ***M***)	$\frac{2}{5}MR^2$
Hollow Sphere (Radius ***R***, Mass ***M***)	$\frac{2}{3}MR^2$
Point Mass ***m***	0

Note: The moment of inertia of a mass M with any shape around a point A different from the center of gravity G is given by:

$J_{/A}=J_{/G}+M(AG)^2$ (Huygens-Steiner Theorem)

I.6.2 Lagrange's Equation

The Lagrangian function (Lagrangian of the system) is the difference between the kinetic energy T and the potential energy U of the system:

$$L = T - U \tag{I.8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{ext,q} \tag{I.9}$$

q: is the generalized coordinate that characterizes the vibratory motion.

$Fext, q$: The generalized external forces.

D: is the dissipation function**:** $= \frac{1}{2}\alpha \dot{x}^2$, where α is the coefficient of viscous friction.

I.7 Free Undamped Systems (Free Oscillators)

An oscillating system in the absence of any excitation force (external forces) is called a free (undamped free oscillator). The number of independent variables describing the system is called the degree of freedom (DOF).

I.8 Harmonic Oscillator

In mechanics, a harmonic oscillator is one that, when displaced from its equilibrium position by a distance x (or angle θ), is subjected to a restoring force that is opposite and proportional to the displacement x (or θ).

I.9 Equation of Motion

The differential equation of an undamped free motion is in the form: $\ddot{x} + \omega_0^2 x = 0$

The Lagrange equation for free oscillations of a conservative system is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \qquad \text{(I.10)}$$

- **Example 1:** (Mass-Spring)

Consider a mass attached to a spring with a spring constant K, applying a displacement to the mass in the x direction.

F: Restoring force of the spring: $F = -Kx$

- Newton's Dynamic Principle: $\sum \vec{F} = m\vec{a}$

Projection on the X-Axis

$$\boldsymbol{m\vec{g} + \vec{T} = m\vec{a}}$$

$$m\ddot{x} = mg - T \Rightarrow m\ddot{x} = mg - (-kx)$$

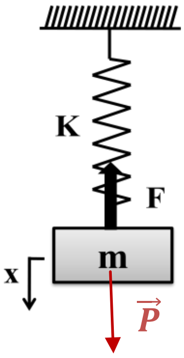


Figure I.6.a Mass-Spring

At Equilibrium:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{P} + \vec{T} = \vec{0} \Rightarrow mg + kx = 0 \Rightarrow m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

$\ddot{x} + \omega_0^2 x = 0$ is the Differential Equation of Motion , $\omega_0^2 = \frac{k}{m}$

- **Calculation of the Lagrangian:** $L = T - U = \frac{1}{2}m\dot{x}^2 - (\frac{1}{2}kx^2)$

Lagrange's Equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$

$$\left.\begin{array}{c} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ddot{x} \\ \frac{\partial L}{\partial x} = -kx \end{array}\right. \Rightarrow m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

The Ratio $\frac{k}{m}$ Being Positive and Setting: $\omega_0 = \sqrt{\frac{k}{m}}$

We obtain the differential equation of a harmonic vibration in the form: $\ddot{x} + \omega_0^2 x = 0$

- **Example 2:** (Simple Pendulum)

$$M = \begin{cases} x = lsion\theta \;\Rightarrow \dot{x} \;= l\dot{\theta}\,cos\theta \\ y = lcos\theta \Rightarrow \dot{y} = -l\dot{\theta}\,sin\theta \end{cases}$$

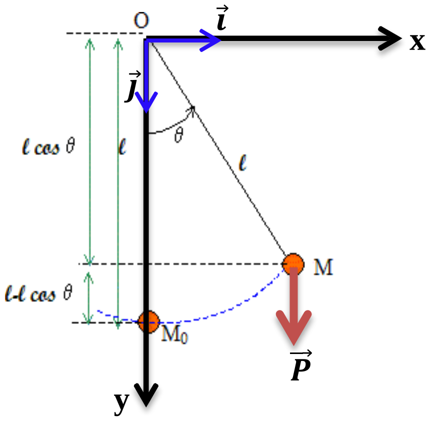


Figure I.6.b Simple Pendulum

The kinetic energy: $T = \frac{1}{2}mv^2$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(lsion\theta - lcos\theta) = l\frac{dsin\theta}{d\theta}\frac{d\theta}{dt} - l\frac{dcos\theta}{d\theta}\frac{d\theta}{dt} = lcos\theta\dot{\theta} + lsin\theta\dot{\theta}$$

So $v^2 = l^2\dot{\theta}^2\,(cos^2 + sin^2)$

$$\Rightarrow T = \frac{1}{2}m(\dot{x} + \dot{y})^2 = \frac{1}{2}ml^2\dot{\theta}^2(cos^2 + sin^2)$$

$$\Rightarrow T = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential Energy of a System: $U = mgh$

$$\Rightarrow h = l - lcos\theta = l(1 - cos\theta)$$

$$U = mgl(1 - cos\theta)$$

Calculation of the Lagrangian: L=T-U=$\frac{1}{2}ml^2\dot{\theta}^2 - (-mgl(1 - cos\theta))$

Lagrange's Equation:$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = 0$

$$\begin{cases} \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\,(ml^2\dot{\theta}) = ml^2\ddot{\theta} \\ \frac{\partial L}{\partial\theta} = -mglsin\theta\ (if\ \ \theta \ll\ \ \Rightarrow\ \ sin\theta = \theta\ ) \end{cases} \Rightarrow ml^2\ddot{\theta} - (-mgl\theta) = 0$$

$$\ddot{x} + \frac{g}{l}x = 0$$

The Ratio $\frac{g}{l}$ Being Positive and Setting: $\omega_0 = \sqrt{\frac{g}{l}}$

We obtain the differential equation of a harmonic vibration in the form: $\ddot{x} + \omega_0^2 x = 0$

This second-order, homogeneous differential equation, with $\omega_0^2 = \frac{g}{l}$ being the natural frequency of the oscillator, must be positive for vibration to occur.

I.10 Solution of the Differential Equation

The solution of the equation $\ddot{x} + \omega_0^2 x = 0$ is in the form: $x(t) = Ae^{rt}$

where r is a real number and A is a positive constant.

$$\dot{x}(t) = Are^{rt}$$

$$\ddot{x}(t) = Ar^2e^{rt}$$

$$\Leftrightarrow (r^2 - \omega_0^2)Ae^{rt} = 0$$

$$r_1 = i\omega_0 \ \ and \ \ r_2 = -i\omega_0$$

We obtain two solutions:

$$x_1(t) = A_1e^{r_1} = A_1e^{i\omega_0 t}$$

$$x_2(t) = A_2e^{r_2} = A_2e^{-i\omega_0 t}$$

The general solution of the equation of motion

$$x(t) = x_1(t) + x_2(t)$$

$$x(t) = A(e^{i\omega_0 t} + e^{-i\omega_0 t})$$

According to Euler's formula: $e^{\pm i\omega_0 t = cos\omega_0 t \pm isin\omega_0 t}$

So ; $x(t) = A_1(cos\omega_0 t + isin\omega_0 t) + A_2(cos\omega_0 t - isin\omega_0 t)$

$$x(t) = (A_1 + A_2)cos\omega_0 t + i(A_1 - A_2)sin\omega_0 t$$

$$x(t) = Bcos\omega_0 t + Csin\omega_0 t$$

Let's suppose: $B = Dcos\theta \; and \; C = Dsin\theta$

So: $x(t) = Dcos\theta cos\omega_0 t + Dsin\theta sin\omega_0 t$

$x(t) = Dcos(\omega_0 + \varphi)$

We find the general solution of equation $\ddot{x} + \ \omega_0^2 x = 0$, is

$$x(t) = Dcos(\omega_0 + \varphi)$$

D and φ these are constants derived from the initial conditions.

I.11 Electrical Harmonic Oscillator

Let’s consider an LC electrical circuit consisting of a capacitor C and an inductor L. This circuit acts as an electrical harmonic oscillator, similar to a mechanical system like a spring with a mass.

- **Kirchhoff's Loop Law**

By applying Kirchhoff's loop law to this circuit, we get the following relation:

$$V_C + V_L = 0V$$

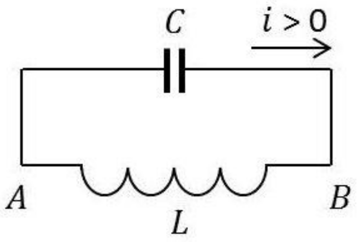


Figure I.6.c LC Circuit

$\frac{q}{C}+L\frac{di}{dt}=0 \Rightarrow \frac{q}{C}+L\ddot{q}=0 \Rightarrow \ddot{q}+\frac{1}{LC}q=0$ $\quad\quad (i=\frac{dq}{dt})$

$\Rightarrow \ddot{q}+\omega_0^2 q=0$ with $\omega_0^2=\frac{1}{LC}$

Calculation of the Lagrangian: $L=T-U$

$T=V_L\, dq=L\frac{di}{dt}\,dq=L\frac{di}{dt}\,i\,dt=L\,i\,di=\frac{1}{2}Li^2=\frac{L}{2}\dot{q}^2$

$U=V_c\, dq=\frac{q}{C}\,dq=\frac{1}{2C}q^2$

$L=T-U=\frac{L}{2}\dot{q}^2-\frac{1}{2C}q^2$

Lagrange's Equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0 \Rightarrow \frac{d}{dt}(L\dot{q})—\frac{1}{C}q.=0$

$$\Rightarrow L\ddot{q}+\frac{1}{C}q=0 \Rightarrow \ddot{q}+\frac{1}{LC}q=0 \Rightarrow \ddot{q}+\omega_0^2 q=0$$

I.12 Mathematical System (Simple Model)

The simplification of a complex vibratory system into a simple model representing the real case is done by determining the equivalent spring from all the existing springs and the equivalent mass from all the masses that make up the system. In what follows, we will provide simple examples to help understand the concept of the equivalent spring and the equivalent mass.

I.12.1 Equivalent Spring

In practice, springs are found arranged either in series or in parallel.

I.12.1.a Springs in Parallel

Would you like to continue from here or adjust the translation further?

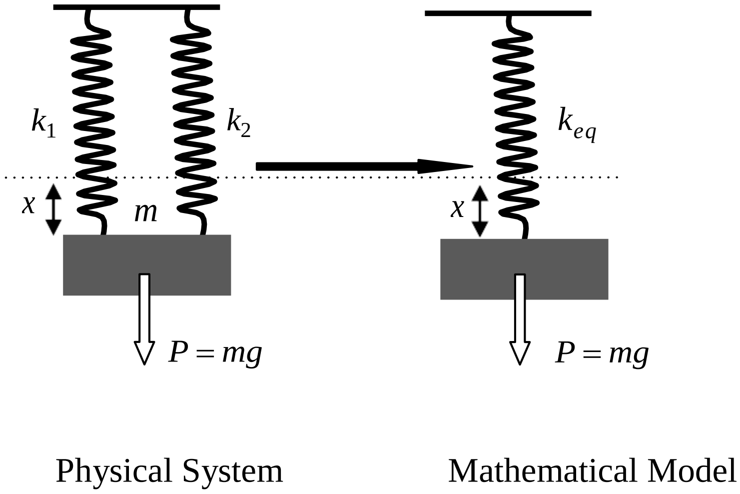


Figure 1.7 Equivalent Spring for Two Springs in Series in a Mass-Spring System

The equilibrium equations for the system are written as follows:

For the real system:

$$p = k_1 x + k_2 x \tag{I.11}$$

$$p = (k_1 + k_2)x \tag{I.12}$$

For the Mathematical Model:

$$p = k_{eqv} x \tag{I.13}$$

From relations (I.12) and (I.13), the stiffness constant of the equivalent spring for two springs in series can be derived using the following relation:

$$k_{eqv} = k_1 + k_2 \tag{I.14}$$

If the system consists of multiple springs in series, the stiffness constant of the equivalent spring is given by the following relation:

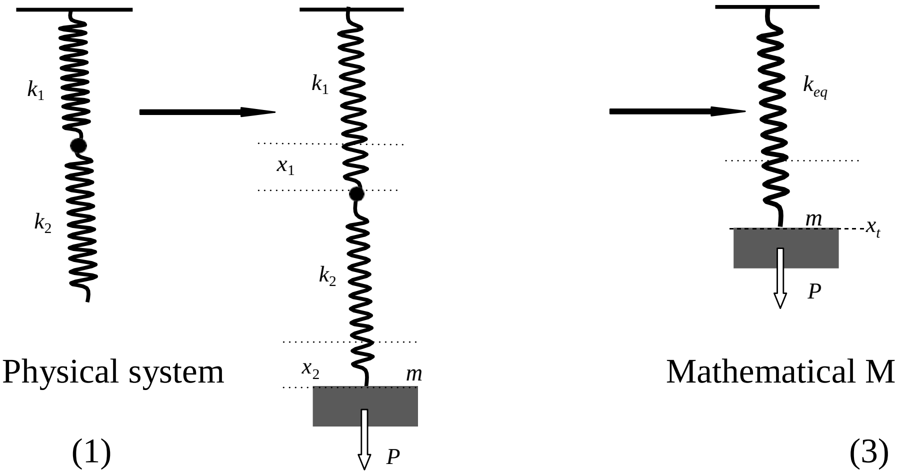
$$k_{eqv} = \sum_i k_i \tag{I.15}$$

I.12.1.b Springs in Series

The suspension of mass m at the free end of two springs k_1 and k_2 causes elongations x_1 and x_2 in k_1 and k_2, respectively. The total elongation is given by:

$$x_{eqv} = x_1 + x_2 \tag{I.16}$$

This relation reflects the fact that for springs in series, the total displacement is the sum of the individual displacements of each spring.



Physical system

(1)

Physical system

(2)

Mathematical Model

(3)

Figure 1.8 Equivalent Spring of a Mass-Spring System with Two Springs in Series

At mechanical equilibrium: If we consider the real system, the following can be written:

$$p = k_1 x \tag{I.17}$$

$$p = k_2 x \tag{I.18}$$

For the mathematical Model:

$$p = k_{eqv} x_t \tag{I.19}$$

From the previous relations, we can write:

$$k_1 x_1 = k_{eqv} x_t \Rightarrow x_t = \frac{k_{eq}}{k_1} x_t \tag{I.20}$$

$$k_2 x_2 = k_{eqv} x_t \Rightarrow x_t = \frac{k_{eq}}{k_2} x_t \tag{I.21}$$

We sum equations (1.8) and (1.9), taking into account the following relation:

$$x_t = x_1 + x_2 \tag{I.22}$$

$$x_1 + x_2 = \frac{k_{eq}}{k_1} x_t + \frac{k_{eq}}{k_2} x_t \Rightarrow x_t = k_{eq} (\frac{1}{k_1} + \frac{1}{k_2}) x_t$$

In the end, we obtain the relation that gives the stiffness constant of the equivalent spring:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{I.23}$$

In the general case where the system consists of multiple springs in parallel, the stiffness constant of the equivalent spring can be expressed as follows:

$$\frac{1}{k_{eq}} = \sum_i \frac{1}{k_i} \tag{I.24}$$

Exercise 1

Solve the following second-order differential equations:

1-) $5x''+4x=0$

2-) $3x''+5x'+6x=0$

3-) $3x''+6x'+6x=0$

4-) $3x''+\sqrt{48}x'+8x=0$

Exercise 2

A vibratory motion is characterized by the following displacement: $x(t)=6\cos(27t+\frac{\pi}{3})$ (cm)

1. Determine the maximum amplitude.

2. Provide the natural angular frequency, the frequency, and the period of the motion.

3. Express the initial phase (the phase shift at the origin).

4. Calculate the displacement, velocity, and acceleration at the moments t=0s and t=0.5s.

Exercise 3

A harmonic motion is described by: $X(t) = X\cos(\omega_0 t + \varphi)$ The initial conditions are: $x(0)=x_0$, $\dot{x}(0) = \dot{x}_0$,

1. Calculate X and φ.

2. Express $x(t)$ in the form of: $x(t)=B\cos(\omega_0 t)+C\sin(\omega_0 t)$ and deduce B et C

Exercise 4

Determine the equivalent spring constants of the systems shown in the figure below:

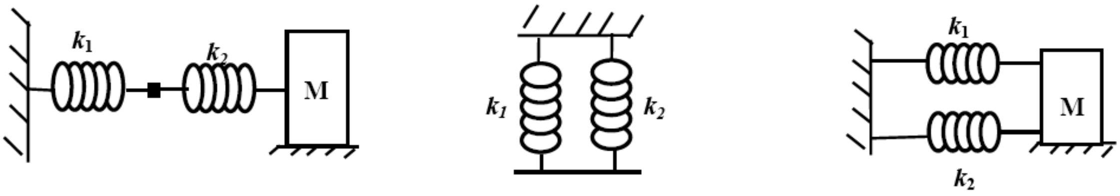


Figure1

Exercise 5

Determine for each system shown below (see fig 2):

1- The number of degrees of freedom, the kinetic energy, and the potential energy.

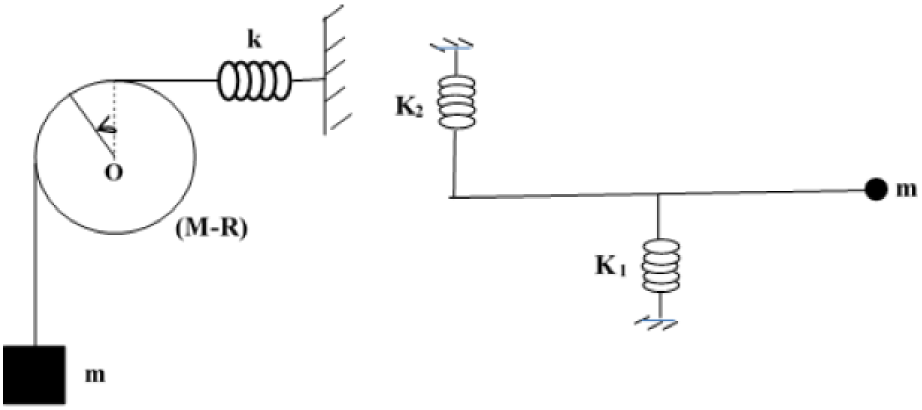


Figure2

Exercise 6

Consider a mechanical system below (Figure 3) consisting of two point masses (*m* and *m')* attached to the free ends of a rod with mass M and length *2L*. This system is in rotational motion about a fixed point *A*. Calculate the kinetic energy and potential energy of the system.

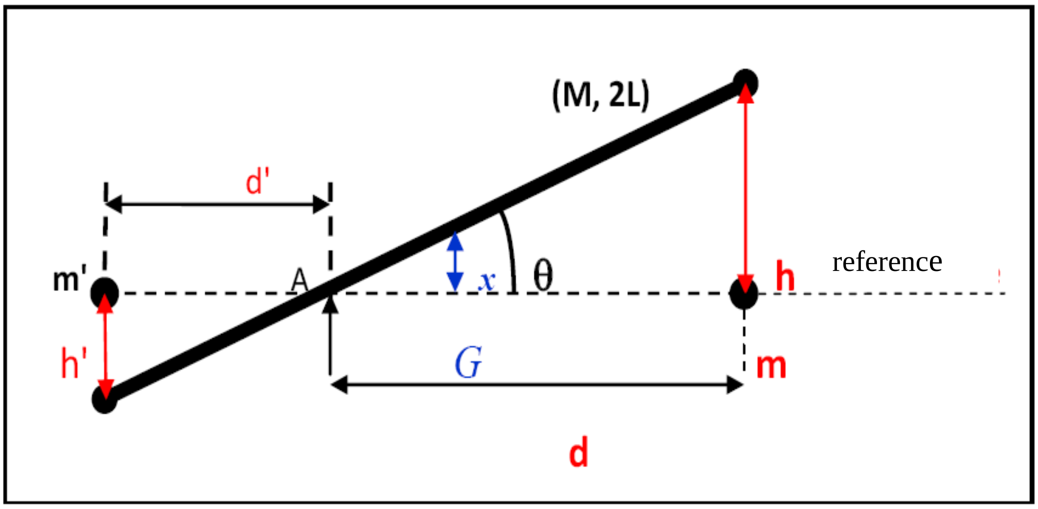


Figure3

Exercise7

A homogeneous pulley (see Figure 4) has a mass *M=3kg* and a radius *R=0.4m*. We denote $I/0=J/0=\frac{1}{2}MR^2$, the moment of inertia of the pulley. The pulley is suspended by an inextensible rope to a fixed frame. At the ends of the pulley, a spring with stiffness *k=62N/m*, and a mass *m=0.6kg* are attached by a mass less inextensible string. We also neglect the mass of the spring and the friction around the axis of the pulley. Let x be the vertical displacement of the mass m.

1-Express the kinetic energy *T* and the potential energy *U* of the system as functions of θ.

2-Find the Lagrangian *L* and derive the equation of motion.

3-Find the natural angular frequency ω_0, the natural period T_0, and the natural frequency f_0.

4-Find the final solution using the initial conditions: $\theta(0) = \frac{\pi}{26}$; $\theta'(0) = 0$.

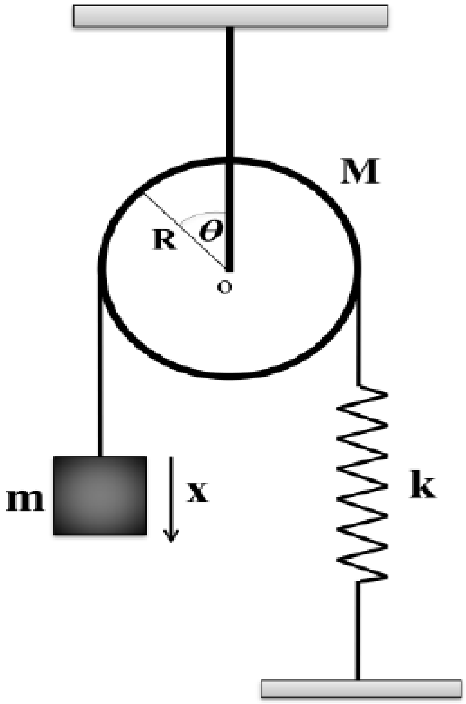


Figure4

Exercise 8

Consider a pulley (hollow cylinder) of mass *M* and radius *R* (Figure 5), which can rotate freely around its fixed axis. The pulley is suspended at its center by an inextensible rope attached to a fixed frame. On both sides of the pulley, two springs with the same stiffness *k* are attached at its periphery. The other ends of the two springs are connected to the ground. The moment of inertia is given as $J_{cylindre/O}=MR^2$.

Given numerical values: $M=2\ kg$, $K=70\ N/m$, $R=0.3\ m$

1-What is the type of the system?

2-Find the kinetic energy T and the potential energy U.

3-Determine the differential equation of motion in terms of θ.

4-Find the final solution using the initial conditions: $\theta(0)=0\ ;\theta'(0)=5\omega_0$

Calculate the total energy.

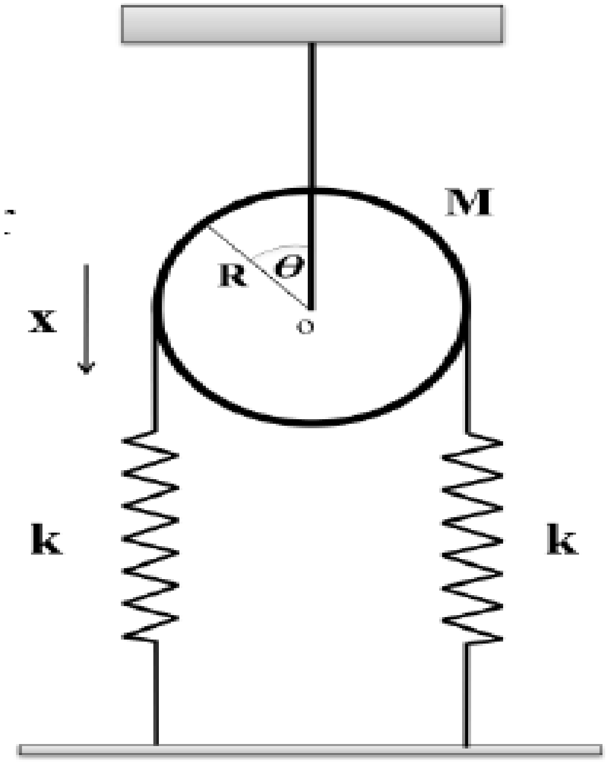


Figure5

Exercise 9

Consider the mechanical system shown in the accompanying figure 6. The rod has a negligible mass and a length of $2L$. The rod undergoes small amplitude oscillations around a fixed axis passing through point O. The middle of the rod (*point b*) is connected to a fixed support by a spring with stiffness K, and its free end holds a mass M.

1-What is the number of degrees of freedom of the studied system?

2-Find the kinetic energy T and the potential energy U of the system (for the variable θ).

3-Derive the differential equation of motion for this system, using the variable θ, and determine its natural frequency f_0.

4-The solution of the differential equation is of the form: $x(t)=A\sin(\omega_0 t+\varphi)$.

Determine the amplitude A and the initial phase φ if at time $t=0$, $(0)=\frac{\pi}{20}$; $\theta'(0)=0$

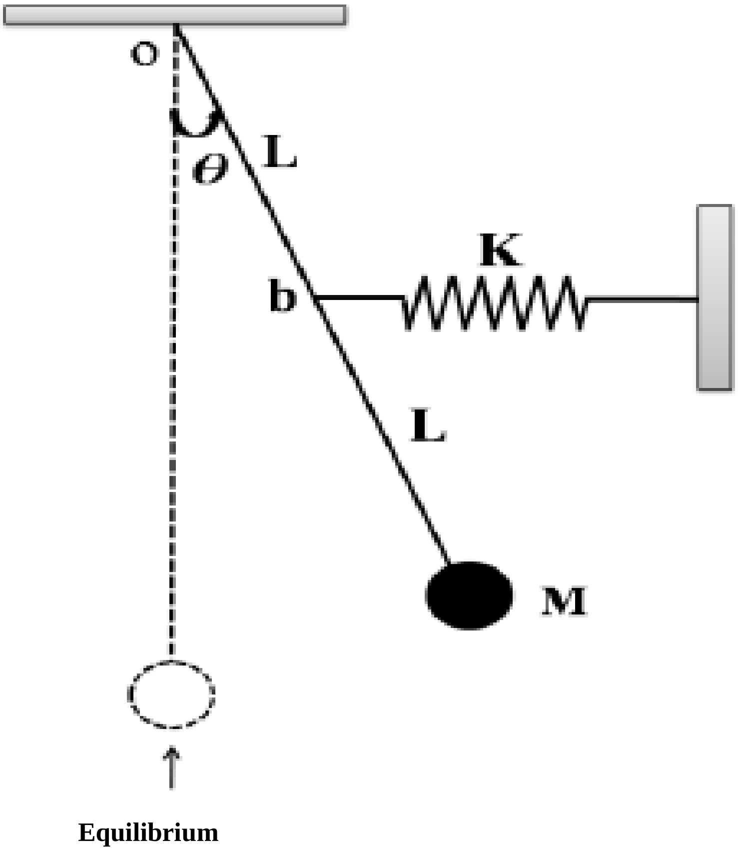


Figure6

Exercise 10

For the mechanical system shown in the accompanying figure 7, consider a bar of negligible mass and length $2L$. At its ends, a mass m and springs with stiffness K are attached. The equilibrium position corresponds to $\theta\,(0)=0$.

1-What is the type of the system?

2-Determine the kinetic energy T and the potential energy U of the system (for the variable θ).

3-Derive the differential equation of motion for free oscillations of small amplitudes.

4-Find the natural angular frequency ω_0, the natural period T_0, and the natural frequency f_0.

5-Find the solution $\theta(t)$.

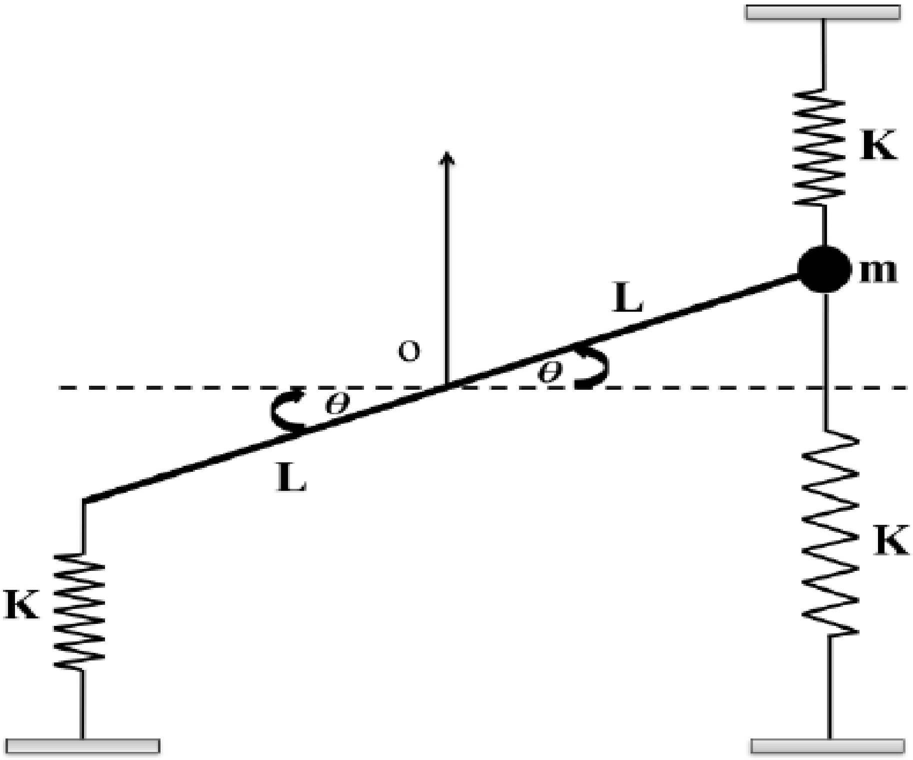


Figure7

Chapter II

Free Damped Linear Systems with One Degree of Freedom

Chapter II

Free Damped Linear Systems with One Degree of Freedom

II.1. Introduction

In damped oscillations, friction forces are taken into account. The friction is viscous and depends on velocity.

II.2 Damped Oscillator

Harmonic oscillatory motion does not exist in reality: every real mechanical system experiences friction, and part of its mechanical energy is dissipated as heat, causing the system's oscillations to be damped.

II.3 Friction and Damping Coefficient

II.3.1 Viscous Friction

The viscous friction force is proportional to the velocity of motion and acts in the opposite direction.

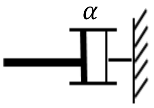
$$\vec{F} = -\alpha\vec{v} \tag{II.1}$$

With:

α : The viscous friction coefficient (N.s/m)

q: The generalized coordinate of the system.

$\dot{q}$: The generalized velocity of the system. In a one-dimensional motion x, the force is written as: $\vec{F} = -\alpha\vec{v} = -\alpha\dot{x}\vec{u}$

In mechanics, the damper is represented by 

II.3.2 Solid Friction

The friction force is constant and opposes the motion.

II.4 Lagrange's Equation

The Lagrange equation for a free damped system is written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) = 0 \qquad \text{(II.2)}$$

The dissipation function is defined as:

- Translational motion)

$$D = \frac{1}{2}\alpha\dot{q}^2 \Rightarrow \frac{\partial D}{\partial \dot{q}} = \alpha.\dot{q}, \dot{q} = \dot{x}_\alpha \qquad \text{(II.3)}$$

$\dot{x}_\alpha$: The displacement of the damper

- The dissipation function in a rotational system

$$D = \frac{1}{2}\alpha I\dot{\theta}^2 \qquad \text{(II.4)}$$

- I is the moment of inertia of the system about the axis of rotation.
- $\dot{\theta}$ is the angular velocity (the derivative of the angle θ with respect to time).

The form of the differential equation is:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0 \qquad \text{(II.5)}$$

Where:

δ is the damping coefficient.

ω_0 is the natural angular frequency.

III.5 Regimes of the Damped Oscillator

The differential equation of a damped oscillator is: $\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0$

There are three possible regimes:

II.5.1 Aperiodic Regime ($\delta > \omega_0$)

In this case, the friction is significant, the value of the damping coefficient is large, and the system slowly returns to its equilibrium position without oscillating. The solution to the differential equation is of the form:

$$\Delta' > 0 \Rightarrow \delta > \omega 0$$

$$r_1 = \frac{-2\delta + \sqrt{\Delta'}}{2} = -\delta + \sqrt{\delta^2 - \omega_0^2} \quad and\ r_2 = \frac{-2\delta - \sqrt{\Delta'}}{2} = -\delta - \sqrt{\delta^2 - \omega_0^2}$$

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{II.6}$$

$$x(t) = A_1 e^{\left[-\delta - \sqrt{\delta^2 - \omega_0^2}\right]t} + A_2 e^{\left[-\delta + \sqrt{\delta^2 - \omega_0^2}\right]t} \tag{II.7}$$

$$x(t) = e^{-\delta t}\left[A_1 e^{\left(\sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\delta^2 - \omega_0^2}\right)t}\right] \tag{II.8}$$

The coefficients A_1 and A_2 are determined by the initial conditions on displacement and velocity.

$x(0) = x_0 \quad \dot{x}(0) = 0$

$A_1 = \frac{x_0(\omega-\delta)}{2\omega}$ and $A_2 \frac{x_0(\omega+\delta)}{2\omega}$ with $\omega = \sqrt{\delta^2 - \omega_0^2}$

$$x(t) = e^{-\delta t}(A_1 e^{\omega t} + A_2 e^{-\omega t}) \qquad \text{(II.9)}$$

By moving the system from its equilibrium position, it no longer oscillates and comes to a complete stop after a certain time, which depends on the damping coefficient. The greater the damping coefficient, the shorter the stopping time; this regime is called aperiodic, and the damping is heavy. The term $\sqrt{\delta^2 - \omega_0^2}$is not considered a frequency, as in the case of a heavily damped regime, there is no oscillation around the equilibrium position.

II.5.2 Critical Regime (δ =ω_0)

$$\Delta' = 0 \Rightarrow \delta = \omega 0$$

The characteristic equation has a real double root: $r_1 = r_2 = -\delta$

In this regime, the system returns to the equilibrium position without oscillating, and it does so quickly. It marks the boundary between the pseudoperiodic and aperiodic regimes. The solution is of the form:

$$x(t) = (A_1 + A_2)e^{-\delta t} \qquad \text{(II.10)}$$

A_1 and A_2 are integration constants defined by the initial conditions.

X(0)= x_0 $\dot{x}(0) = 0$

$A_1 = x_0$ and $A_2 = \delta x_0$

$x(t) = x_0 e^{-\delta t}(1 + \delta t)$

II.5.3 Pseudoperiodic Regime ($\delta<\omega_0$)

Corresponds to the case where the reduced discriminant is negative:

$\Delta' < 0 \Rightarrow \delta < \omega_0$

This is the regime of weak damping. In this case, the amplitude of the oscillations decreases over time, and it is modulated by a decaying exponential factor that depends on the friction. The solution to the differential equation is of the form:

$$q(t) = Ae^{-\delta t}\cos\,(\omega t + \varphi) \qquad \text{(II.11)}$$

A and φ are integration constants determined from the initial conditions. ω is the pseudo frequency defined by:

$$\omega = \sqrt{\omega_0^2 - \delta^2} \qquad \text{(II.12)}$$

$$\Delta' = (-1)(\omega_0^2 - \delta^2) = i^2(\omega_0^2 - \delta^2) \qquad \text{(II.13)}$$

$$x(t) = A_1 e^{(-\delta - i\sqrt{\Delta'})t} + A_2 e^{(-\delta + i\sqrt{\Delta'})t} \qquad \text{(II.14)}$$

$$x(t) = e^{-i\omega t}(A_1 e^{(-i\sqrt{\Delta'})t} + A_2 e^{(i\sqrt{\Delta'})t} \qquad \text{(II.15)}$$

The solution of this equation is:

$$x(t) = Ae^{-\delta t}\cos\,(Acos\omega t + Bsin\omega t) \qquad \text{(II.16)}$$

Or:

$$x(t) = Acos(\omega t - \varphi) \qquad \text{(II.17)}$$

The constants A and φ are determined by the initial conditions.

The system undergoes oscillations with decreasing amplitudes and a (pseudo-period) given by:

$$T_p = \frac{2\pi}{\omega_a} \Rightarrow T_p = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}} \tag{II.18}$$

$$\omega = \omega_0^2 \sqrt{1 - \delta^2} \tag{II.19}$$

$$T_p = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}} = \frac{T_0}{\sqrt{1 - \xi^2}} \Rightarrow T_0 < T_p \tag{II.20}$$

If $\delta \ll \omega_0 \Rightarrow \xi^2 = 1 \; so \; T_p \cong T_0$

The curve x(t) is enveloped by the two exponentials $Ae^{-\delta t}$and $-Ae^{-\delta t}$as the modulus of $\cos(\omega t + \varphi$)cannot exceed unity. We observe that x(t) approaches zero as ttt tends toward the equilibrium position (see Figure II.3). There is a pseudo-frequency, and the motion is referred to as pseudo-periodic. The damping is weak.

It should be noted that the pseudo-frequency ω is lower than the natural frequency ω_0, and the pseudo-period T is longer than the period T_0 of the corresponding undamped oscillator.

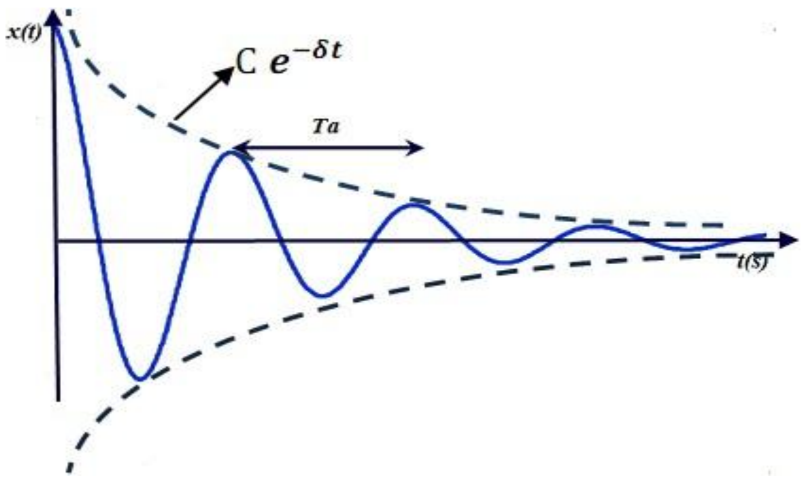


Figure II.1 Weakly Damped System (1st Case)

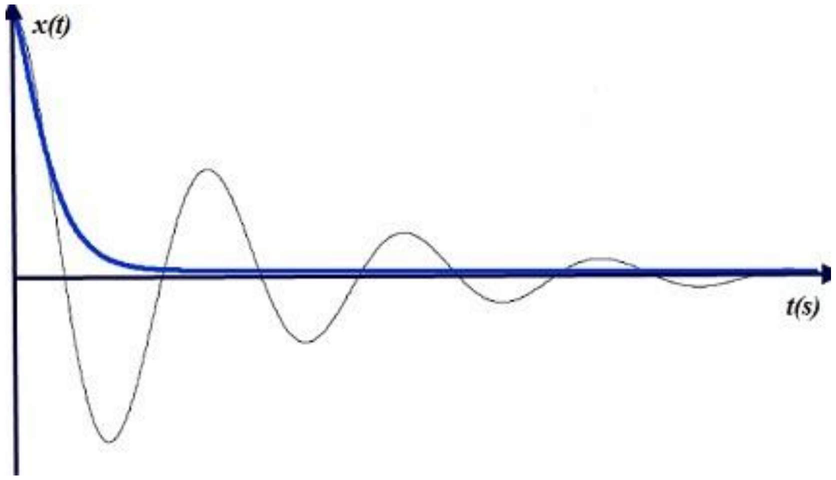


Figure II.2 Critical Damping(2nd Case)

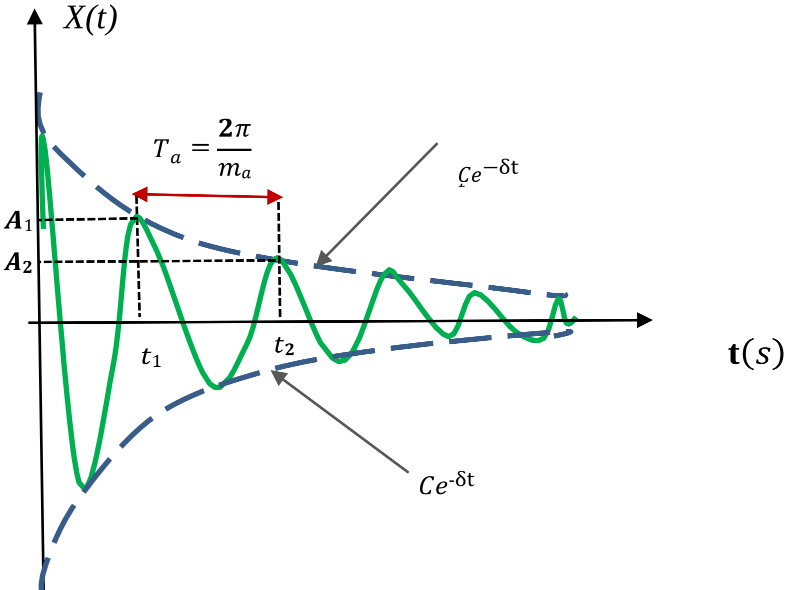


Figure II.3 Heavily Damped System (3rd Case)

II.6 Total Energy of a Damped Harmonic Oscillator

we consider that:

$$x(t) = Ae^{-\delta t}\sin(\omega t + \varphi) \qquad \text{(II.21)}$$

$$\dot{x}(t) = Ae^{-\delta t}\omega\cos(\omega t + \varphi) - A\delta e^{-\delta t}\sin(\omega t + \varphi) \qquad \text{(II.22)}$$

In the case of very weak damping ($\delta \to 0$), the pseudo-angular frequency is approximately equal to the natural angular frequency of the system, i.e.:

$$\omega = \omega_0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

The total energy is written as:

$$E_T(t) = U + T \qquad \text{(II.23)}$$

$$E_T(t) = \frac{1}{2}kA^2e^{-i\omega t}sin^2(\omega t + \varphi)\frac{1}{2}mA^2[e^{-2\delta t}\omega^2cos^2(\omega t + \varphi) + e^{-2\delta t}sin^2(\omega t +$$

$$\varphi - 2\delta\omega\sin\omega t + \varphi\cos(\omega t + \varphi) \qquad \text{(II.24)}$$

If we let the second and third terms of the kinetic energy approach zero, we obtain the following three expressions for the total energy:

$$E_T(t) = \frac{1}{2}kA^2e^{-2\delta t} \qquad \text{(II.25)}$$

or

$$E_T(t) = \frac{1}{2}m\omega_0^2e^{-2\delta t} \qquad \text{(II.26)}$$

II.7 Quality Factor

In a damped harmonic oscillator, there is dissipation of mechanical energy. This dissipation is characterized by the coefficient or quality factor Q, which accounts for the efficiency or quality of the oscillator.

It is given by the following ratio:

$$Q = 2\pi \frac{E_m}{\Delta E} \tag{II.27}$$

$$Q = 2\pi \frac{E_T(t)}{[E_T(t) - E_T(t+T)]} \tag{II.28}$$

$$Q = 2\pi \frac{\frac{1}{2}kA^2 e^{-2\delta t}}{[\frac{1}{2}kA^2 e^{-i\omega t} - \frac{1}{2}kA^2 e^{-2\delta(t+T)}]} \tag{II.29}$$

$$Q = 2\pi \frac{1}{[1 - e^{-2\delta t}]} \tag{II.30}$$

We assume that the damping is very weak.

Let :

$\delta \to 0 \qquad e^{-2\delta t} = 1 - 2\delta T$

$$Q = \frac{\pi}{\delta T} \tag{II. 31}$$

$Q = \frac{\omega}{2\delta}$ where $Q = \frac{\omega_0}{2\delta}$

$$\Delta \geq 0 \Rightarrow \delta \geq \omega_0$$

Let: $Q \leq \frac{1}{2}$

There are no oscillations; the system is in an aperiodic regime. The pseudo-period is written as:

$T = \frac{T_0}{\sqrt{1 - \frac{1}{4}Q^2}}$, $T = \frac{T_a}{\sqrt{1 - \frac{1}{4Q^2}}}$

II.8 Differential Equation: Mass-Spring-Damper System

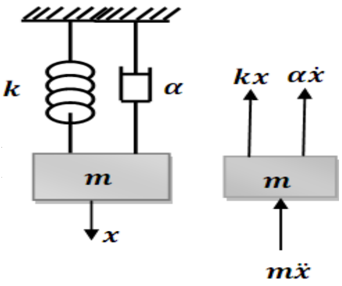
Let’s revisit the case of the elastic pendulum (vertical, for example). The study of the damped oscillator is done in the same way as before, but by adding the viscous friction force.

In one dimension, the Lagrange equation is written as: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) + \left(\frac{\partial D}{\partial \dot{x}}\right) = 0$

The kinetic energy of the system: it is the kinetic energy of the mass m: $T = \frac{1}{2} m\dot{x}^2$

The potential energy: it is the energy stored in the spring: $U = \frac{1}{2} kx^2$

The dissipation function**:** $D = \frac{1}{2}\alpha \dot{x}^2$



Calculation of the Lagrangian: $L = T - U = \frac{1}{2} m\dot{x}^2 - (\frac{1}{2} kx^2$

Lagrange's Equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) + \left(\frac{\partial D}{\partial \dot{x}}\right) = 0$

$$\begin{cases} \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) = m\ddot{x} \\ \dfrac{\partial L}{\partial x} = -kx \qquad \Rightarrow\; m\ddot{x} + kx + \alpha\dot{x} = 0 \Rightarrow \ddot{x} + \dfrac{\alpha}{m}\dot{x} + \dfrac{k}{m}x = 0 \\ \dfrac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{cases}$$

This is the differential equation of motion for a damped free system.

The general form:

- Compared to undamped free oscillations, we recognize a new term ($\frac{\alpha}{m}\dot{x}$), which results from energy dissipation.

- The general form: $\ddot{q}+\frac{\alpha}{m}\dot{q}+\frac{k}{m}q=0$

- Often, the differential equation is written in what is called a reduced form :

$$\ddot{q}+2\delta\dot{q}+\omega_0^2q=0$$

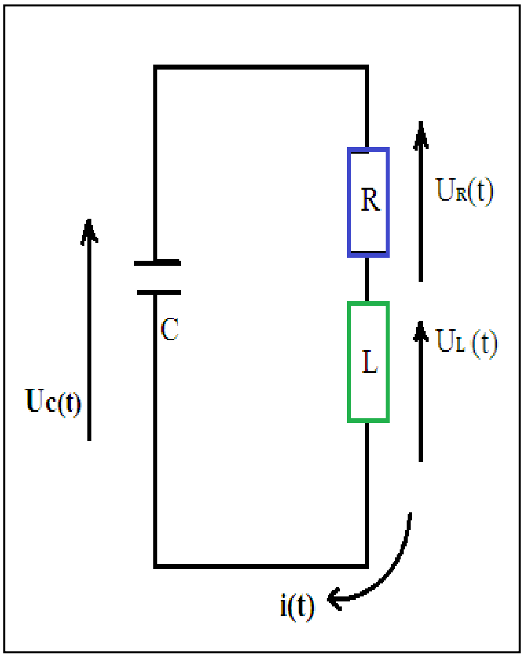
such that $\begin{cases}\delta=\frac{\alpha}{2m}\left[\frac{1}{s}\right] & \text{Damping factor.}\\ \omega_0^2=\frac{k}{m}\left[\frac{rad}{s}\right] & \text{Natural frequency of the system}\end{cases}$

In one dimension, the form becomes: $\ddot{x}+2\delta\dot{x}+\omega_0^2x=0$

II.9 Electrical Harmonic Oscillator

The oscillating circuit, in addition to the inductance L and capacitance C, includes an ohmic resistance R.

According to Kirchhoff's law:



$$U_R+U_C+U_L=0$$

$$Ri(t)+\frac{1}{C}q+L\frac{di}{dt}=0$$

$$R\frac{dq}{dt}+\frac{1}{C}q+L\frac{d^2q}{dt^2}=0$$

$$R\dot{q}+\frac{1}{C}q+L\ddot{q}=0$$

$$\ddot{q}+2\delta\dot{q}+\omega_0^2q=0$$

With,

$$\begin{cases}\delta=\frac{R}{2L}\\ \omega_0^2=\frac{1}{LC}\end{cases}\text{ so, }\ddot{q}+2\delta\dot{q}+\omega_0^2=0\Rightarrow\begin{cases}\delta=\frac{R}{2L}\\ \omega_0=\sqrt{\frac{1}{LC}}\end{cases}$$

II.10 Logarithmic Decrement

The logarithmic decrement D is the natural logarithm of the ratio of the amplitudes x(t) and x(t+T) of two successive elongations. It thus characterizes the decay of the amplitude over a period.

$$D = ln\left[\frac{x(t)}{x(t_1+T_a)}\right] \qquad \text{(II. 32)}$$

For a damped system:

$$x(t) = Ce^{-\delta t}\sin(\omega_a t + \varphi) \Rightarrow D = ln\frac{Ce^{-\delta t}\sin(\omega_a t+\varphi)}{Ce^{-\delta(t_1+T_a)}\sin(\omega_a(t_1+T_a)+\varphi)} \qquad \text{(II. 33)}$$

$$D = \ln\left(e^{\delta T_a}\right) = \delta T_a \qquad \text{(II. 34)}$$

$$\delta T_a = \delta\frac{T_a}{\sqrt{1-\xi^2}} = \xi\omega_0\frac{T_a}{\sqrt{1-\xi^2}} = 2\pi\frac{\xi}{\sqrt{1-\xi^2}} \qquad \text{(II. 35)}$$

$$D = ln\frac{x(t_1)}{x(t_2)} = 2\pi\frac{\xi}{\sqrt{1-\xi^2}} = \delta T_a \qquad \text{(II. 36)}$$

Note :

For several periods:$T = nT_a(t_1 = t_2 + nT_a)$

$$D = ln\frac{x(t_1)}{x(t_1+nT_a)} = 2\pi\frac{n\xi}{\sqrt{1-\xi^2}} \qquad \text{(II. 37)}$$

The pseudo-period and the logarithmic decrement only make sense if the regime is pseudo-periodic.

Exercise 1

In the system shown, a homogeneous disk of mass M and radius R can rotate freely with an angle θ around its fixed axis. The mass m on the horizontal plane is connected to a damper with coefficient α and to the disk by an inextensible, non-slipping string. (See Fig 1).

Numerical values: $M=3kg$, $m=2kg$, $R=0.3\ m$, $\alpha=1.5\ kg/s$ $K=75N/m$.

1- Find the relationship between x and θ.

2-Determine the kinetic energy T, potential energy U, and dissipation function D in terms of the variable x, as well as the solution in the weak damping regime ($\delta<\omega_0$). The moment of inertia of the disk around its axis is: $J_{/0}=1/2\ MR^2$.

3-Find the Lagrangian and derive the equation of motion as a function of the variable x.

4-Calculate the quality factor Q of the mechanical system.

5-Define and calculate the logarithmic decrement D.

6-Find the number of periods n required for the amplitude of the motion to become *60%* of its initial value.

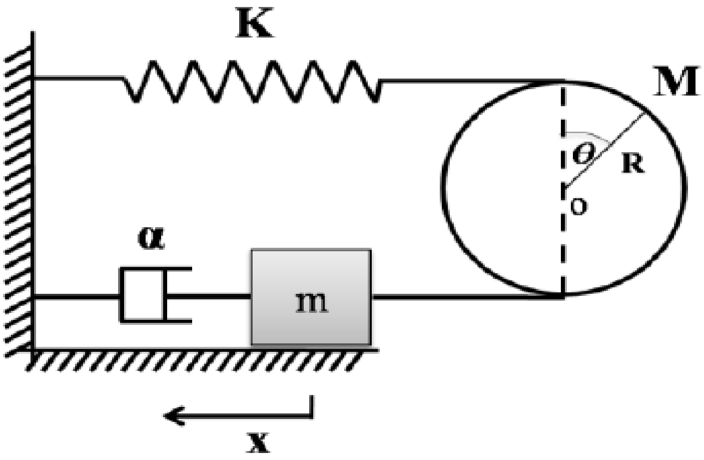


Figure 1

Exercise 2

A homogeneous disk with mass M and radius $3R$ is connected at its periphery to a spring with stiffness K and a damper with friction coefficient. A mass $3m$ is suspended by a string wound around the periphery of the disk, and another mass m is suspended by a string wound around a groove of radius R carved into the surface of the disk. (see Fig2) The strings are assumed to be inextensible and non-slipping. The disk can rotate freely around its fixed axis. The moment of inertia of the disk about its axis is given as $J_{/0}=1/3\ MR^2$.

Given: $\alpha=9\ N.s/m,\ K=3\ N/m,\ M=5m=4kg,\ m=0.7kg.$

1-Find the kinetic energy T, potential energy U, and the dissipation function D for $\theta \ll 1$ (at equilibrium, the spring was not deformed).

2-Establish the differential equation of motion.

3-Derive the natural frequency ω_0 and the damping coefficient δ.

4-Determine the nature of the motion.

5-What is the maximum value of α that must not be exceeded to allow oscillations?

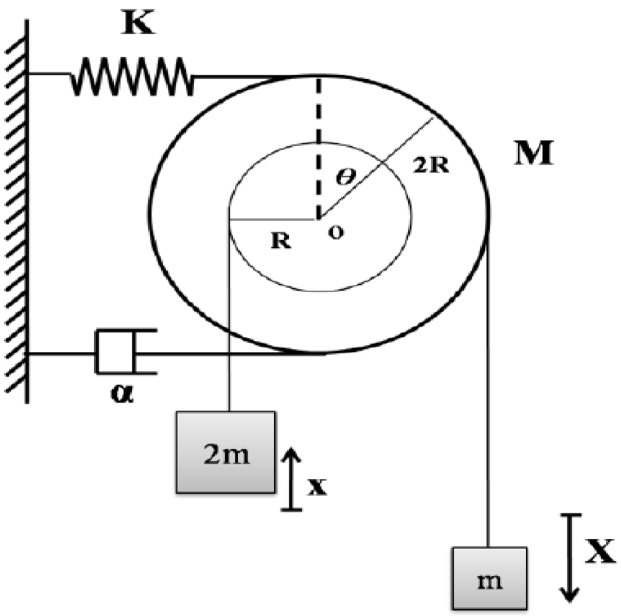


Figure 2

Exercise 3

We consider the mechanical system shown, consisting of a rod of length L and negligible mass, which can rotate in a vertical plane around its fixed axis O. Point A is connected to a fixed frame by a damper with viscous friction coefficient α. At the other end of the rod, a point mass m is attached, connected to a second fixed frame by a spring with stiffness K. We consider the case of free oscillations of small amplitude. (See Fig3)

1-What is the number of degrees of freedom of the studied system? Justify your answer.

2-Calculate the kinetic energy T, potential energy U, and the dissipation function D as a function of the variable θ.

3-Establish the differential equation of motion in the weak damping regime and solve it.

4-After *18 periods*, the amplitude of the motion decreases by *30%* of its initial value. Calculate the logarithmic decrement D.

5-Deduce the number of periods required for the total energy to decrease by the same percentage, and conclude.

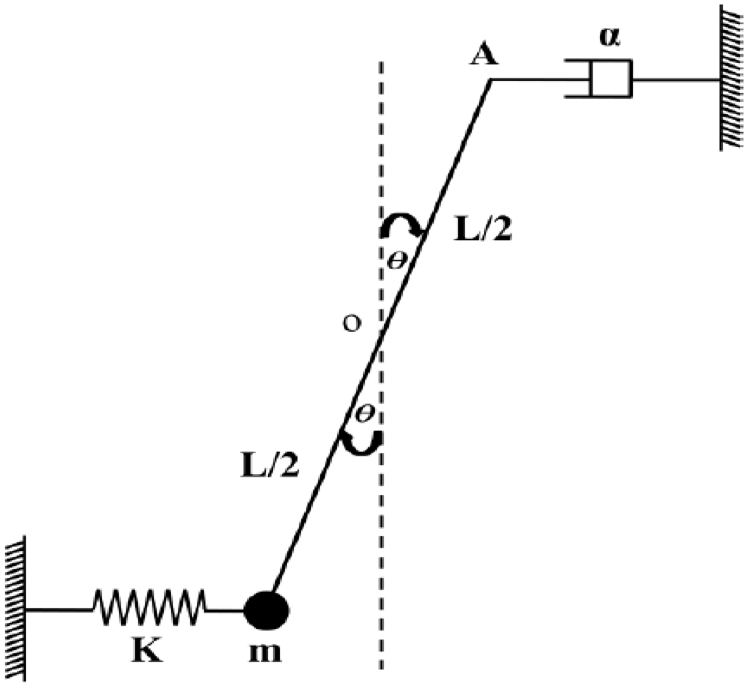


Figure 3

Exercise 4

Consider the damped mechanical system with one degree of freedom shown in the figure. Let θ represent the angular displacement of the disk in the clockwise direction. (See Fig4)

1-Calculate the kinetic energy T and the potential energy U of the system.

2-Using the Lagrange equation, find the differential equation of motion.

3-Find the natural frequency ω_0 of the system's free oscillations.

4-What is the value of δ if the damping ratio α of the system is equal to 1.25?

Numerical values: $R_1=12cm$, $R_2=33cm, J=1.23Km$, $m_1=12kg$, $m_2=28kg$, $k_1=1.150N/m$, $k_2=1.121N/m$

5-Write the general solution x(t).

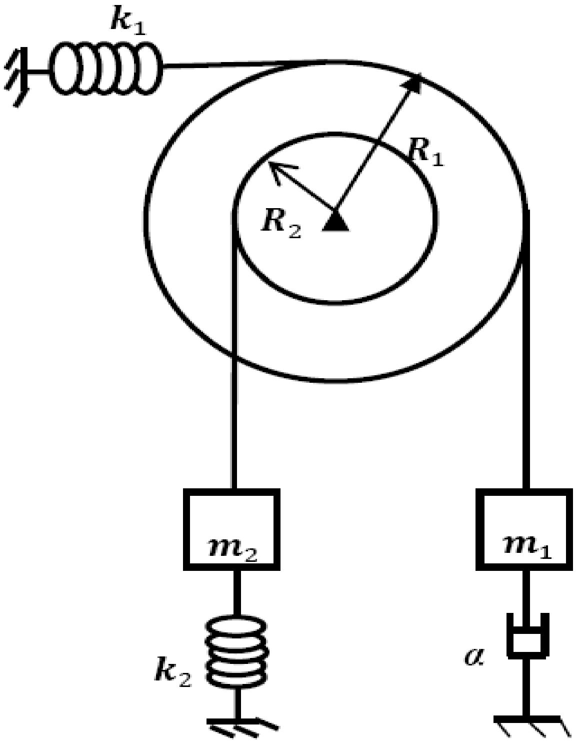


Figure 4

Chapter III

Damped Forced Oscillations: Systems with One Degree of Freedom

Chapter III

Damped Forced Oscillations: Systems with One Degree of Freedom

III.1Introduction

we have seen that the damping of oscillations is due to a reduction in mechanical energy, dissipated as heat. To compensate for this energy loss and sustain the oscillations, an external source of energy is required, typically provided through an external force. This external force, often referred to as the driving force, is introduced.

Thus, an additional force is applied, and it is preferable that this force be collinear with the motion and aligned in the same direction as the movement as much as possible.

In this chapter, we study the response of a damped system with one degree of freedom (DOF) to a sinusoidal harmonic excitation produced by an external force acting on the system. This type of excitation is commonly encountered in industries (rotating machines, fans, motors, pumps, etc.).

III.2 Differential Equation of Motion

- For translational motion, we write: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)+(\frac{\partial D}{\partial \dot{x}})=F_{ext}$ (III.1)
- For rotational motion, the equation is written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)+(\frac{\partial D}{\partial \dot{\theta}})=M(F_{ext}) \quad \text{(III.2)}$$

$F_{ext:}$ is the generalized force due to an external force.

$M(F_{ext})$ is the moment of the applied force.

$$M(F_{ext}) = F_{ext} \times L \qquad \text{(III.3)}$$

The Moment: characterizes the ability of a force to rotate an object around a point.

L: is the straight-line distance of the force's action.

The differential equation of forced vibrational motion is:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = A(t) \qquad \text{(III.4)}$$

III.3 Example of a Damped Forced System (Mass-Spring-Damper System)

In the figure shown, the mass m is attached to a spring K and a damper α.

A force $F_{ext} = F_0 sin\omega t$is applied to the mass m.

The kinetic energy of the system: $T = \frac{1}{2}m\dot{x}^2$

The potential energy of the system: $U = \frac{1}{2}kx^2$

The dissipation function: $D = \frac{1}{2}\alpha\dot{x}^2$

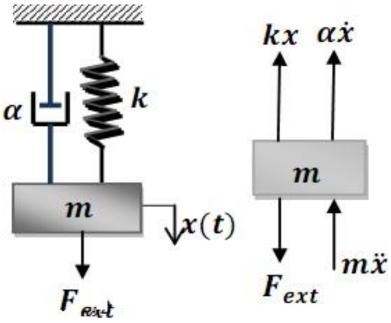


Figure III.1 Mass-Spring System

$$\begin{cases} \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) = m\ddot{x} \\ \qquad \dfrac{\partial L}{\partial x} = -kx \\ \qquad \dfrac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{cases} \Rightarrow \; m\ddot{x} + kx + \alpha\dot{x} = F_{ext} \Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m} sin\omega t$$

This is the differential equation of motion for a damped forced system with one degree of freedom (1 DOF).

It takes the form: $\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = A_0 sin\omega t$

$$\begin{cases} \delta = \dfrac{\alpha}{2m}\left[\dfrac{1}{s}\right] \quad \text{Damping factor.} \\ \omega_0^2 = \dfrac{k}{m}\left[\dfrac{rad}{s}\right] \quad \text{Natural frequency of the system} \\ \mathrm{A}_0 = \dfrac{\mathrm{F}_0}{\mathrm{m}} \end{cases}$$

III.4 Solution of the Differential Equation of Motion

The differential equation of forced oscillations is a second-order differential equation with a non-homogeneous term. The solution to this second-order differential equation is equal to the sum of the solution of the equation without the non-homogeneous term (or homogeneous/transient solution) $q_H(t)$and a particular (steady-state) solution of the equation with the non-homogeneous term $q_P(t)$.

$$q(t) = q_H(t) + q_P(t) \tag{III.5}$$

The equation without the non-homogeneous term $q_H(t)$ has already been treated, and this solution always contains the exponential term $e^{-\delta t}$. It is important to note that at the beginning of the motion, *x(t)* represents the transient regime. Over time, the homogeneous solution $q_H(t)$becomes negligible, and the particular solution $q_P(t)$, which defines the steady-state regime, dominates. Thus, the total solution in this case is approximately:

$$q(t) \approx q_P(t) \tag{III.6}$$

When the homogeneous solution is non-negligible and non-zero, the regime is called transient.

Important remarks

1. The general solution is the sum of two solutions:
    - A homogeneous solution whose amplitude and energy decrease over time until they become zero. This is a transient solution.
    - A particular solution with frequency ω and amplitude A equal to:

$$A = \frac{A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2}} \tag{III.7}$$

This is a steady-state or stationary solution.

2. After a certain time, the transient regime disappears, and the oscillatory motion is solely due to the steady-state motion $x_{p}(t)$.

For $t < t_0, x(t) = x_p(t) + x_H(t)$ (III.8)

For $t > t_0, x(t) = x_p(t)$ (III.9)

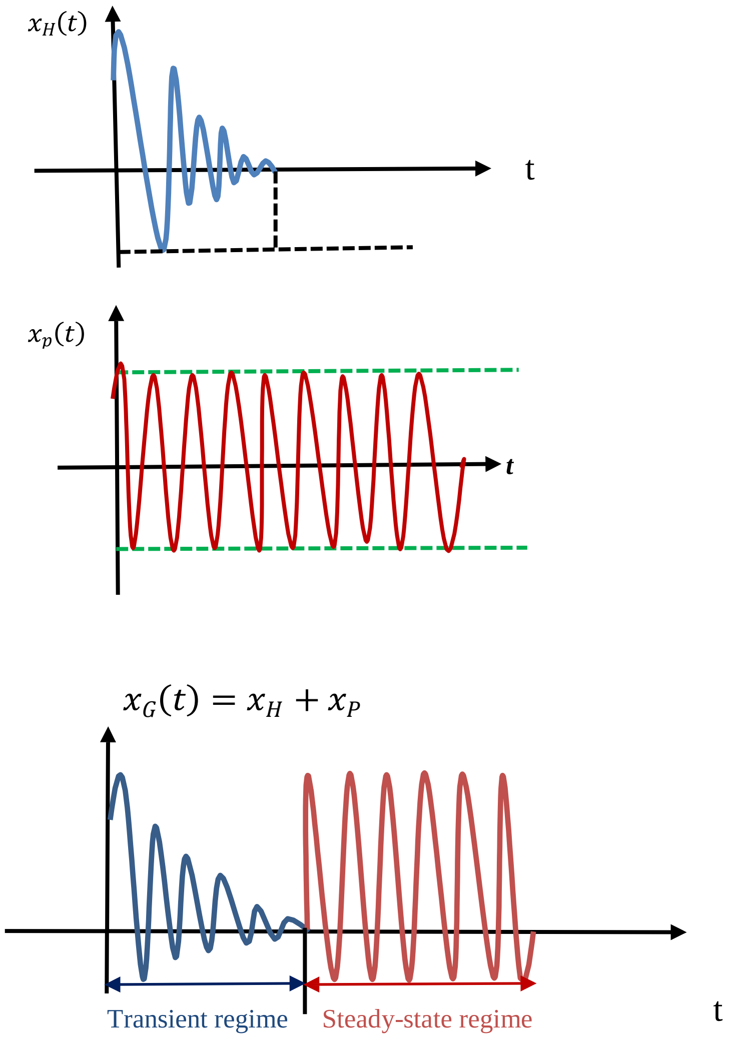


Figure III.2 Solution of the Differential Equation of Motion

III.4.1 Sinusoidal Excitation

In the case where the excitation is a sinusoidal function of the form $A(t)=A_0\cos(\omega t)$, the total solution is written as follows:

$$q(t) = q_P(t) = A\cos(\Omega t + \varphi) \tag{III.10}$$

Where the constant A represents the amplitude of the total solution, and ϕ is the phase shift.

III.4.2 Calculation of the Amplitude A

The equation of motion becomes:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = A_0 sin\Omega t \tag{III.11}$$

The excitation A(t) in complex form is equal to:

$$A(t) = A_0 e^{j\Omega t} \tag{III.12}$$

We seek the solution of the differential equation in complex form:

$$q(t) = q_P(t) = Ae^{j(\Omega t+\varphi)}$$

$$\dot{q_P}(t) = Aj\Omega e^{j(\Omega t+\varphi)} = j\Omega q_P(t)$$

$$\ddot{q_P}(t) = Aj^2\Omega^2 e^{j(\Omega t+\varphi)} = -\Omega^2 q_P(t)$$

$$Aj^2\Omega^2 e^{j(\Omega t+\varphi)} + 2\delta Aj\Omega e^{j(\Omega t+\varphi)} + \omega_0^2 Ae^{j(\Omega t+\varphi)} = A_0 e^{j\Omega t}$$

$$[(\omega_0^2 - \Omega^2) + 2\delta\Omega j]Ae^{j(\Omega t+\varphi)} = A_0 e^{j\Omega t}$$

$$[(\omega_0^2 - \omega^2) + 2\delta\omega j]Ae^{j\varphi} = A_0$$

We divide by $e^{j\varphi}$ and find:

$$[(\omega_0^2 - \Omega^2) + 2\delta\Omega j]A = A_0 e^{-j\varphi} \tag{III.13}$$

The conjugate of this equation is as follows:

$$[(\omega_0^2 - \Omega^2) + 2\delta\Omega j]A = A_0 e^{j\varphi} \tag{III.14}$$

$$(\text{III}.13) \times (\text{III}.14) \Rightarrow [(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega)^2]A^2 = A_0^2$$

$$\Rightarrow A = \frac{A_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega\omega)^2}} \tag{III.15}$$

III.4.3 Calculation of $\boldsymbol{\phi}$

$$[(\omega_0^2 - \Omega^2) + 2\delta\Omega j]A = \begin{cases} A_0 e^{-j\varphi} \\ A_0(cos\varphi - jsin\varphi) \end{cases} \tag{III.16}$$

$$A(\omega_0^2-\Omega^2)+2A\delta\Omega j$$

$$=A_0 cos\varphi - jA_0 sin\varphi$$

$$\Rightarrow\begin{cases}A(\omega_0^2-\Omega^2)=A_0 cos\varphi\\ 2A\delta\Omega=-A_0 sin\varphi\end{cases}\Rightarrow tan\varphi=\frac{-2\delta\Omega}{(\omega_0^2-\Omega^2)}\Rightarrow\varphi$$

$$=-Arctg\frac{2\delta\Omega}{(\omega_0^2-\Omega^2)}$$

So,

$$q_{P(t)}=\frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\delta\Omega)^2}}\cos\left(\Omega t+Arctg\frac{-2\delta\Omega}{(\omega_0^2-\Omega^2)}\right) \qquad \text{(III.17)}$$

III.5 Resonance Phenomenon

III.5.1 Amplitude Resonance

We will study the variation of amplitude A as a function of the excitation frequency.

$$A(\Omega)=\frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\delta\Omega)^2}} \qquad \text{(III.18)}$$

When, $\Omega=0, \mathrm{A}(0)=\frac{\mathrm{A}_0}{\sqrt{(\omega_0^2)^2}}=\frac{\mathrm{A}_0}{\omega_0^2}=\frac{\frac{\mathrm{F}_0}{\mathrm{m}}}{\frac{\mathrm{k}}{\mathrm{m}}}=\frac{\mathrm{F}_0}{\mathrm{k}}$

$$\pi\to\infty, A(\infty)=0$$

The maximum amplitude is obtained for: $\frac{dA}{d\Omega}=0$

$$\frac{dA}{d\Omega}=0\Rightarrow\frac{d}{d\Omega}\left[\frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\delta\Omega)^2}}\right]=0\Rightarrow\frac{d}{d\Omega}[(\omega_0^2-\Omega^2)^2+(2\varepsilon\Omega)^2]=0\Rightarrow$$

$$\Omega[(\omega_0^2-\Omega^2)-2\delta^2]=0\Rightarrow\begin{cases}\Omega_1=0\\ \Omega_2=\sqrt{\omega_0^2-2\delta^2}\end{cases} \qquad \text{(III.19)}$$

The resonance frequency is given by:

$$\Omega_R=\sqrt{\omega_0^2-2\delta^2} \qquad \text{(III.20)}$$

In the case of weak damping ($\delta \ll \omega_0$) the resonance frequency is very close to the natural frequency $\omega_R \approx \omega_0$, and we get:$A(\Omega) = \frac{A_0}{2\delta\omega_0}$

For resonance to occur: $\delta < \frac{\omega_0}{\sqrt{2}}$, we say that the system is in resonance ($\Omega = \Omega_R$) and the amplitude A is at its maximum:

$$A_{max} = A(\Omega_R) = \frac{A_0}{2\delta\sqrt{\omega_0^2 - \delta^2}} \tag{III.21}$$

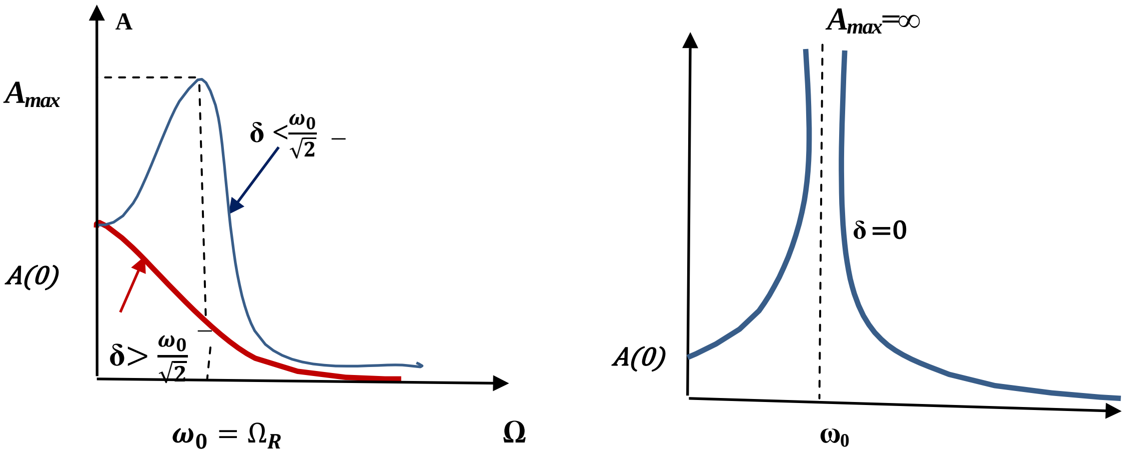
$$\lim_{\omega \to \infty} A(\Omega) = 0$$

Note:

For resonance to occur, we need

$\omega_0^2 - 2\delta^2 > 0 \Rightarrow 1 - \frac{1}{2Q^2} > 0 \Rightarrow Q > \frac{1}{\sqrt{2}} \Rightarrow$ meaning the damping must be weak.

- The vibration amplitude reaches a maximum when $\Omega_R \cong \omega_0$.
- If δ=0 (undamped system), the amplitude tends to infinity. However, in reality, all systems have damping, so the amplitude is never infinite.



Case of the conservative system

FigureIII.3 Phase variation as a function of Ω

III.5.2 Phase Variation as a Function of Ω

We found:

$$tan\varphi = \frac{-2\delta\Omega}{(\omega_0^2-\Omega^{\,2})} \qquad \text{(III.22)}$$

$$\frac{d\varphi}{d\Omega} = cos^2\varphi\,\frac{d(tan\varphi)}{d\Omega}\ and\ cos\varphi = \frac{1-\Omega^2}{\sqrt{(\omega_0^2-\Omega^2)^2+\frac{(2\delta\Omega)^2}{\omega_0^2}}} \qquad \text{(III.23)}$$

$$\frac{d\varphi}{d\Omega} = \frac{(1-\Omega^2)^2}{(\omega_0^2-\Omega^2)^2+\frac{(2\delta\Omega)^2}{\omega_0^2}}\left(-2\,\frac{\delta}{\omega_0}\right)\frac{1+\frac{\Omega^2}{\omega_0^2}}{(1-\frac{\Omega^2}{\omega_0^2})} \qquad \text{(III.24)}$$

$$\frac{d\varphi}{d\Omega} = \frac{-2\frac{\delta}{\omega_0}(1+\frac{\Omega^2}{\omega_0^2})}{\left(1+\frac{\Omega^2}{\omega_0^2}\right)+4\frac{\delta^2\,(\Omega)^2}{\omega_0^2\ \omega_0^2}} \qquad \text{(III.25)}$$

$$\frac{d\varphi}{d\Omega} = \frac{-2\delta\omega_0(\omega_0^2+\Omega^2)}{(\omega_0^2+\Omega^2)+4\delta^2\Omega^2} \qquad \text{(III.26)}$$

- If $\Omega \rightarrow 0 \Rightarrow \tan\varphi \rightarrow 0$ and $\frac{d\varphi}{d\Omega} \cong -2\,\frac{\omega_0}{\delta}$
- If $\Omega \rightarrow \omega_0 \Rightarrow \tan\varphi \rightarrow \infty$ and $\frac{d\varphi}{d\Omega} \cong \frac{\omega_0}{\delta}$
- $\varphi = -\frac{\pi}{2}$The displacement is ahead of the excitation by $\frac{\pi}{2}$
- When, $\Omega \rightarrow \infty, tan\varphi\ and\ \frac{d\varphi}{d\Omega} \rightarrow 0$

Since $\frac{d\varphi}{d\Omega}$ is always < 0 , The displacement is $-\pi$.

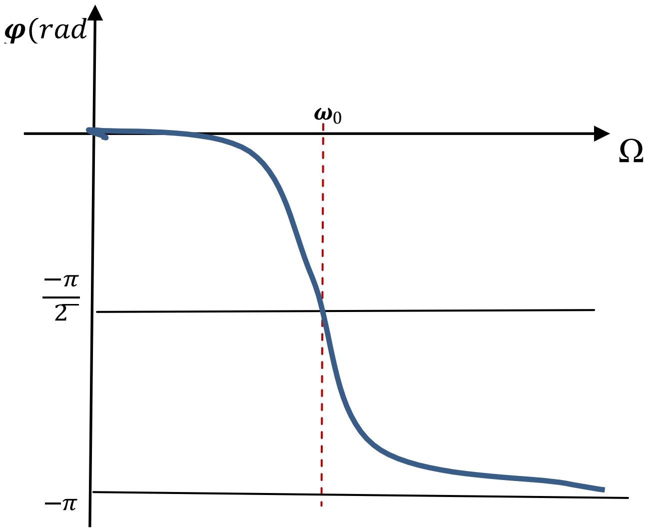


Figure III.4 Variation of phase as a function of Ω

III.5.3 Bandwidth

We define bandwidth as the range of frequencies around $\omega_R \cong \omega_o$ for which

$$A(\Omega) > \frac{A_{max}(\Omega_R)}{\sqrt{2}} \tag{III.27}$$

The bandwidth B is written as:

$$B = \Omega_2 - \Omega_1 \tag{III.28}$$

The two frequencies ω_2 and ω_1 located on either side of the frequency ω_0 and for which

$A(\Omega) = \frac{A_{max}(\Omega_R)}{\sqrt{2}}$, are called cutoff frequencies.

The calculation of B involves finding the two frequencies for which $A(\Omega) = \frac{A_{max}(\Omega_R)}{\sqrt{2}}$.

We obtain the expression for the bandwidth B as:

$$B = \Omega_2 - \Omega_1 = 2\delta \tag{III.29}$$

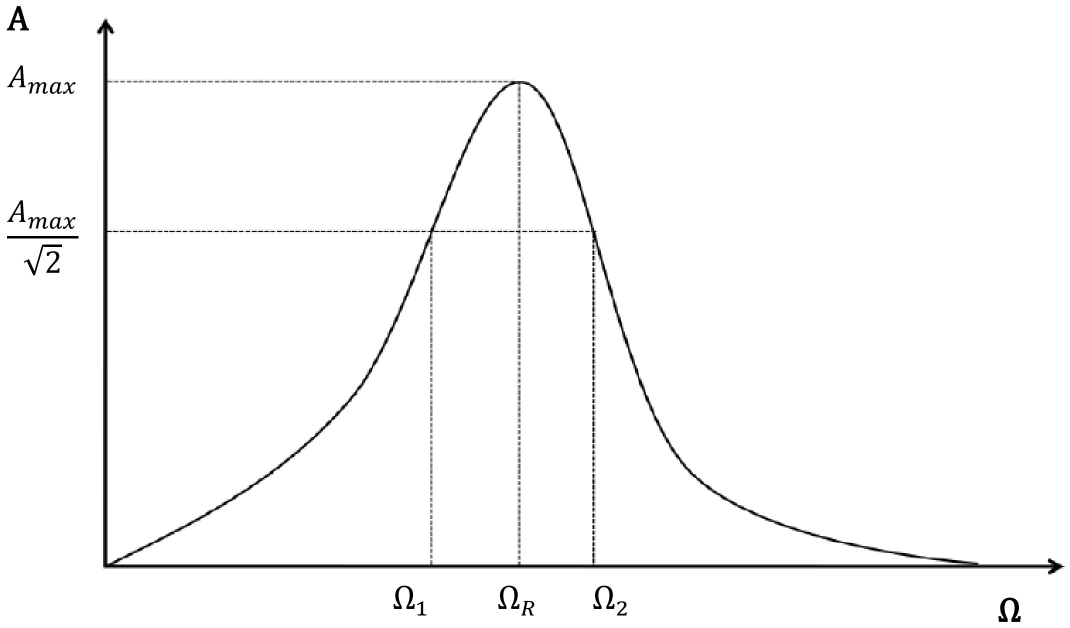


Figure III.5 Amplitude A en fonction de Ω (bande passante)

III.5.4 Quality Factor

The quality factor is defined as the ratio of the natural frequency ω_0 to the bandwidth B.

$$Q = \frac{\omega_0}{B} = \frac{\omega_0}{2\delta} \qquad \text{(III.30)}$$

III.5.5 Periodic Excitation

Let there be a periodic excitation applied to a damped single-degree-of-freedom system. The differential equation governing this system is written as:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = A(t) \qquad \text{(III.31)}$$

The function A(t) being periodic with period T, its Fourier series expansion is written as:

$$A(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + b_n \sin\,(n\Omega t) \qquad \text{(III.32)}$$

The differential equation is then written as:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + b_n \sin(n\Omega t) \tag{III.33}$$

The steady-state response can be calculated by:

$$q(t) = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} \frac{a_n \cos(n\Omega t+\varphi) + b_n \sin(n\Omega t+\varphi)}{\sqrt{(\omega_0^2-\Omega^2)^2+(2\delta\Omega)^2}} \tag{III.34}$$

III.5.6 Mechanical Impedance

A mechanical system subjected to a sinusoidal force $F(t)=F_0\cos(\Omega t)$, where the point of application of this force moves with a velocity $v(t)=V_0\cos(\Omega t+\varphi)$. The input mechanical impedance of the system is the ratio of the complex amplitudes of the force F and the velocity v.

$$Z_E = \frac{F}{v} \tag{III.35}$$

- **Mass**

  The fundamental relationship of dynamics is written as:

$$F = m\frac{dv}{dt} \tag{III.36}$$

The complex impedance of a mass is: $Z_m = jm\Omega = m\Omega e^{j\frac{\pi}{2}}$ (III.37)

The applied force f on the spring is expressed in terms of the extension as: $F = kx$ (III.38)

k: The spring constant

The complex impedance of a spring is given by: $Z_k = \frac{k}{j\Omega} = -j\frac{k}{\Omega} = \frac{k}{\Omega}e^{-j\frac{\pi}{2}}$ (III.39)

- **Damping Element**

The applied force is related to the velocity by:

$$F=\alpha v \tag{III.40}$$

From this, we deduce the complex impedance of a damper:

$$Z_{\alpha}=\alpha \tag{III.41}$$

Exercise 1

The mass *m* is welded to the end of a rod of negligible mass and length *L*. This mass is subjected to a sinusoidal perpendicular force with angular frequency *Ω*. The other end of the rod is articulated at point *O*. The rod is connected at point *A* to a fixed support B_1 by a spring with stiffness coefficient k_1 and a damper with a damping coefficient α. Additionally, it is connected to another fixed support B_2 by a spring with stiffness coefficient k_2. The distance *OA* is equal to *1/2* (see the figure1).

1- Using the method of moments or the Lagrangian of the free system, derive the equation of motion of the mass m, deduce the natural angular frequency ω_0 , and write the solution of the equation of motion.

2- Rewrite the equation of motion of this damped vibrating system.

3- Determine the expressions for the amplitude and phase shift of the forced oscillations as a function of the angular frequency *Ω*. Deduce the equivalent electrical circuit.

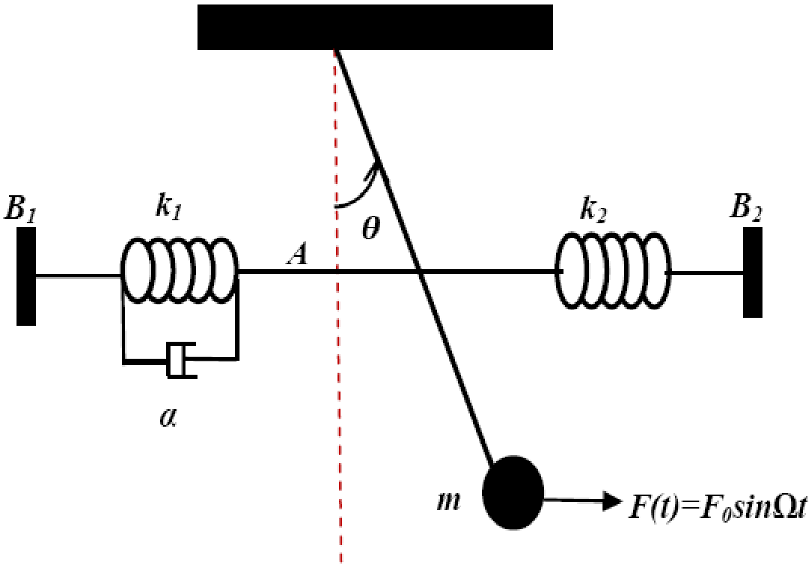


Figure 1

Exercise 2

In the system shown in figure2, an inextensible string moves without slipping around a disk of mass m and radius R, which rotates freely around its fixed axis. At one end of the string, there is a mass $2m$; a rod of mass m and length $2L$; and a spring with stiffness k, one end of which is fixed, and the other is connected to the middle of the rod with a damper of coefficient α. At equilibrium (represented by the dashed line), the rod was vertical, and the spring experienced an initial deformation.

1- Derive the differential equation of motion.

2- If $m=2Kg$, $L=3m$, $N=24N/m$, find the maximum value of α that the system can have while still oscillating.

3- Determine the nature of the motion when $\alpha=25N.s/m$, and solve the equation.

The system is subjected to an external force $F(t)=F_0 sin(\Omega t)$ applied to the mass $3m$.

1- Establish the differential equation of the forced damped motion and provide the expression for the steady-state solution.

2- Write the expressions for the amplitude A and the phase shift φ.

3- Provide the general solution and plot $\theta(t)$ (showing both transient and steady-state regimes).

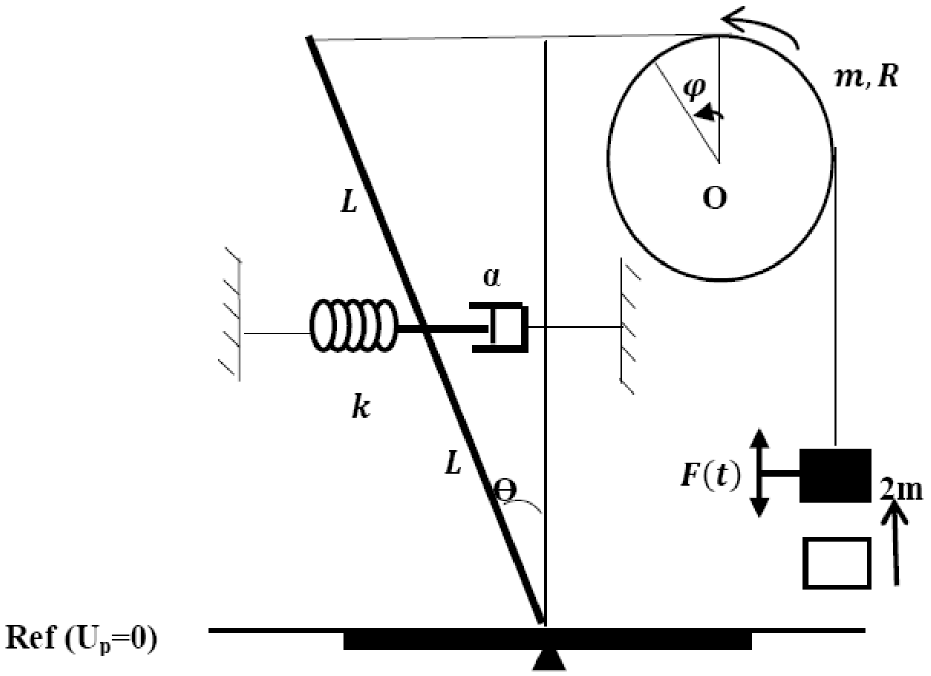


Figure 2

Chapter IV

Oscillations of multi-degree-of-freedom systems

Chapter IV

Oscillations of multi-degree-of-freedom systems

IV.1Introduction

A multi-degree-of-freedom (MDOF) system is a mechanical or structural system requiring multiple independent coordinates to describe its motion. These systems exhibit complex oscillatory behavior with multiple natural frequencies and mode shapes**.**

IV.2 Free Systems with Two Degrees of Freedom

A two-degree-of-freedom (2DOF) system is a mechanical system that requires two independent coordinates to describe its motion completely. These systems are fundamental in structural dynamics, mechanical vibrations, and control systems.

Such a set of coordinates, which leads to an uncoupled system of equations, is called p rinciple copordinates.

IV.2.1 Types of Coupling

There are three types of coupling: elastic, inertial, and viscous.

IV.2.1.1 Elastic Coupling

In mechanical systems, coupling is achieved through elasticity (a spring). In electrical systems, we find circuits coupled by capacitance, which is equivalent to elastic coupling.

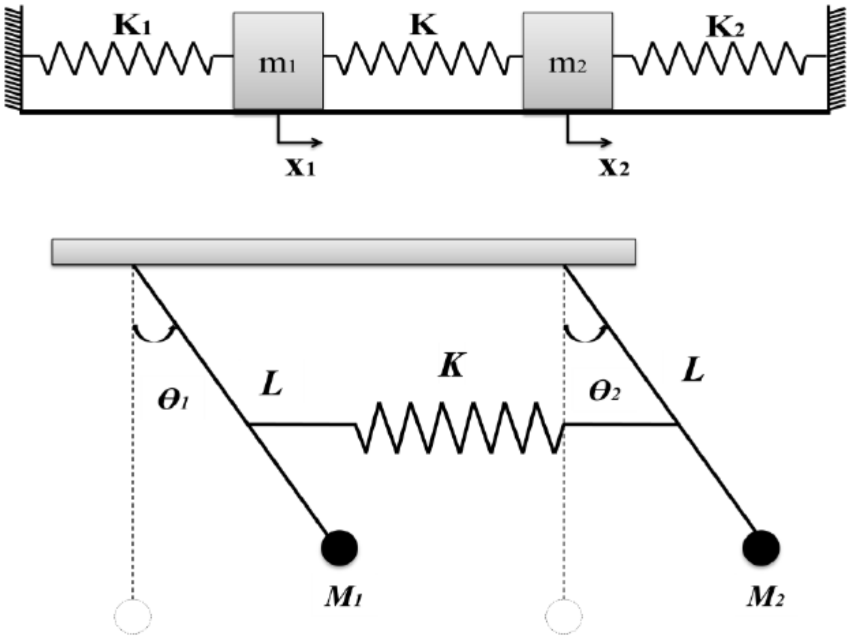


Figure IV.1 Examples of Two-Degree-of-Freedom Systems Coupled by Elasticity

The corresponding differential equations are:

$$\begin{cases}\ddot{q}_1 + 2\delta\dot{q}_1 + \omega_0^2 q_1 = a_1 q_2 \\ \ddot{q}_2 + 2\delta\dot{q}_2 + \omega_0^2 q_2 = a_2 q_1\end{cases} \quad \text{(IV.1)}$$

Where: a_1x_2 and a_2x_1 are the coupling terms. a_1 and a_2 are constants.

IV.2.1.2 Inertial coupling

In mechanical systems, coupling is achieved through inertia (a mass).

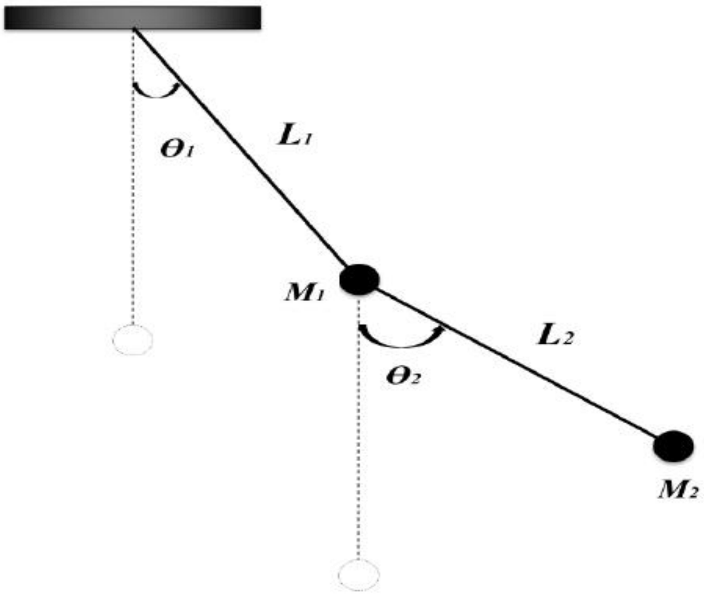


Figure IV.2 Example of a Two-Degree-of-Freedom System Coupled by Inertia

The corresponding differential equations are:

$$\begin{cases}\ddot{q}_1 + 2\delta\dot{q}_1 + \omega_0^2 q_1 = b_1 q_2 \\ \ddot{q}_2 + 2\delta\dot{q}_2 + \omega_0^2 q_2 = b_2 q_1\end{cases} \qquad \text{(IV.2)}$$

Where: b_1x_2 and b_2x_1 are the coupling terms. $b_1$and b_2 are constants.

IV.2.1.3 Viscous coupling

In mechanical systems, coupling is achieved through a damper. In electrical systems, we find circuits coupled by inductance, which is equivalent to inertial coupling.

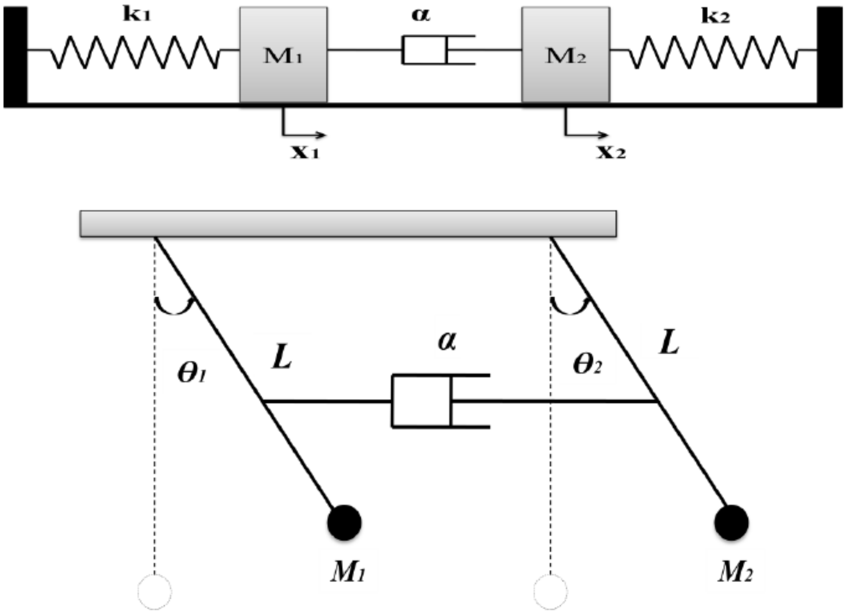


Figure IV.3 Viscous Coupling of Different Mechanical Systems

The corresponding differential equations are:

$$\begin{cases} \ddot{q}_1 + 2\delta\dot{q}_1 + \omega_0^2 q_1 = c_1 q_2 \\ \ddot{q}_2 + 2\delta\dot{q}_2 + \omega_0^2 q_2 = c_2 q_1 \end{cases} \qquad \text{(IV.3)}$$

Where: c_1x_2 and $_{c2x1}$ are the coupling terms. c_1 and c_2 are constants.

IV.2.2 Differential Equations of Motion

To study two-degree-of-freedom systems, it is necessary to write two differential equations of motion, which can be obtained from the Lagrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = 0 \end{cases} \qquad \text{(IV.4)}$$

q_1 and q_2: the two generalized coordinates that characterize the two-degree-of-freedom system.

The Lagrangian is:

$$L = T - U \tag{IV.5}$$

IV.2.3 General Method for Solving the Equations of Motion

For a mechanical system, the formulation of the coupled system equations follows this method:

1. Write the two differential equations in terms of the generalized coordinates.
2. Assume that the system admits harmonic solutions, meaning that the system can oscillate with the same frequency for all the oscillators.
3. Solving the system of equations gives two specific frequencies ω_1 and ω_2: these are the natural frequencies.
4. Then substitute ω_1 into one of the two equations to obtain the first normal mode.
5. Then substitute ω_2 into one of the two equations to obtain the second normal mode.
6. Write the two general solutions of the differential equations of motion.

IV.2.4 Study of a Free Mechanical System with Two Degrees of Freedom

IV.2.4.1 Complex System (Masses-Springs)

Consider the mechanical system shown in Figure IV.4, consisting of two harmonic oscillators (m1,K_1) and (m_2,K_2) coupled by a spring with stiffness constant K. The two masses are assumed to move frictionlessly on a horizontal plane, and their displacements from their equilibrium positions are denoted by x_1 and x_2.

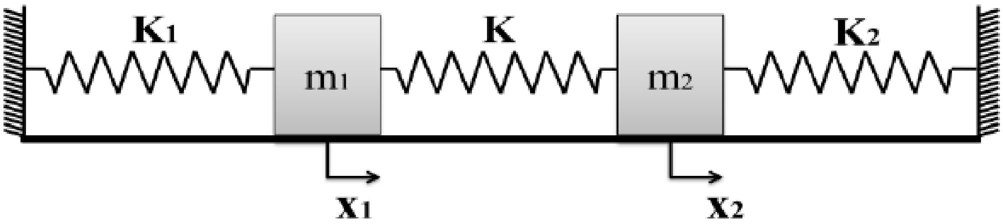


Figure IV.4: Oscillatory Motion of a Two-Degree-of-Freedom Coupled System

When this system is displaced from its equilibrium position and then left to itself, it undergoes free vibratory motion.

The kinetic energy of the system T:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad \text{(IV.6)}$$

The potential energy of the system U:

$$U = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_1 (x_2 - x_1)^2 + \frac{1}{2} K_2 (-X_2)^2 \qquad \text{(IV.7)}$$

The Lagrangian of the system is expressed as follows:

$$L = T - U \qquad \text{(IV.8)}$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} K_1 X_1^2 - \frac{1}{2} K_1 (X_2 - X_1)^2 - \frac{1}{2} X_2^2 \qquad \text{(IV.9)}$$

The Lagrange equations in this case are written as:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \end{cases} \qquad \text{(IV.10)}$$

$$\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 \qquad \text{(IV.11)}$$

$$\left(\frac{\partial L}{\partial x_1}\right) = -(K_1 + K) x_1 + K x_2 \qquad \text{(IV.12)}$$

$$\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 \qquad \text{(IV.13)}$$

$$\left(\frac{\partial L}{\partial x_2}\right) = -(K_2 + K)x_2 + Kx_1 \tag{IV.14}$$

The equations describing the variation of displacements x_1 and x_2 as a function of time are written as follows:

$$\begin{cases} m_1\ddot{x_1} + (K_1 + K)x_1 - Kx_2 = 0 \\ m_2\ddot{x_2} + (K_2 + K)x_2 - Kx_1 = 0 \end{cases} \tag{IV.15}$$

The terms $-Kx_1$ and $-Kx_2$ are called coupling terms.

It is assumed that the system admits harmonic solutions:

$$so: \begin{cases} x_1(t) = A_1 \cos(\omega t + \varphi) \Rightarrow \ddot{x_1} = -\omega^2 x_1 \\ x_2(t) = A_2 \cos(\omega t + \varphi) \Rightarrow \ddot{x_2} = -\omega^2 x_2 \end{cases} \tag{IV.16}$$

By substituting the solutions into the differential system, we obtain the following linear system:

$$-\omega^2 m_1\ddot{x_1} + (K_1 + K)x_1 - Kx_2 = 0 \tag{IV.17}$$

$$\begin{cases} -\omega^2 m_1\ddot{x_1} + (K_1 + K)x_1 - Kx_2 = 0 \\ -\omega^2 m_2\ddot{x_2} + (K_2 + K)x_2 - Kx_1 = 0 \end{cases} \Rightarrow \begin{cases} (-\omega^2 m_1 + K_1 + K)x_1 - Kx_2 = 0 \\ -Kx_1 + (-\omega^2 m_2 + K_2 + K)x_2 = 0 \end{cases} \tag{IV.18}$$

$$\begin{pmatrix} -\omega^2 m_1 + K_1 + K & -K \\ -K & -\omega^2 m_2 + K_2 + K \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{IV.19}$$

The system admits non-zero solutions if and only if the determinant equals zero.

$$\Delta(\omega) = \begin{vmatrix} -\omega^2 m_1 + K_1 + K & -K \\ -K & -\omega^2 m_2 + K_2 + K \end{vmatrix} = 0 \tag{IV.20}$$

The determinant Δ(ω) is called the characteristic determinant. The equation Δ(ω)=0 is called the characteristic equation or the equation of natural frequencies. It is written as follows:

$$\Delta(\omega) = (-\omega^2 m_1 + K_1 + K) \times (-\omega^2 m_2 + K_2 + K) - K^2 = 0$$

$$\omega^4 - (\Omega_1^2 + \Omega_2^2)\omega^2 + \Omega_1^2\Omega_2^2\left(1 - K'^2\right) = 0 \qquad \text{(IV.21)}$$

The following constants are defined as follows:

$$\Omega_1^2 = \frac{K_1}{m_1}, \Omega_2^2 = \frac{K_2}{m_2}, K' \frac{K^2}{(K_1+K)(K_2+K)} \qquad \text{(IV.22)}$$

K′ is called the coupling coefficient.

The two natural frequencies are:

$$\begin{cases} \omega_1^2 = \frac{\Omega_1^2+\Omega_2^2}{2} - \frac{1}{2}\sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4K\Omega_1^2\Omega_2^2} \\ \omega_2^2 = \frac{\Omega_1^2+\Omega_2^2}{2} + \frac{1}{2}\sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4K\Omega_1^2\Omega_2^2} \end{cases} \qquad \text{(IV.23)}$$

The general solution of the system is written as a superposition of the two normal modes, as follows:

$$\begin{cases} x_1(t) = A_1\cos(\omega_1 t + \varphi_1) + A_2\cos(\omega_2 t + \varphi_2) \\ x_2(t) = B_1\cos(\omega_1 t + \varphi_1) + B_2\cos(\omega_2 t + \varphi_2) \end{cases} \qquad \text{(IV.24)}$$

A_1, A_2, B_1, B_2,ϕ_1, and ϕ_2 are constants of integration determined from the initial conditions.

To simplify the number of unknowns, the amplitude ratios for the normal modes are determined:

$$\begin{cases} \omega = \omega_1; \frac{A_1}{A_2} \\ \omega = \omega_2; \frac{B_1}{B_2} \end{cases} \qquad \text{(IV.25)}$$

In the case of a symmetric system where m_1=m_2=m and K_1=K_2=k, the relations (IV.18) become:

$$\begin{cases} (-\omega^2 m + k + K)x_1 - Kx_2 = 0 & \text{(IV.19)} \\ -Kx_1 + (-\omega^2 m + K + k)x_2 -= 0 & \text{(IV.20)} \end{cases}$$

$$\begin{pmatrix}-\omega^2 m+k+K & -K\\ -K & -\omega^2 m+k+K\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{(IV.21)}$$

The system admits non-zero solutions if and only if the determinant equals zero.

$$\Delta(\omega)=\begin{vmatrix}-\omega^2 m+k+K & -K\\ -K & -\omega^2 m+k+K\end{vmatrix}=0$$

$$\Delta(\omega)=(-\omega^2 m+k+K)^2-K^2=0$$

$$\begin{cases}-\omega^2 m+k+K=K\\ -\omega^2 m+k+K=-K\end{cases}\Rightarrow \omega_1^2=\frac{k}{m},\omega_2^2=\frac{k+2K}{m}$$

The two natural frequencies are:

$$\omega_1=\sqrt{\frac{k}{m}}\ and\ \omega_2=\sqrt{\frac{k+2K}{m}}$$

The solutions of the system:

The general solution is then written as a linear combination of the two solutions.

$$\begin{cases}x_1(t)=A_1\sin(\omega_1 t+\varphi_1)+B_1\sin(\omega_2 t+\varphi_2)\\ x_2(t)=A_2\sin(\omega_1 t+\varphi_1)-B_2\sin(\omega_2 t+\varphi_2)\end{cases} \qquad \text{(IV.22)}$$

IV.2.4.2 Study of Eigen modes

- **First mode:** Substitute into (1) or (2) with: $\omega_1^2=\frac{k}{m}$

We obtain after calculation: $x_2=x_1$.

In this mode: $\omega=\omega_1\Rightarrow x_2=x_1\Rightarrow A_1=A_2\Rightarrow\vec{v}_1\begin{pmatrix}1\\ 1\end{pmatrix}$, $\vec{v}_1$ is the first eigenvector.

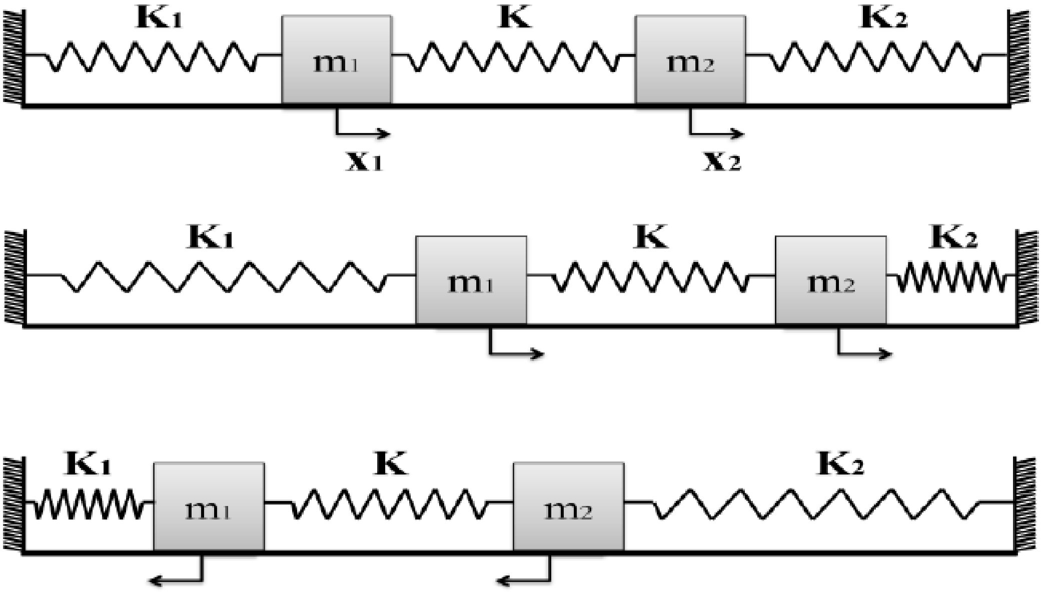


Figure IV.5 State of the system for the first mode, "In Phase".

In this mode, both masses oscillate with the same angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$, with the same amplitude A, and in phase.

- **Second mode:** Substitute into (1) or (2) with $\omega_2^2 = \frac{k+2K}{m}$
- After calculation, we obtain: $x_2=-x_1$

In this mode: $\omega=\omega_2 \Rightarrow x_2=x_1 \Rightarrow B_1=B_2 \Rightarrow \vec{v}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2$ is the second eigenvector.

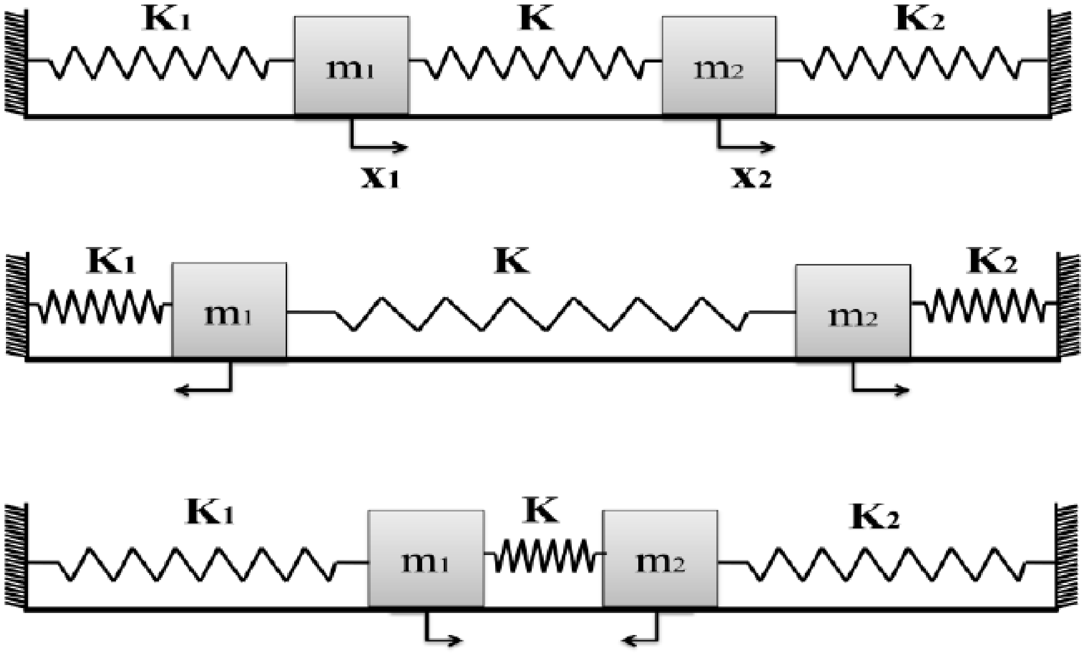


Figure IV.6 State of the system for the second mode, "Out of Phase".

In this mode, both masses oscillate with the same angular frequency $\omega_2=\sqrt{\frac{k+2K}{m}}$, with the same amplitude B, and out of phase.

Thus, the general solutions become:

$$\begin{cases} x_1(t) = A\sin(\omega_1 t+\varphi_1) + B\sin(\omega_2 t+\varphi_2) \\ x_2(t) = A\sin(\omega_1 t+\varphi_1) - B\sin(\omega_2 t+\varphi_2) \end{cases} \qquad \text{(IV.23)}$$

Where the constants A, B, ϕ_1 and ϕ_2 will be defined by the initial conditions.

IV.2.4.3 Beat Phenomenon

The general solutions are then given by:

$$\begin{cases} x_1(t) = A\cos(\omega_1 t+\varphi_1) + B\cos(\omega_2 t+\varphi_2) \\ x_2(t) = A\cos(\omega_1 t+\varphi_1) - B\cos(\omega_2 t+\varphi_2) \end{cases} \qquad \text{(IV.24)}$$

For well-chosen initial conditions, we can write:

$$\begin{cases} x_1(t) = A\,(\cos\omega_1 t + cos\omega_2 t) = 2Acos\left(\frac{\omega_2-\omega_1}{2}t\right)\cos\,(\frac{\omega_2+\omega_1}{2}t) \\ x_2(t) = A\,(\cos\omega_1 t - cos\omega_2 t) = 2Acos\left(\frac{\omega_2-\omega_1}{2}t\right)\sin\,(\frac{\omega_2+\omega_1}{2}t) \end{cases} \quad \text{(IV.25)}$$

When the coupling is weak (K is small), the natural frequencies of the two oscillators (ω_1 and ω_2) are close ($\omega1\approx\omega2\Rightarrow\Delta\omega=\omega_2-\omega_1$ is small), leading to a phenomenon of beats. The two oscillators exchange energy between them and vibrate with a frequency ω, which is the average of the two natural frequencies: $\omega = \frac{(\omega_2+\omega_1)}{2}$ with a period equal to: $T = \frac{2\pi}{\omega} = \frac{4\pi}{\omega_2+\omega_1}$

Meanwhile, the beat frequency is $\omega_B=\frac{(\omega_2-\omega_1)}{2}$

with a period equal to: $T_B=\frac{4\pi}{\omega_2-\omega_1}$

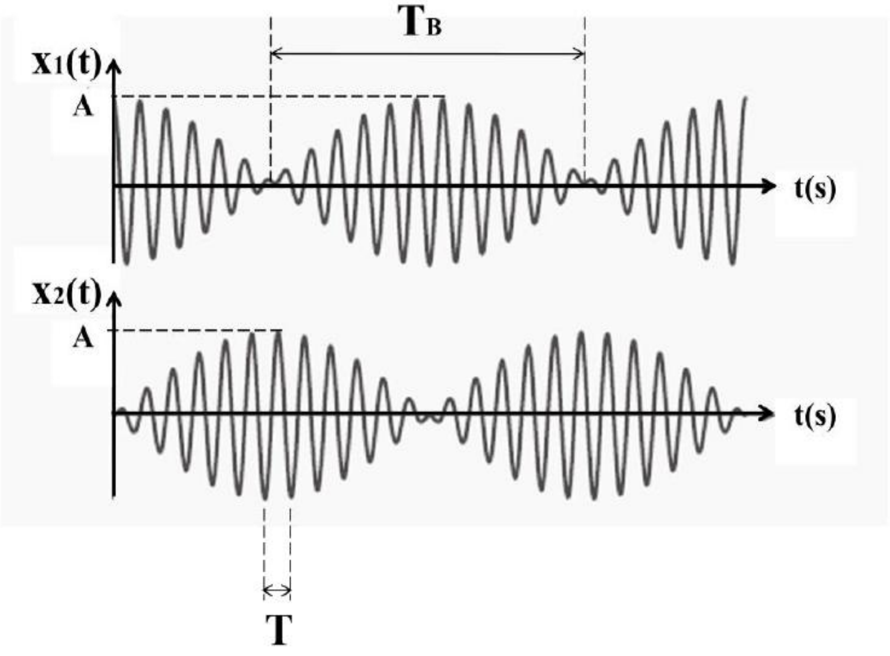


Figure IV.7 Phenomenon of beats or amplitude modulation.

A beat period is thus the time it takes for the vibrational energy to complete a round trip.

IV.2.5 Coupled Pendulums (Inertial Coupling)

Consider two pendulums that are coupled by a mass m_1, located at a distance l_1 from the axis of rotation.

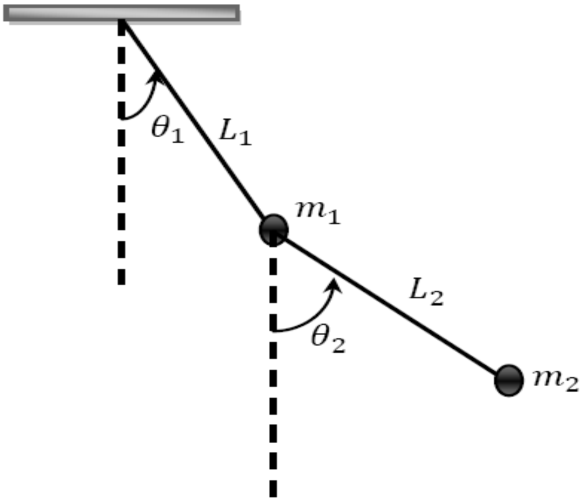


Figure IV.8 Inertial Coupling

Differential Equations of Motion

• Coordinates of the system elements

The mass m1 is located at a distance l_1 from point O

$$m_1\begin{cases} x_{m_1} = l_1.sin\theta_1 \\ y_{m_1} = -l_1.cos\theta_1 \end{cases} \Rightarrow \begin{cases} \dot{x}_{m_1} = l_1\dot{\theta}_1 cos\theta_1 \\ \dot{y}_{m_1} = l_1\dot{\theta}_1 sin\theta_1 \end{cases} \Rightarrow v_{m_1}^2 = l_1^2\dot{\theta}_1^2 \qquad \text{(IV.26)}$$

The mass m_2 is located at a distance (l_1+l_2) from O

$$m_2\begin{cases} x_{m_2} = l_1.sin\theta_1 + l_2.sin\theta_2 \\ y_{m_2} = -l_1.cos\theta_1 - l_2.cos\theta_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_{m_2} = l_1\dot{\theta}_1 cos\theta_1 + l_2\dot{\theta}_2 cos\theta_2 \\ \dot{y}_{m_2} = l_1\dot{\theta}_1 sin\theta_1 + l_2\dot{\theta}_2 sin\theta_2 \end{cases} \quad \text{(IV.27)}$$

$$\Rightarrow v_{m_2}^2 = (l_1\dot{\theta}_1 cos\theta_1 + l_2\dot{\theta}_2 cos\theta_2)^2 + (l_1\dot{\theta}_1 sin\theta_1 + l_2\dot{\theta}_2 sin\theta_2)^2$$

$$\Rightarrow v_{m_2}^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 cos\theta_1 l_2\dot{\theta}_2 cos\theta_2 + 2l_1\dot{\theta}_1 sin\theta_1 l_2\dot{\theta}_2 sin\theta_2$$

$$\Rightarrow v_{m_2}^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2)$$

We have: $\cos(\theta_1 - \theta_2) = (cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2)$

$$\Rightarrow v_{m_2}^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2 cos(\theta_1 - \theta_2) \quad \text{(IV.28)}$$

In the case of small oscillations, where the angles are very small, we have: $\cos(\theta_1 -$

$\theta 2 \approx 1$

$$v_{m_2}^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2 = (l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2 \quad \text{(IV.29)}$$

The kinetic energy of the system:

$$T = T_{m_1} + T_{m_2} = \frac{1}{2}m_1 v_{m_1}^2 + \frac{1}{2}m_2 v_{m_2}^2 \quad \text{(IV.30)}$$

$$T_{m_1} = \frac{1}{2}m_1 v_{m_1}^2 = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 \quad \text{(IV.31)}$$

$$T_{m_2} = \frac{1}{2}m_2 v_{m_2}^2 = \frac{1}{2}m_2(l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2 \quad \text{(IV.32)}$$

$$\Rightarrow T = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2 \quad \text{(IV.33)}$$

The potential energy of the system: $U = U_{m_1} + U_{m_2}$

If we choose the (ox) axis as the origin of potential energies, we have for the two masses:

$$U = U_{m_1} + U_{m_2} = -m_1 g l_1 cos\theta_1 - m_2 g(l_1 cos\theta_1 + l_2 cos\theta_2) \tag{IV.34}$$

(The negative sign comes from the fact that the mass m is below the chosen axis).

$$U = -g l_1(m_1 + m_2)cos\theta_1 - m_2 g l_2 cos\theta_2 \tag{IV.35}$$

The Lagrangian function will therefore be:

$$L = T - U = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2 + g l_1(m_1 + m_2)cos\theta_1 + m_2 g l_2 cos\theta_2$$

We clearly observe two generalized coordinates describing the motion, so we will have two Lagrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \end{cases} \tag{IV.36}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = -g l_1(m_1 + m_2) sin\theta_1 \end{cases} \tag{IV.37}$$

$$\Rightarrow m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + g l_1(m_1 + m_2)\theta_1 = 0$$

$$\Rightarrow l_1^2 \ddot{\theta}_1 (m_1 + m_2) + g l_1 (m_1 + m_2)\theta_1 = -m_2 l_1 l_2 \ddot{\theta}_2$$

We divide by $l_1(m_1 + m_2)$ and find:

$$l_1 \ddot{\theta}_1 + g\theta_1 = -\frac{m_2 l_2}{m_1 + m_2} \ddot{\theta}_2 \tag{IV.38}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 = m_2 l_2 (l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_2}\right) = -g l_2 m_2 sin\theta_2 \end{cases} \tag{IV.39}$$

$$\Rightarrow m_2 l_2 \left(l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2\right) + g l_2 m_2 \theta_2 = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + g l_2 m_2 \theta_2 = -m_2 l_2 l_1 \ddot{\theta}_1$$

We divide by $l_2 m_2$ and find: $l_2 \ddot{\theta}_2 + g\theta_2 = -l_1 \ddot{\theta}_1$

Thus, the two differential equations of motion are:

$$\begin{cases} l_1 \ddot{\theta}_1 + g\theta_1 = -\dfrac{m_2 l_2}{m_1 + m_2} \ddot{\theta}_2 & \qquad (\text{IV}.40) \\ l_2 \ddot{\theta}_2 + g\theta_2 = -l_1 \ddot{\theta}_1 & \qquad (\text{IV}.41) \end{cases}$$

1. We assume that the system admits harmonic solutions:

$$\begin{cases} \theta_1(t) = A_1 \sin(\omega t + \varphi_1) \Rightarrow \ddot{\theta}_1 = -\omega^2 \theta_1 \\ \theta_2(t) = A_2 \sin(\omega t + \varphi_2) \Rightarrow \ddot{\theta}_2 = -\omega^2 \theta_2 \end{cases} \qquad (\text{IV}.42)$$

Such that: A_1, A_2, φ_1, φ_2, and ω are one of the natural frequencies of the system.

We substitute into equations $(\text{IV}.40)$ and $(\text{IV}.41)$, thus:

$$\begin{cases} -l_1 \omega^2 \theta_1 + g\theta_1 = \frac{m_2 l_2}{m_1 + m_2} \omega^2 \theta_2 \\ \quad -l_2 \omega^2 \theta_2 + g\theta_2 = l_1 \omega^2 \theta_1 \end{cases} \qquad (\text{IV}.43)$$

$$\begin{cases} (g - l_1 \omega^2)\theta_1 - \dfrac{m_2 l_2}{m_1 + m_2} \omega^2 \theta_2 = 0 & \qquad (\text{IV}.44) \\ (g - l_2 \omega^2)\theta_2 - l_1 \omega^2 \theta_1 = 0 & \qquad (\text{IV}.45) \end{cases}$$

2. Calculation of the natural frequencies: We assume that $m_1 = m_2 = m$ and $l_1 = l_2 = l$

$$\begin{cases} (g - l\omega^2)\theta_1 - \dfrac{l}{2} \omega^2 \theta_2 = 0 & \qquad (\text{IV}.46) \\ (g - l\omega^2)\theta_2 - l\omega^2 \theta_1 = 0 & \qquad (\text{IV}.47) \end{cases}$$

$$\begin{pmatrix} g - l\omega^2 & -\frac{l}{2}\omega^2 \\ -l\omega^2 & g - l\omega^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\text{IV}.48)$$

These two equations will accept a solution if the determinant equals 0.

$\begin{vmatrix} g-l\omega^2 & -\frac{l}{2}\omega^2 \\ -l\omega^2 & g-l\omega^2 \end{vmatrix} = 0 \Leftrightarrow (g-l\omega^2)^2 - l\omega^2\frac{l}{2}\omega^2 = 0$;This is the eigenvalue equation.

ω_i eigen values

$$(g-l\omega^2)^2 - \frac{1}{2}(l\omega^2)^2 = 0 \Rightarrow g-l\omega^2 = \begin{cases} +\frac{1}{\sqrt{2}}l\omega^2 \\ -\frac{1}{\sqrt{2}}l\omega^2 \end{cases} \qquad (\text{IV}.49)$$

$$\Rightarrow \begin{cases} \omega_1^2 = \frac{g}{l(1+\frac{1}{\sqrt{2}})} \\ \omega_2^2 = \frac{g}{l(1-\frac{1}{\sqrt{2}})} \end{cases} \qquad (\text{IV}.50)$$ These are the eigen values

3. Calculation of the oscillation modes or eigenvectors:

   First mode: substitute into $(\text{IV}.46)$ or $((\text{V}.47)$ by $\omega_1^2 = \frac{g}{l(1+\frac{1}{\sqrt{2}})}$

$$\left(g - l\frac{g}{l(1+\frac{1}{\sqrt{2}})}\right)\theta_1 - \frac{l}{2}\frac{g}{l(1+\frac{1}{\sqrt{2}})}\theta_2 = 0 \qquad (\text{IV}.51)$$

$\left(1 - \frac{1}{1+\frac{1}{\sqrt{2}}}\right)\theta_1 = \frac{1}{2(1+\frac{1}{\sqrt{2}})}\theta_2 \Rightarrow \theta_2 = \sqrt{2}\,\theta_1$; This is the first mode.

Second mode: substitute into $(\text{IV}.46)$ or $(\text{IV}.47)$ by: $\omega_2^2 = \frac{g}{l(1-\frac{1}{\sqrt{2}})}$

We find: $\theta_2 = -\sqrt{2}\,\theta_1$; This is the second mode.

IV.2.6 Forced Oscillations of Systems with Two Degrees of Freedom

IV.2.6.1 Lagrange Equations

The differential equations describing the forced oscillatory motion of systems with two degrees of freedom are:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right) - \left(\frac{\partial L}{\partial q_1}\right) + \frac{\partial D}{\partial \dot{q_1}} = F_{q_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_2}}\right) - \left(\frac{\partial L}{\partial q_2}\right) + \frac{\partial D}{\partial \dot{q_2}} = F_{q_2} \end{cases} \qquad (\text{IV}.52)$$

F_{q_1} and F_{q_2} are the generalized forces conjugate to the respective generalized coordinates q_1 and q_2.

In the case where the coordinate represents a rotation ($q=\theta$), the force $F_q(t)$ is replaced by the moment of this force $M(F_q(t))$.

IV.2.6.2 Differential Equation of a Forced System with Two Degrees of Freedom

Consider the system with two degrees of freedom (x_1,x_2) shown in Figure IV.9, composed of masses, springs, and dampers, with their characteristics illustrated in the figure. This system is subjected to an external force $F_{q_1}=F=F_0\cos(\Omega t)$.

- K: coupling spring constant.

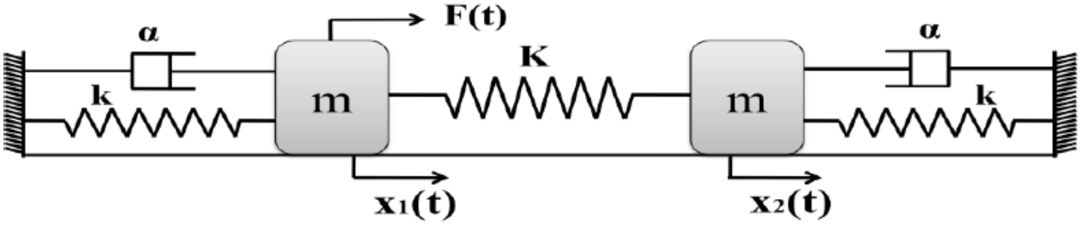


Figure IV.9 System with Two Degrees of Freedom Coupled by a Spring

In this case, the Lagrange equations for the system can be written as:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \left(\frac{\partial L}{\partial x_1}\right) + \frac{\partial D}{\partial \dot{x_1}} = F_{q_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \left(\frac{\partial L}{\partial x_2}\right) + \frac{\partial D}{\partial \dot{x_2}} = F_{q_2} \end{cases} \qquad (\text{IV}.53)$$

The kinetic energy of the system L: $L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2$

The potential energy of the system T: $T = \frac{1}{2}kx_1^2 + \frac{1}{2}K(x_1 - x_2)^2 + \frac{1}{2}kx_2^2$

The dissipation energy D :$D = \frac{1}{2}\alpha\dot{x}_1^2 + \frac{1}{2}\alpha\dot{x}_2^2$

Whence the Lagrangian : L=T-U=$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}K(x_1 - x_2)^2 - \frac{1}{2}kx_2^2$

The Lagrange equations in this case are written as:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \left(\frac{\partial L}{\partial x_1}\right) + \frac{\partial D}{\partial \dot{x_1}} = F_{x_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \left(\frac{\partial L}{\partial x_2}\right) + \frac{\partial D}{\partial \dot{x_2}} = 0 \end{cases} \qquad (\mathrm{IV}.54)$$

The equations describing the variation of the elongations x_1 and x_2 as a function of time are written as follows:

$$\begin{cases} m_1\ddot{x}_1 + (k + K)x_1 + \alpha\dot{x}_1 - Kx_2 = F \\ m_2\ddot{x}_2 + (k + K)x_2 + \alpha\dot{x}_2 - Kx_1 = 0 \end{cases} \qquad (\mathrm{IV}.55)$$

IV.2.6.3 Study of the sinusoidal steady-state (Resolution of the differential equations)

The general solution of the system of differential equations is equal to the solution of the homogeneous system plus a particular solution. The solution of the homogeneous equation, due to damping, tends to zero as time increases. When the steady-state is reached, the solution becomes equal to the steady-state solution and is then written as:

$$\begin{cases} x_1(t) \cong x_{1_p}(t) = x_1(\Omega t + \varphi_1) \\ x_2(t) \cong x_{2_p}(t) = x_2(\Omega t + \varphi_2) \end{cases} \qquad (\mathrm{IV}.56)$$

To calculate the amplitudes X_1 and X_2, as well as the phases φ_1 and φ_2, we use the methods of complex numbers. We can thus write:

$$\begin{cases} x_1(t) = x_1 e^{j(\Omega t + \varphi_1)} = \overline{x_1}e^{j\Omega t} \Rightarrow \ddot{x}_1(t) = -\Omega^2\overline{x_1}e^{j\Omega t} \\ x_2(t) = x_2 e^{j(\Omega t + \varphi_1)} = \overline{x_2}e^{j\Omega t} \Rightarrow \ddot{x}_2(t) = -\Omega^2\overline{x_2}e^{j\Omega t} \end{cases} \qquad (\mathrm{IV}.57)$$

IV.2.6.4 Calculation of X₁ and X₂ in the case of low damping

IV.2.6.4.1 Negligible damping

First, let's consider the case of sufficiently low damping so that we can assume $\alpha \approx 0$.

The system of differential equations is then written as:

$$\begin{cases} m_1\ddot{x}_1 + (k+K)x_1 + \alpha\dot{x}_1 - Kx_2 = F \\ m_2\ddot{x}_2 + (k+K)x_2 + \alpha\dot{x}_2 - Kx_1 = 0 \end{cases} \qquad (\text{IV}.58)$$

The frequencies $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{k+2K}{m}}$ are the natural frequencies calculated in the previous chapter.

$$F(t) = F_0 cos\Omega t = F_0 e^{j\Omega t} \qquad (\text{IV}.59)$$

$$\begin{cases} -m_1\Omega^2\overline{x_1} + (k+K)\overline{x_1} - K\overline{x_2} = F_0 e^{j\Omega t} \\ \quad -m_2\Omega^2\overline{x_2} + (k+K)\overline{x_2} - K\overline{x_1} = 0 \end{cases} \qquad (\text{IV}.60)$$

The solutions of this system are:

$$\overline{x_1} = \frac{F_0}{m}\frac{(\Omega_A^2 - \Omega^2)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} \qquad (\text{IV}.61)$$

$$\overline{x_2} = \frac{KF_0}{m^2}\frac{1}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} \qquad (\text{IV}.62)$$

The value of the frequency Ω_A is: $\Omega_A = \sqrt{\frac{k+K}{m}}$

$\overline{x_1} = x_1 e^{j\Omega t}$ and $\overline{x_2} = x_2 e^{j\Omega t} \Rightarrow \varphi_1 and \varphi_{21}$ They can only have values of 0 and/or π, since the imaginary part of $\overline{x_1}$ and $\overline{x_2}$ is zero

$$x_1 = |\overline{x_1}| \; and \; \varphi_1 = -arctg\frac{Im(\overline{x_1})}{Re\,(\overline{x_1})} \qquad (\text{IV}.63)$$

We will have finally: the amplitudes of the displacements X₁ and X₂

$$x_1 = \frac{F_0}{m}\frac{|\Omega_A^2 - \Omega^2|}{|\omega_1^2 - \Omega^2||\omega_2^2 - \Omega^2|} \qquad (\text{IV}.64)$$

$$x_2 = \frac{KF_0}{m^2}\frac{1}{|\omega_1^2 - \Omega^2||\omega_2^2 - \Omega^2|} \qquad (\text{IV}.65)$$

IV.2.6.4.2 Variations of the amplitudes X_1 and X_2

The variation of the amplitudes X_1 and X_2 is a function of the excitation force frequency Ω, as shown in figures IV.9 and IV.10.

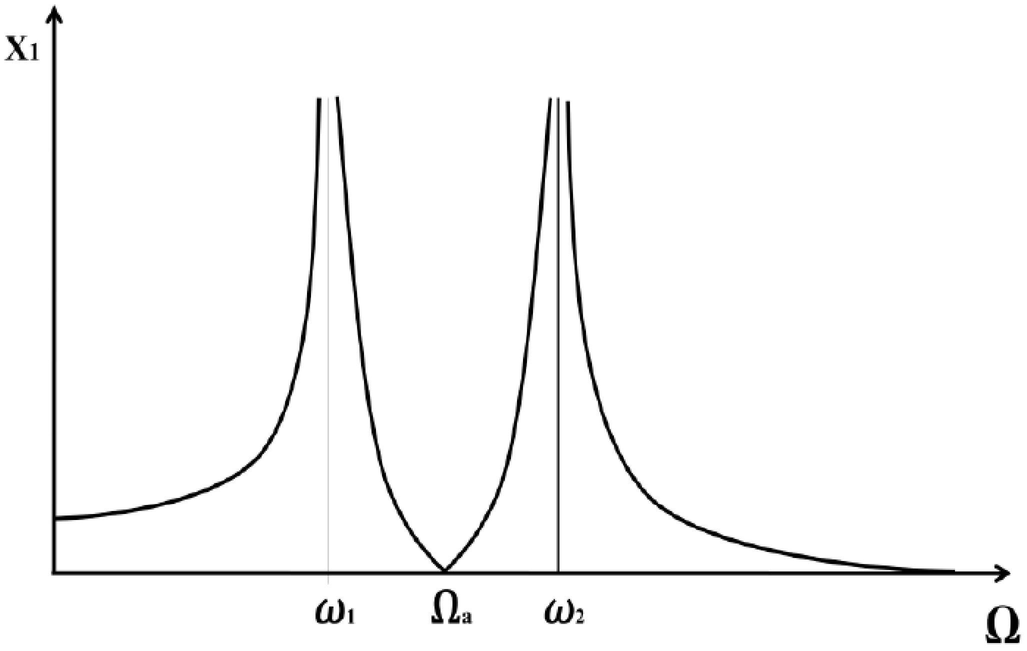


Figure IV.10: Variation of X_1 as a function of Ω

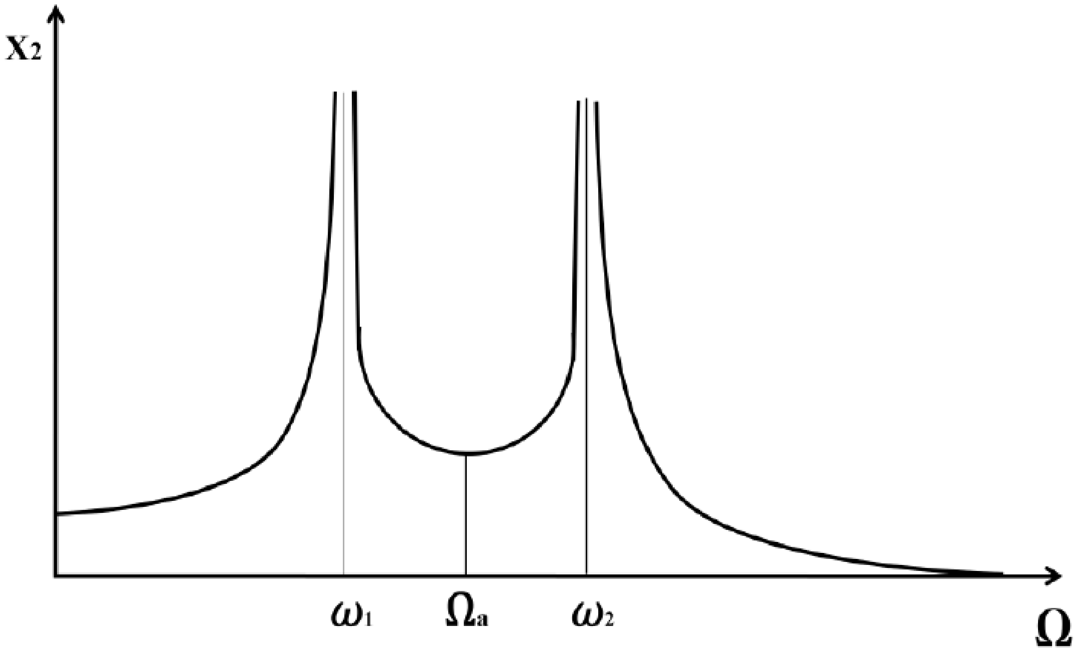


Figure IV.11: Variation of X_2 as a function of Ω

Ω=0⇒$x_1 = \frac{F_0}{m}\frac{|\Omega_A^2|}{|\omega_1^2||\omega_2^2|}$ and $x_2 = \frac{KF_0}{m^2}\frac{1}{|\omega_1^2||\omega_2^2|}$

- It is noted that the resonance phenomenon occurs for both X_1 and X_2 when the excitation frequency Ω is equal to one of the natural frequencies ω_1 and ω_2 of the system (when $\Omega = \omega_1$ and $\Omega = \omega_2$ ⇒ Resonance).

$\Omega = \omega_1 = \omega R_1$ (Called the first resonance frequency)

$\Omega = \omega_2 = \omega R_2$ (Called the second resonance frequency)

- When the damping is very low, the amplitudes at resonance are very large.
- When the frequency Ω becomes very large, these amplitudes tend to zero.
- When $\Omega = \Omega_A$ ⇒The amplitude X_1 is equal to zero ($A_1 = 0$) ⇒ Ω_A is called the anti-resonance frequency.

Exercise 1

Let the two mechanical subsystems *A* and *B* (see figure1): The mechanical system *A* consists of a horizontal bar of negligible mass and length 2*L*, carrying masses m at its ends. Friction is represented by a damper with a coefficient α. The mechanical system B consists of a spring with a stiffness constant K connected to a point mass m. The two subsystems *A* and *B* are coupled by a spring with stiffness *2K*. The new system is described at time t by the generalized coordinates $\theta(t)$ and $x(t)$, which are assumed to have small amplitudes.

1. What is the number of degrees of freedom? Specify the type of coupling.
2. Derive the differential equations of motion for the coupled system.
3. Neglecting the damping effect ($\alpha=0$), write the equations of motion in matrix form$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \theta \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
4. Find the values of a, b, c, and d.

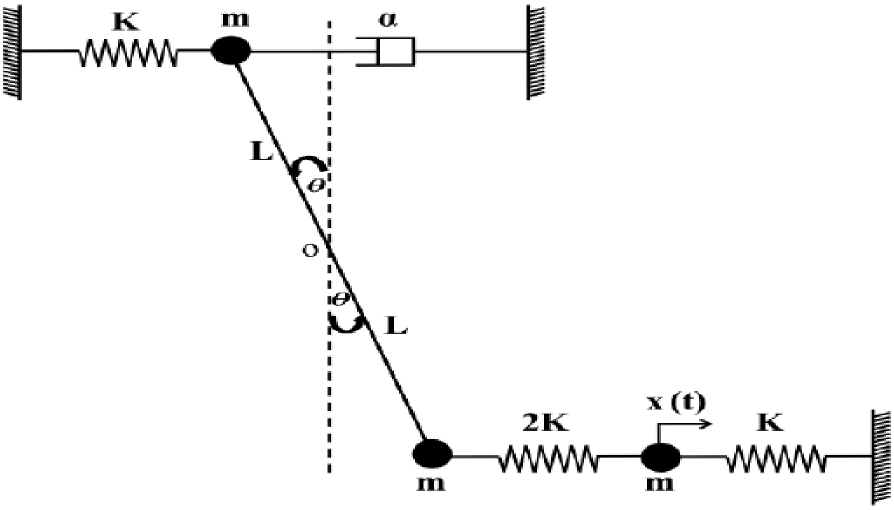


Figure 1

Exercise 2

Let us consider the two-degree-of-freedom system shown in the adjacent figure2. Let x_1 and x_2 be the dynamic displacements of m and M, respectively, relative to their equilibrium positions.

1. Describe the system and specify the type of coupling.
2. Determine the kinetic energy T, potential energy U, and dissipation function D of the system.
3. Find the steady-state solutions, given that $F(t)=KA\cos(\Omega t)$.
4. If $\alpha=0$, for which value of Ω does the system enter resonance? In this case, provide the condition under which the excited mass m remains stationary.

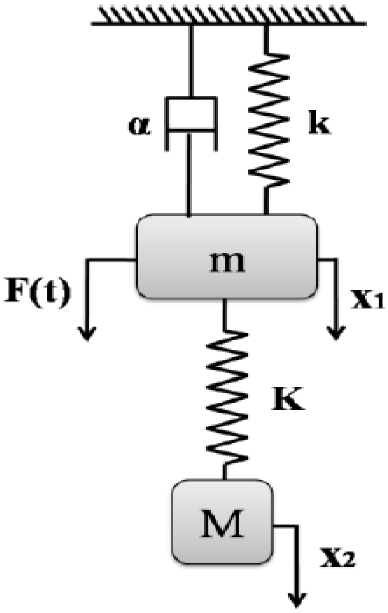


Figure 2

Exercise 3

We consider the system represented by the adjacent figure, composed of two harmonic oscillators (M_1, K) and (M_2, K), coupled by a damper α.

1. Describe the system and determine the type of coupling.
2. Determine the kinetic energy T and potential energy U of the system.
3. Determine the Lagrangian L of the system.
4. Derive the differential equations of motion in terms of the variables $x_1(t)$ and $x_2(t)$.
5. Given the parameters $M_1=M_2=M$, establish the new differential equations of motion.

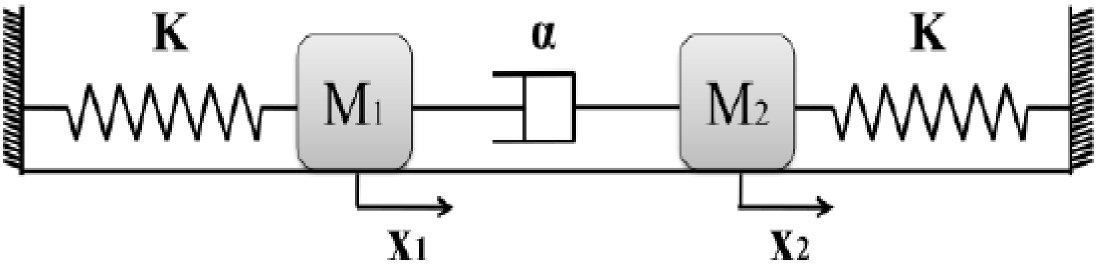


Figure 3

PART II: MECHANICAL WAVES

Chapter V

Wave propagation

Chapter V

Wave propagation

V.1 Definition of Waves

In physics, mathematics, engineering, and related fields, a wave is a propagating dynamic disturbance (change from equilibrium) of one or more quantities. Periodic waves oscillate repeatedly about an equilibrium (resting) value at some frequency. When the entire waveform moves in one direction, it is said to be a travelling wave; by contrast, a pair of superimposed periodic waves traveling in opposite directions makes a standing wave. In a standing wave, the amplitude of vibration has nulls at some positions where the wave amplitude appears smaller or even zero. There are several sources of waves, including:

- Light Waves (Vision and Light)
- Sound Waves (Hearing)
- Mechanical Waves
- Hertzian Waves (Radio, TV, etc.)
- Nuclear Waves
- Seismic Waves

Based on their physical nature, waves can be classified into two categories:

- **Mechanical Waves:** Waves that require a material medium (solid, liquid, or gas) to propagate. These waves transfer energy through the vibrations of particles in the medium.
- **Electromagnetic Waves:** Waves that can propagate in a vacuum or through a transparent medium (air, glass, water). They consist of oscillating electric and magnetic fields, which do not need a material medium to transfer energy.

Both mechanical and electromagnetic waves follow a common equation, ensuring a uniform velocity for all frequencies.

V.2 Electromagnetic Waves

Electromagnetic waves are also known as EM waves. Electromagnetic radiations are composed of electromagnetic waves that are produced when an electric field comes in contact with the magnetic field. It can also be said that electromagnetic waves are the composition of oscillating electric and magnetic fields. Electromagnetic waves are solutions of Maxwell’s equations, which are the fundamental equations of electrodynamics.

Generally, an electric field is produced by a charged particle. A force is exerted by this electric field on other charged particles. Positive charges accelerate in the direction of the field and negative charges accelerate in a direction opposite to the direction of the field.

The Magnetic field is produced by a moving charged particle. A force is exerted by this magnetic field on other moving particles. The force on these charges is always perpendicular to the direction of their velocity and therefore only changes the direction of the velocity, not the speed.

So, the electromagnetic field is produced by an accelerating charged particle. Electromagnetic waves are nothing but electric and magnetic fields travelling through free space with the speed of light c. An accelerating charged particle is when the charged particle oscillates about an equilibrium position. If the frequency of oscillation of the charged particle is f, then it produces an electromagnetic wave with frequency f. The wavelength λ of this wave is given by λ = c/f. Electromagnetic waves transfer energy through space.

For example, the energy-carrying particle of electromagnetic light waves is the **photon**, which has no mass but possesses energy given by:

$$E = h\nu = \frac{hc}{\lambda} \tag{V.1}$$

Where:

- $\boldsymbol{\nu}$: Frequency, representing the number of cycles per unit of time, expressed in Hertz (Hz).
- $\boldsymbol{c}$: The speed of light in a vacuum, measured in meters per second (m/s).
- $\boldsymbol{\lambda}$: Wavelength, which characterizes the periodic oscillatory nature of the wave in space, expressed in angstroms (Å) or meters (m).
- $\boldsymbol{h}$: Planck's constant.

V.3 Mechanical Waves

A mechanical wave is a temporary local disturbance that propagates through an elastic, homogeneous, and isotropic material medium, carrying energy without transporting matter. Since the medium consists of multiple interacting particles, internal forces are responsible for the displacement of the disturbance. The propagation of mechanical waves depends on the physical properties of the medium in which they travel.

Mechanical waves can be:

- **Unidimensional** (e.g., a vibrating string)
- **Bidimensional** (e.g., water waves)
- **Tridimensional** (e.g., sound waves)

V.4 Wave Equation

The propagation of waves is governed by a second-order partial differential equation known as the d'Alembert equation, or simply the wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = V^2\frac{\partial^2\psi}{\partial x^2} \qquad \text{(V.2)}$$

Where:

- ψ is a physical quantity that depends on both position and time, propagating along the x-axis with a constant velocity $\boldsymbol{V}$.
- The wave velocity $\boldsymbol{V}$ remains constant in a linear, homogeneous, isotropic, and non-dispersive medium.
- It depends on the inertia, rigidity, and temperature of the medium.

- The velocity varies from one medium to another.

V.5 Solution of the Wave Equation

Given **d'Alembert's equation**:

$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2} \tag{V.3}$$

We use the method of change of variables to solve this equation by introducing new variables:

$$\begin{cases} p = t + \frac{x}{v} \\ q = t - \frac{x}{v} \end{cases} , \tag{V.4}$$

This transformation simplifies the equation into a more manageable form.

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\psi}{\partial q}\frac{\partial q}{\partial t} \tag{V.5}$$

with :

$$\begin{cases} \frac{\partial p}{\partial t} = 1 \\ \frac{\partial q}{\partial t} = 1 \end{cases} \tag{V.6}$$

We obtain:

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial p} + \frac{\partial\psi}{\partial q} \tag{V.7}$$

For the second order, we obtain:

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial\psi}{\partial t}\left[\frac{\partial\psi}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial\psi}{\partial p} + \frac{\partial\psi}{\partial q}\right] = \frac{\partial}{\partial p}\left[\frac{\partial\psi}{\partial p} + \frac{\partial\psi}{\partial q}\right]\frac{\partial p}{\partial t} + \frac{\partial}{\partial q}\left[\frac{\partial\psi}{\partial p} + \frac{\partial\psi}{\partial q}\right]\frac{\partial q}{\partial t} \tag{V.8}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial p\partial q} + \frac{\partial^2\psi}{\partial q^2} + \frac{\partial^2\psi}{\partial p\partial q} + \frac{\partial^2\psi}{\partial p^2} \tag{V.9}$$

The first term of the equation becomes:

First-order derivatives:

Using the variable change:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \psi}{\partial q}\frac{\partial q}{\partial x} \tag{V.10}$$

with :

$$\begin{cases} \dfrac{\partial p}{\partial x} = \dfrac{1}{v} \\ \dfrac{\partial q}{\partial x} = -\dfrac{1}{v} \end{cases} \tag{V.11}$$

We obtain :

$$\frac{\partial \psi}{\partial x} = \frac{1}{v}\left[\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q}\right] \tag{V.12}$$

Second-order derivatives:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial x}\left[\frac{\partial \psi}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q}\right] = \frac{\partial}{\partial p}\left[\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q}\right]\frac{\partial p}{\partial x} + \frac{\partial}{\partial q}\left[\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q}\right]\frac{\partial q}{\partial x} \tag{V.13}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial^2 \psi}{\partial p \partial q} + \frac{\partial^2 \psi}{\partial q^2} - \frac{\partial^2 \psi}{\partial p \partial q} + \frac{\partial^2 \psi}{\partial p^2} \tag{V.14}$$

We obtain:

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial p \partial q} = 0 \\ \dfrac{\partial^2 \psi}{\partial q \partial p} = 0 \end{cases} \tag{V.15}$$

$$\begin{array}{l} \dfrac{\partial \psi}{\partial p} = F'(q)\text{: Does not depend on q.} \\ \dfrac{\partial \psi}{\partial q} = F'(p)\text{: Does not depend on q.} \end{array} \quad \text{so} \quad \begin{cases} \psi = \psi_1(q) = F(q) + C_1 \\ \psi = \psi_2(p) = G(p) + C_2 \end{cases} \tag{V.16}$$

The integration constants C_1 and C_2 vanish at the point. $x = 0 \ and \ t = 0$

The total solution will take the form:

$$\psi_T = \psi_-(p) + \psi_+(q) \tag{V.17}$$

The general one-dimensional solution as a function of x and t is given by:

$$\psi_T = \psi_-(x - vt) + \psi_+(x + vt) \tag{V.18}$$

V.6 Sinusoidal Progressive Wave

V.6.1 Properties of the Functions ψ_- and ψ_+

ψ_- and ψ_+ are functions whose nature is determined by boundary conditions.

The function ψ_- takes the value $\psi_-\ (x_1 - vt_1)$ at a point x_1 and instant t_1. At a later instant $t_2 > t_1$, the function ψ_- takes the same value at a point x_2 such that:

$$x_1 - vt_1 = x_2 - vt_2 \tag{V.19}$$

$$\Rightarrow x_2 = x_1 + v(t_2 - t_1) \tag{V.20}$$

That is, $x_2 > x_1$:

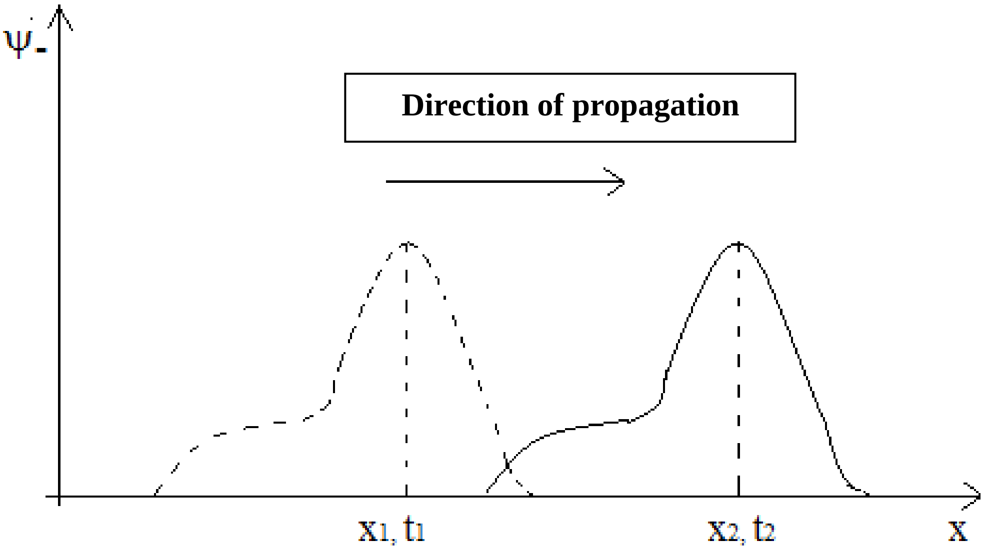


Figure V.1 Functions ψ_- as a function of x

One notice that the wave propagates without deformation in the direction of increasing x. his is the definition of a progressive wave. If in this case ψ_- is created by a vibration at x = 0, ψ_- is called the incident wave.

The same applies to the function ψ_+ which undergoes a translation from x$_1$ to x$_2$ between the instants t$_1$ and t$_2$:

$$x_1 + vt_1 = x_2 + vt_2 \tag{V.21}$$

$$\Rightarrow x_2 = x_1 - v(t_2 - t_1) \tag{V.22}$$

C'est-à-dire $x_2 < x_1$:

We observe that the wave propagates without deformation in the direction of increasing x. This corresponds to the definition of a progressive wave.

If ψ_- is generated by a vibration at x = 0, then ψ_- is called the incident wave.

Similarly, for the function ψ_+, which undergoes a translation from x$_1$ to x$_2$ between the instants t$_1$ and t$_2$ we have:

$$x_1 + vt_1 = x_2 + vt_2 \tag{V.23}$$

$$\Rightarrow x_2 = x_1 - v(t_2 - t_1) \tag{V.24}$$

That is to say $x_2 < x_1$:

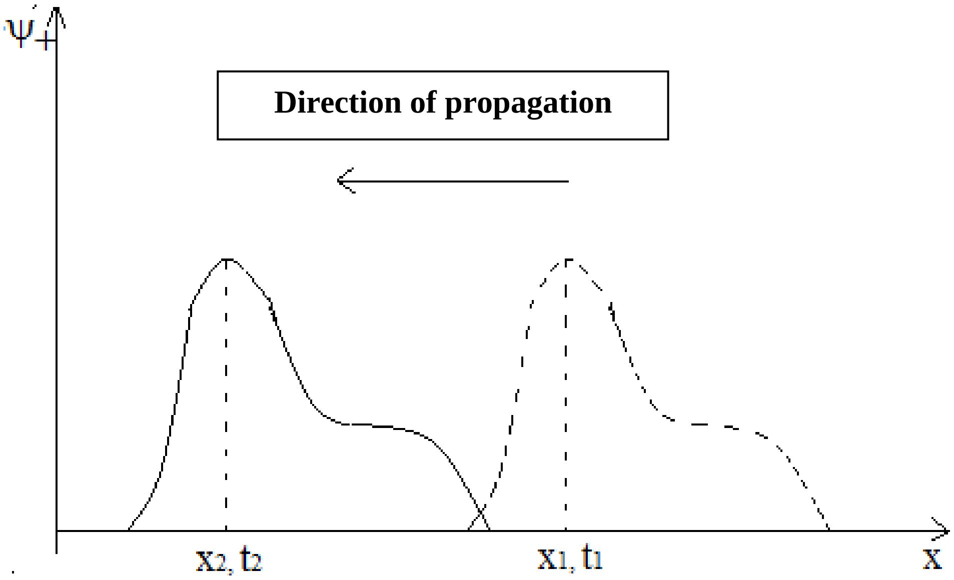


Figure V.2 Functions ψ_+ as a function of x

In this case, the propagation occurs towards decreasing x. If ψ_+ is generated by a vibration at x = L, then ψ_+ is called a reflected wave.

There are two types of signals written as:

$\psi_-\left(t-\frac{x}{v}\right)$ → Incident wave

$\psi_+\left(t+\frac{x}{v}\right)$ → Reflected wave

In the sinusoidal regime, the solution is written as:

$$\psi(x,t)_T = A\cos\left[\omega\left(t-\frac{x}{v}\right)\right] + B\cos\left[\omega\left(t+\frac{x}{v}\right)\right] \tag{V.25}$$

Thus, we obtain the sum of two types of signals, written as:

$\psi_-\left(t-\frac{x}{v}\right)$ → Incident wave

$\psi_-\left(t-\frac{x}{v}\right)$ → Reflected wave

V.7 Vibrating String

V.7.1 Tension of the String

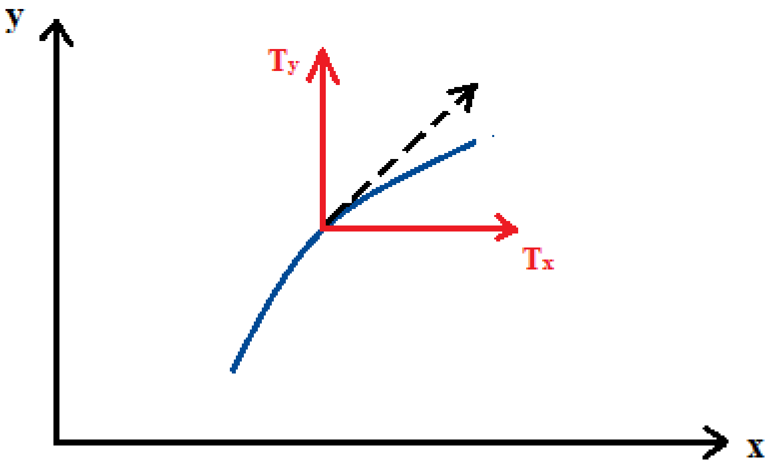


Figure V.3 Tension Y as a function of x

The string is stretched horizontally with a tension Y. Only small vertical displacements are observed, resulting in slight inclinations, even though these effects are exaggerated in the figure.

The effect of gravity is neglected.

Since there is no horizontal displacement and the angles are small, we obtain:

$$T_x = T = C^{te} \tag{V.26}$$

Since the tension is parallel to the tangent of the curve, we have: $\frac{T_y}{T_x} = \frac{d_y}{d_x}$

If y also depends on time, we must convert this into partial derivatives:

$$T_y(x) = T\frac{\partial y(x,t)}{\partial x} \tag{V.27}$$

This represents the vertical force applied by the left part on the right part at x. Conversely, the left part applies a vertical force $-Ty(x)$ on the right part.

V.7.2 Equation of Motion

By applying Newton's law to an element of the string between the positions $x \; and \; x + dx$, with a mass of μdx (where μ\muμ is the linear mass density of the string, assumed to be homogeneous), we obtain the following equation in the vertical projection:

$$\mu dx\ddot{y} = T_y(x + dx) - T_x(x) = T\left[\frac{\partial y(x+dx,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x}\right] \tag{V.28}$$

Dividing by dx, we obtain:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \; \frac{\partial^2 y(x,t)}{\partial x^2} \quad and \quad v^2 = \frac{T}{\mu} \tag{V.29}$$

This equation is the wave equation for vibrating strings.

Exercise 1

Consider a string vibrating transversely in the *Oxy* plane. The equation of motion is given by $y= (x, t)$. Let T and μ\muμ be the tension and the linear mass density of the string at equilibrium.

Write the wave propagation equation and deduce the velocity V of the oscillations.

Suppose the initial disturbance is sinusoidal. Determine the solutions of the wave equation using the method of separation of variables.

Now, the string is fixed at both ends at a distance a and is released without an initial velocity. Determine the form of the general solution.

Show that the vibration frequencies of the string are integer multiples of a fundamental frequency f_1.

Numerical application:

For the third string of a guitar, made of nylon, with a length $a=63cm$, a volumetric mass density $\rho = 1200\,\mathrm{kg/m^3}$, and a cross-sectional area $S = 0{,}42mm^2$

Calculate the tension in the string so that it produces the fundamental frequency $f_1 = 147Hz$ (corresponding to the note *D*).

Exercise 2

Let U be a longitudinal mechanical wave propagating along the *Ox* axis in a homogeneous, undeformable cylindrical rod with mass density ρ, Young's modulus E, length l, and cross-sectional area S.

Write the d'Alembert wave equation for the system.

Determine the wave solution $U(x,t)$.

Determine the mechanical impedance *Z(x)* at position *x*. Deduce the characteristic impedance Z_c of the medium.

Exercise 3

An elastic string is stretched between two points *O* and *A*, with a transverse wave propagation speed of *10 m/s*. At point *O*, a positive displacement in meters is given by: $y = -125(t - 0.02)^2 + 0.05$, where t is in seconds and $y \geq 0$ (parabolic arc).

Find the wave equation.

Write the expression of the deformation at *t=0.08 s.*

Determine the position of the crest at that moment.

Illustrate the wave.

Calculate the maximum transverse velocity of a point on the string.

Hint:

Use the function $f(t - \frac{x}{v})$; 5) $\partial f\left(t - \frac{x}{v}\right) / \partial t$;

Exercise 4

Recalculate the speed of sound in liquids. The transverse speed $v_l$and the longitudinal speed $v_l$of sound are given by:

$$v_t = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{\text{shear modulus}}{\text{density}}}$$

$$v_l = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{\text{Young's modulus}}{\text{density}}}$$

The speed of sound in solids.

Solids	Y (N m^{-2})	G (N m^{-2})	ρ (Kg/m^{-3})	v_t (m/s)	v_l (m/s)
iron	$20*10^{10}$	$8*10^{10}$	7850	3240	5200

Exercise 5

A man generates surface waves on a calm lake by rocking a boat. He observes that the boat completes 12 oscillations in 20 seconds, with each oscillation producing a wave. It takes 6 seconds for the crest of a wave to reach the shore, which is 12 meters away.

1- Calculate the wavelength of the surface waves.

Exercise 6

A gunshot is fired at a vertical rock located at a certain distance. If the average speed of the bullet is 600 m/s, the air temperature is 20°C, and the time interval between the echo and the detonation is 1.25 seconds.

1- Calculate the distance from the rock.

Exercise 7

Calculate the wavelength of a sound of frequency 5000 Hz in air at 20°C. Find the wave function that describes it if the amplitude of the displacement is 1 micron and if the latter is zero at the origin.

1- Calculate the corresponding pressure amplitude.

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