Applied Mathematical Sciences, Vol. 4, 2010, no. 66, 3251 - 3260

Spectral Properties of a New Generation of Multiplication Operators

Mohand Ould-Ali

University of Mostaganem
Department of Mathematics
27000, Algeria
mohand_ouldalidz@yahoo.fr

B. Messirdi

University of Oran Es-Senia
Department of Mathematics
31000, Algeria
messirdi.bekkai@univ-oran.dz

Abstract

In this paper, we etablish some spectral properties for a general class of multiplication type-operators containing n involutions on the Hilbert space $L^2(X,\mu)$. We construct the resolvent and we investigate the spectrum and the spectral functions of these operators.

Keywords: Involution, Multiplication-type operator, Resolvent, spectrum and spectral functions

1 Introduction

Every Hermitian matrix is unitarily equivalent to a diagonal one. The spectral theorem is regarded as the generalization of this assertion to operators on Hilbert spaces. More precisely, a version of the spectral theorem identifies a class of linear operators that can be modelled by multiplication operators, which are as simple as one can hope to find. Examples of operators to which the spectral theorem applies are selfadjoint operators or more generally normal operators on Hilbert spaces.

Then any selfadjoint operator is unitarily equivalent to a multiplication operator. This version of the spectral theorem is usually avoided, since most technical problems can be dealt with by the functional calculus [1], [5].

Although the development of the spectral theory of selfadjoint operators can be regarded as complete, there exist only a few operators for which the resolvent and spectral functions can be given in explicit form.

In this paper, we give some spectral properties for a class of multiplication-type operators $(TM)_n$, on the Hilbert space $L^2(X,\mu)$, containing n involutions. We construct the resolvent and we investigate the spectrum and the spectral functions of these operators.

An operator from this class has the form :

$$Tf = \theta f + \sum_{i=1}^{n} \eta_i U_i \ f$$

where θ and $\eta_i\,(i = 1, ..., n)$ are measurable functions defined on a measurable space (X,μ) with σ-finite measure μ and U_i are involutions on $L^2(X,\mu)$.

Particularly, if $X = \mathbb{R}^p$ and μ is the Lebesgue measure on $\mathbb{R}^p$, the operator T is unitarily equivalent to convolution type operator on $L^2(\mathbb{R}^p)$ which represents a large class of operators containing the differential and integrodifferential operators with kernels depending on the difference and sum of the arguments and other operators that appears in many fields of physics and mathematics. As an example, we refer to the dissipation of energy in the atmosphere of stars and planets which can be described by integral equations with kernels depending on the difference and sum of the arguments [3].

2 $(TM)_n$ operators

2.1 Multiplication operators and involutions

Let (X,μ) be a measurable space with σ-finite measure μ such that the Hilbert space $L^2(X,\mu)$ of square integrable function is separable. The measure μ is defined on a $\sigma-$algebra of subsets of X which is not specified here.

We use the following notation: $\langle , \rangle$ for the inner product and $\|.\|$ for the $L^2(X,\mu)$-norm; I_E is the characteristic function of a set $E \subset X$; and $\overline{\theta}$ is the complex conjugate of θ; $\mathcal{M}(X,\mu)$ is the set of measurable functions on (X,μ). T_φ is the multiplication operator in $L^2(X,\mu)$ by the function φ :

$$T_\varphi f = \varphi f \ \text{ with dense domain } D(T_\varphi) = \left\{ f \in L^2(X,\mu) \ ; \varphi f \in L^2(X,\mu) \right\}$$

By U we denote the involution operator on $L^2(X,\mu)$ $(U^2 = I,\ I$ is the operator identity). We recall the following properties of such operators [2]:

1) The involution U can be extented as an operation U on the set $\mathcal{M}(X,\mu)$ perserving the multiplication : $U(f.g) = U(f).U(g)$.

2) There exists a measurable function φ which is positive μ-almost everywhere on X and satisfies the conditions $U\varphi = \varphi^{-1}, U^* = T_\varphi U$

where U^* is the adjoint operator of U. U^* is defined on

$$D(U^*) = \left\{ f \in L^2\left(X,\mu\right)\ ;\ \varphi U f \in L^2\left(X,\mu\right) \right\}$$

3) The maximal operator $V = T_{\sqrt{\varphi}}U$ is unitary and selfadjoint on $L^2(X,\mu)$.

2.2 Adjoint of $(TM)_n$ operators

Let θ and $(\eta_i)_{1\leq i\leq n}$ be $(n+1)$ measurable functions on (X,μ) and n involutions U_i on $L^2(X,\mu)$. Consider on $L^2(X,\mu)$ the so-called $(TM)_n$ operator :

$$Tf = T_\theta f + \sum_{i=1}^{n} \eta_i \overset{\vee_i}{f},\quad \overset{\vee_i}{f} = U_i f\ ,\ i = 1,...,n$$

defined on $D(T) = \left\{ f \in L^2\left(X,\mu\right)\ ;\ \theta f + \sum_{i=1}^{n} \eta_i \overset{\vee_i}{f} \in L^2\left(X,\mu\right) \right\}$.

Accordingly to property 2) of the opertors U_i, there exists measurable positive functions φ_i such that $U_i^* = T_{\varphi_i} U_i$ for all $i = 1,...,n$; therfore the adjoint T^* of T is given by

$$T^* = T_{\overline{\theta}} + \sum_{i=1}^{n} T_{\varphi_i} \overset{\vee_i}{\overline{\eta}} U_i$$

with domain

$$D(T^*) = \left\{ f \in L^2\left(X,\mu\right)\ ;\ \overline{\theta} f + \sum_{i=1}^{n} \varphi_i \overset{\vee_i}{\overline{\eta}} \overset{\vee_i}{f} \in L^2\left(X,\mu\right) \right\}$$

where $\overset{\vee_i}{\overline{\eta}} = U_i\overline{\eta}, i = 1,...,n$. Thus, the operator T is selfadjoint if and only if the function θ is real and $\eta_i = \varphi_i \overset{\vee_i}{\overline{\eta}},\ i = 1,...,n$.

Theorem 1 *([4]) The operator T is bounded if and only if the functions θ and $\sqrt{\varphi_i}\overset{\vee_i}{\overline{\eta}_i}$ are μ-essentially bounded on X, for all $i = 1,...,n$. In this case*

$$\|T\| \leq \|\theta\|_\infty + \sum_{i=1}^{n} \left\| \sqrt{\varphi_i}\overset{\vee_i}{\overline{\eta}_i} \right\|_\infty$$

where $\|f\|_\infty = \sup_{x\in X} ess\, |f(x)|$. T is compact if and only if θ and $(\eta_i)_{1\leq i\leq n}$ vanishes μ-almost everywhere on X.

2.3 Resolvent of $(TM)_n$ operators

To illustrate the resolvent of $(TM)_n$ operators, we consider the case $n = 2$.

Let a $(TM)_2$ operator T defined on
$D(T) = \{f \in L^2(X,\mu)\ ;\ \theta f + \eta_1 U_1 f + \eta_2 U_2 f \in L^2(X,\mu)\}$ by

$$Tf = \theta f + \eta_1 U_1 f + \eta_2 U_2 f$$

where $\theta,\ \eta_1,\ \eta_2 \in L^2_{loc}(X,\mu)$ and the involutions $U_1$and U_2 commutes. We note $U_j f = \overset{\vee_j}{f},\ j = 1,2,\ U_1U_2 f = U_2U_1 f = \overset{\vee_{12}}{f}$.

If $\Delta \neq 0$ with

$$\Delta = \begin{vmatrix} (\theta - \lambda) & \eta_1 & \eta_2 & 0 \\ \overset{\vee_1}{\eta_1} & (\overset{\vee_1}{\theta} - \lambda) & 0 & \overset{\vee_1}{\eta_2} \\ \overset{\vee_2}{\eta_2} & 0 & (\overset{\vee_2}{\theta} - \lambda) & \overset{\vee_2}{\eta_1} \\ 0 & \overset{\vee_{12}}{\eta_2} & \overset{\vee_{12}}{\eta_1} & (\overset{\vee_{12}}{\theta} - \lambda) \end{vmatrix} \tag{1}$$

we obtain the first main result :

Theorem 2 *If the functions*

$$\begin{aligned} R_0(\lambda) &= \tfrac{1}{\Delta}\left[\left(\overset{\vee_1}{\theta} - \lambda\right)\left(\overset{\vee_2}{\theta} - \lambda\right)\left(\overset{\vee_{12}}{\theta} - \lambda\right) - \overset{\vee_2}{\eta_1}\overset{\vee_{12}}{\eta_1}\left(\overset{\vee_1}{\theta} - \lambda\right) - \overset{\vee_1}{\eta_2}\overset{\vee_{12}}{\eta_2}\left(\overset{\vee_2}{\theta} - \lambda\right)\right] \\ R_1(\lambda) &= \tfrac{1}{\Delta}\left[\overset{\vee_2}{\eta_1}\left(\eta_1\overset{\vee_{12}}{\eta_1} - \eta_2\overset{\vee_{12}}{\eta_2}\right) - \eta_1\left(\overset{\vee_2}{\theta} - \lambda\right)\left(\overset{\vee_{12}}{\theta} - \lambda\right)\right] \\ R_2(\lambda) &= \tfrac{1}{\Delta}\left[\overset{\vee_1}{\eta_1}\left(\eta_2\overset{\vee_{12}}{\eta_2} - \eta_1\overset{\vee_{12}}{\eta_1}\right) - \eta_2\left(\overset{\vee_1}{\theta} - \lambda\right)\left(\overset{\vee_{12}}{\theta} - \lambda\right)\right] \\ R_3(\lambda) &= \tfrac{1}{\Delta}\left[\eta_1\overset{\vee_1}{\eta_2}\left(\overset{\vee_2}{\theta} - \lambda\right) + \eta_2\overset{\vee_{12}}{\eta_2}\left(\overset{\vee_1}{\theta} - \lambda\right)\right] \end{aligned} \tag{2}$$

are μ-essentially bounded on X then the resolvent of the $(TM)_2$ operator T is an $(TM)_3$ opertor defined on $L^2(X,\mu)$ by

$$R_\lambda(T)f = R_0(\lambda)f + R_1(\lambda)\overset{\vee_1}{f} + R_2(\lambda)\overset{\vee_2}{f} + R_3(\lambda)\overset{\vee_{12}}{f} \tag{3}$$

The resolvent set of T is given by

$$\rho(T) = \left\{\lambda \in \mathbb{C}\ ;\ \sum_{i=0}^{3} |R_i(\lambda)| \in L^\infty(X,\mu)\right\}$$

Proof. By applying the operators U_1, U_2 and U_1U_2 to the booth parts of the equation $Tg- \lambda g = f,\ f \in L^2(X,\mu)$, we obtain the following system :

$$\begin{cases} (\theta-\lambda)\, g + \eta_1 \overset{\vee_1}{g} + \eta_2 \overset{\vee_2}{g} + 0 \overset{\vee_{12}}{g} = f \\ \overset{\vee_1}{\eta_1} g + (\overset{\vee_1}{\theta} - \lambda) \overset{\vee_1}{g} + 0 \overset{\vee_2}{g} + \overset{\vee_1}{\eta_2} \overset{\vee_{12}}{g} = \overset{\vee_1}{f} \\ \overset{\vee_2}{\eta_2} g + 0 \overset{\vee_1}{g} + (\overset{\vee_2}{\theta} - \lambda) \overset{\vee_2}{g} + \overset{\vee_2}{\eta_1} \overset{\vee_{12}}{g} = \overset{\vee_2}{f} \\ 0g + \overset{\vee_{12}}{\eta_2} \overset{\vee_1}{g} + \overset{\vee_{12}}{\eta_1} \overset{\vee_2}{g} + (\overset{\vee_{12}}{\theta} - \lambda) \overset{\vee_{12}}{g} = \overset{\vee_{12}}{f} \end{cases}$$

with nonvanishing determinant Δ. Thus, $g = \frac{1}{\Delta}\Delta_1 f$ where

$$\Delta_1 f = \begin{vmatrix} f & \eta_1 & \eta_2 & 0 \\ \overset{\vee_1}{f} & (\overset{\vee_1}{\theta} - \lambda) & 0 & \overset{\vee_1}{\eta_2} \\ \overset{\vee_2}{f} & 0 & (\overset{\vee_2}{\theta} - \lambda) & \overset{\vee_2}{\eta_1} \\ \overset{\vee_{12}}{f} & \overset{\vee_{12}}{\eta_2} & \overset{\vee_{12}}{\eta_1} & (\overset{\vee_{12}}{\theta} - \lambda) \end{vmatrix}$$

then

$$g = (T-\lambda)^{-1} f = R_0(\lambda) f + R_1(\lambda) \overset{\vee_1}{f} + R_2(\lambda) \overset{\vee_2}{f} + R_3(\lambda) \overset{\vee_{12}}{f}$$

where $(R_i(\lambda))_{0 \leq i \leq 3}$ are given by (2). ■

3 Resolvent of involution-type operators on $L^2(\mathbb{R}^p)$

3.1 $(TM)_n$ involution-type operators

Let $S(\mathbb{R}^p)$ stands for the Schwartz space of infinitely differentiable complex functions $\varphi(x)$ such that

$$\sup_{x\in\mathbb{R}^p} \left| x^\beta D^\alpha \varphi(x) \right| < +\infty \ \text{ for all } \alpha, \beta \in \mathbb{N}^p$$

with the topology determined by this semi-norm for any α and β from $\mathbb{N}^p$,where $\mathbb{N}^p$ is the set of multiindices $\alpha = (\alpha_1, ..., \alpha_p)\,,\ \alpha_i \in \mathbb{N}$. $x^\beta = x_1^{\beta_1}.x_2^{\beta_2}...x_p^{\beta_p}$ if $x = (x_1, ..., x_p) \in \mathbb{R}^p$, $\beta = (\beta_1, ..., \beta_p) \in \mathbb{N}^p$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_p^{\alpha_p}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_p$.

By $S'(\mathbb{R}^p)$ we denote the dual space of $S(\mathbb{R}^p)$, its elements are called tempered distributions. Let us denote by $\mathcal{F}$ the Fourier transform on $S(\mathbb{R}^p)$:

$$(\mathcal{F}\varphi)(x) = \int\limits_{\mathbb{R}^p} e^{-2i\pi xt} \varphi(t) dt\ , \quad \varphi \in S\left(\mathbb{R}^p\right)$$

extented by continuity to all $f \in S'(\mathbb{R}^p)$, or equivalently $\langle \mathcal{F}f, \varphi \rangle = \langle f, \overline{\mathcal{F}}\varphi \rangle$ where $\langle f, \varphi \rangle$ is the value of the distribution f on φ and $\overline{\mathcal{F}}\varphi = \mathcal{F}\varphi(-x)$ is the inverse Fourier transform. It is well known that the Fourier transform acts as an isomorphism on the spaces $S(\mathbb{R}^p)$ and $S'(\mathbb{R}^p)$ and it is an unitary operator on $L^2(\mathbb{R}^p)$ (see e.g.[5]).

We recall that any measurable bounded function on $\mathbb{R}^p$ is an element of $S'(\mathbb{R}^p)$. By $S'_1(\mathbb{R}^p)$ we denote the set of all distribution $f \in S'(\mathbb{R}^p)$ that have locally square integrable Fourier transfom $\mathcal{F}f$ on $\mathbb{R}^p$ or $\mathcal{F}f \in L^2_{loc}(\mathbb{R}^p)$.

Definition 3 *A $(TM)_n$ involution-type operator on $L^2(\mathbb{R}^p)$ is defined by*

$$Af = \xi * f + \sum_{i=1}^{n} K_i * \overset{\vee_i}{f} \tag{4}$$

with domain $D(A) = \{f \in L^2(\mathbb{R}^p)\ ;\ Af \in L^2(\mathbb{R}^p)\}$, where $\xi,\ K_i \in S'(\mathbb{R}^p)$, $i = 1, ..., n$ and $*$ is the convolution product on $S'(\mathbb{R}^p)$.

By using the properties of $\mathcal{F}$ we obtain our second main result :

Theorem 4 *The $(TM)_n$ involution-type operator A defined on $L^2(\mathbb{R}^p)$ by (4) is unitarily equivalent to the $(TM)_n$ operator*

$$Tf = T_\theta f + \sum_{i=1}^{n} T_{\eta_i} V_i f$$

where $\theta = \mathcal{F}\xi,\ \eta_i = \mathcal{F}K_i$ and $V_i = \mathcal{F}U_i\overline{\mathcal{F}}$, $i = 1, ..., n$. The resolvent $R_\lambda(A)$ of A is given by $R_\lambda(A)f = \overline{\mathcal{F}}R_\lambda(T) * f$ where the resolvent $R_\lambda(T)$ is given by the relation (3).

3.2 Important cases

1. If $K_i = 0$, for all $i = 1, \ldots, n$ and $\xi \in S'_1(\mathbb{R}^p)$, then $Af = \xi * f$.

   The resolvent $R_\lambda(A)$ and the spectrum $\sigma(A)$ of A are respectively given by

   $$R_\lambda(A)f = \overline{\mathcal{F}}\left[(\mathcal{F}\xi - \lambda)^{-1}\right] * f \tag{5}$$

   and

   $$\sigma(A) = \overline{\{\mathcal{F}\xi(x)\ ;\ x \in \mathbb{R}^p\}}$$

2. If $Af = \xi * f + K * Uf$ is defined by one involution U, $(n = 1)$, then A is unitarily equivalent to the operator

$$Tf = \theta f + \eta U f \tag{6}$$

with $\theta = \mathcal{F}\xi$ and $\eta = \mathcal{F}K$. We have

$$R_\lambda(A)f = \overline{\mathcal{F}}\left[R_\lambda(T)\right] * f \tag{7}$$

where

$$\left\{\begin{array}{c} R_\lambda(T)f = \left(\overset{\vee}{\theta} - \lambda\right) P^{-1}(\lambda)f - \eta P^{-1}(\lambda)\overset{\vee}{f} \\ P(\lambda) = (\theta - \lambda)\left(\overset{\vee}{\theta} - \lambda\right) - \eta\overset{\vee}{\eta},\ \overset{\vee}{\theta} = U\theta,\ \overset{\vee}{\eta} = U\eta \end{array}\right.$$

The resolvent set is then :

$$\rho(T) = \left\{\lambda \in \mathbb{C}\ ;\ \left(\overset{\vee}{\theta} - \lambda\right) P^{-1}(\lambda) \text{ and } \sqrt{\varphi}\overset{\vee}{\eta} P^{-1}(\lambda) \in L^\infty(\mathbb{R}^p)\right\} \tag{8}$$

Furthermore, if $\theta = \overline{\theta}$ and $\eta = \overline{\eta}$, then A is selfadjoint and his spectral function is given by

$$E(J)f = \overline{\mathcal{F}}(\Phi(J)) * f + \overline{\mathcal{F}}(\Psi(J)) * \overset{\vee}{f} \tag{9}$$

for all interval $J \subset \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$, where

$$\begin{array}{rcl} \Phi(J) & = & \frac{1}{2}\left[1_{E_1(J)}(1+\rho) + 1_{E_2(J)}(1-\rho)\right] \\ \Psi(J) & = & \frac{1}{2}\left[1_{E_1(J)} + 1_{E_2(J)}\right]\eta^0 \end{array}$$

with

$$\begin{array}{rcl} E_i(J) & = & \left\{x \in \mathbb{R}^p\ ;\ [\sqrt{\eta\overset{\vee}{\eta} + \theta\overset{\vee}{\theta} + \frac{1}{2}\left(\theta - \overset{\vee}{\theta}\right)} + (-1)^{1-i}\frac{1}{2}\left(\theta + \overset{\vee}{\theta}\right)](x) \in J\right\},\ i = 1, 2 \\ \rho(x) & = & \frac{1}{2}(\overset{\vee}{\theta}(x) - \theta(x)) \\ \eta^0(x) & = & \left\{\begin{array}{ll} \frac{\eta(x)}{\sqrt{|\eta(x)|^2 + |\rho(x)|^2}} & \text{if } \sqrt{|\eta(x)|^2 + |\rho(x)|^2} > 0 \\ 0 & \text{if } \eta(x) = \rho(x) = 0 \end{array}\right. \end{array}$$

4 Example

Consider in $L^2(\mathbb{R}^2)$ the $(TM)_2$ involution type-operator A defined by

$$\begin{aligned} Af(x,y) &= \frac{\partial f}{\partial x}(x,-y) + \frac{\partial f}{\partial y}(-x,y) + \frac{\partial^2 f}{\partial x \partial y} \\ &= D^{(1,0)}\delta * \overset{\vee_1}{f} + D^{(0,1)}\delta * \overset{\vee_2}{f} + D^{(1,1)}\delta * f \end{aligned} \tag{10}$$

where δ is the Dirac distribution on $\mathbb{R}^2$ concentrated at the origin $(0,0)$ and $D^{(\alpha,\beta)} = \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}$, $\alpha, \beta \in \mathbb{N}$. Let

$$\overset{\vee_1}{f}(x,y) = U_1 f(x,y) = f(x,-y) \text{ and} \overset{\vee_2}{f}(x,y) = U_2 f(x,y) = f(-x,y)$$

Then, $U_1^2 = U_2^2 = I$ and $U_1U_2 = U_2U_1$.

Consequently, A is unitarily equivalent to the $(TM)_2$ operator $Tf = \theta f + \eta_1 \overset{\vee_1}{f} + \eta_2 \overset{\vee_2}{f}$, where $\theta = \mathcal{F}\left(D^{(1,1)}\delta\right) = -4\pi^2 xy$, $\eta_1 = \mathcal{F}\left(D^{(1,0)}\delta\right) = 2i\pi x$ and $\eta_2 = \mathcal{F}\left(D^{(,0,1)}\delta\right) = 2i\pi y$. Thus,

$$Tf(x,y) = -4\pi^2 xyf(x,y) + 2i\pi xf(x,-y) + 2i\pi yf(-x,y) \tag{11}$$

By the relation (1), we have

$$\begin{aligned} \Delta &= \begin{vmatrix} (-4\pi^2 xy - \lambda) & 2i\pi x & 2i\pi y & 0 \\ 2i\pi x & (4\pi^2 xy - \lambda) & 0 & -2i\pi y \\ 2i\pi y & 0 & (4\pi^2 xy - \lambda) & -2i\pi x \\ 0 & -2i\pi y & -2i\pi x & (-4\pi^2 xy - \lambda) \end{vmatrix} \\ &= \left(4\pi^2 x^2 + \lambda\right)^4 + 16\pi^4 y^2 \neq 0 \end{aligned}$$

for all $(x,y) \in \mathbb{R}^2$ if $x \neq 0$ or $y \neq 0$. Then, the resolvent of T is given by

$$\begin{aligned} R_\lambda(T)f &= \frac{1}{\Delta}\{\left[\left(4\pi^2x^2 + \lambda\right)\left(4\pi^2x^2 + \lambda\right) - 4\pi^2\left(x^2 + y^2\right)\right] f \\ &\quad + \left[\left(4\pi^2x^2 + \lambda\right) 2i\pi y - 8i\pi^3 y\left(y^2 + x^2\right)\right] \overset{\vee_1}{f} \\ &\quad + \left[\left(-8i\pi^3x^2\left(x^2 + y^2\right) - 2i\pi x\left(4\pi^2x^2 + \lambda\right)\right)\right] \overset{\vee_2}{f} - 8\pi^2 xy\left(4\pi^2x^2 + \lambda\right) \overset{\vee_{12}}{f}\} \end{aligned}$$

Remark 5 *Generally, if $n \geqslant 2$ and T is selfadjoint operator on $L^2(\mathbb{R}^p)$ then T is unitarily equivalent to an operator defined by a $(2^n \times 2^n)$ matrix. We know that his spectral function exist and can be for example determined by Matlab resolution.*

For the precedent example we obtain the following eigenvalues :

$$\begin{aligned}\lambda_1 &= -2\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}\ ;\ \lambda_2 = 2\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}\\ \lambda_3 &= -2\sqrt{4\pi^4x^2y^2-\pi^2x^2-\pi^2y^2}\ ;\ \lambda_4 = 2\sqrt{4\pi^4x^2y^2-\pi^2x^2-\pi^2y^2}\end{aligned}$$

respectively associated to the eingenfunctions :

$$F_1(x,y) = \begin{pmatrix} 0 \\ -\frac{1}{\pi y^2-\pi x^2}(-2i\pi^2xy^2 + iy\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ -\frac{1}{\pi y^2-\pi x^2}(2i\pi^2x^2y - ix\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ 1 \end{pmatrix}$$

$$F_2(x,y) = \begin{pmatrix} 0 \\ -\frac{1}{\pi y^2-\pi x^2}(-2i\pi^2xy^2 - iy\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ -\frac{1}{\pi y^2-\pi x^2}(2i\pi^2x^2y + ix\sqrt{\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ 1 \end{pmatrix} \tag{12}$$

$$F_3(x,y) = \begin{pmatrix} -\frac{x}{y} \\ -\frac{1}{\pi y}(-2i\pi^2xy + i\sqrt{-\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ 0 \\ 1 \end{pmatrix}$$

$$F_4(x,y) = \begin{pmatrix} -\frac{x}{y} \\ -\frac{1}{\pi y}(-2i\pi^2xy - i\sqrt{-\pi^2x^2-\pi^2y^2+4\pi^4x^2y^2}) \\ 0 \\ 1 \end{pmatrix}$$

T is diagonal in the basis $\{F_1, F_2, F_3, F_4\}$ and by using the relation $A = \mathcal{F}T\overline{\mathcal{F}}$, we directly obtain the spectral function of A.

References

[1] N.I. Akhieser and I.M. Glazman. Theory of linear Operators in Hilbert-spaces. Nauka, Moscow, 1966.

[2] E.L. Alexandrov, Operators in $L^2\,(X,\mu)$-spaces and their applications [In Russian]. VINITI 14.04.97, No242 1245-B97, Saratov, 1997.

[3] E.L. Alexandrov. Spectral functions of selfadjoint and symmetric multiplication operators in $L^2(X,\mu)$-spaces. Mathematical Notes, Vol. 67, No 6, 2000.

[4] M. Djaa and B. Messirdi. Etude spectrale des opérateurs de multiplication. Maghreb Mathematical Review, Vol. 8, No 1-2, June and December 1997.

[5] M. Reed, B. Simon, " Methods of Modern Mathematical Physics". Vol. 1, Functional Analysis. Vol. 2, Fourier Analysis, Self-adjointness. Academic Press, 1978.

Received: June, 2010