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UNIVERSITÉ ABDELHAMID IBN BADIS DE MOSTAGANEM  
Faculté des Sciences Exactes et Informatique  
Département de Mathématiques et Informatique



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Abdelhamid Ibn Badis  
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THÈSE PRÉSENTÉE POUR L'OBTENTION DU DIPLÔME DOCTORAT EN SCIENCE  
EN MATHÉMATIQUES

**Option:**

Analyse Fonctionnelle

**Intitulée:**

**Étude quantitative pour des problèmes concernant  
les équations et inclusions différentielles impulsives  
fractionnaires**

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Présentée par:

**Mme Amouria HAMMOU.**

Soutenue le: 06/09/2020, devant le jury composé de:

**Président:** Pr. Saada HAMOUDA, Professeur à l'Université de Mostaganem.

**Examineur:** Pr. Mouffak BENCHOHRA, Professeur à l'Université de Sidi Bel-Abbès.

**Examineur:** Pr. Zoubir DAHMANI, Professeur à l'Université de Mostaganem.

**Examineur:** Dr. Jamal Eddine LAZREG, MCA à l'Université de Sidi Bel-Abbès.

**Examineur:** Dr. Mohammed BENHARRAT, MCA à l'École Nationale Polytechnique Maurice Audin d'Oran.

**Rapporteuse:** Pr. BELARBI née HAMANI Samira, Professeur à l'Université de Mostaganem.

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## Abstract

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Quantitative study for problems with fractional impulsive differential equations and inclusions

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This thesis deals with the quantitative study for problems with fractional impulsive differential equations and inclusions. The purpose of this thesis is to provide sufficient conditions for the existence of solutions for Cauchy problems concerning fractional impulsive differential equations and inclusions with the derivation of Hadamard and the derivation of Caputo-Hadamard . For this, we will use topological methods such as the theorems of fixed points (Banach, Schaefer and the nonlinear alternative from Leray-Schauder), and Mönch's theory combined with Kuratowski's the measure of noncompactness.

**Key words and phrases:** Fractional differential equation, fractional differential inclusion , fractional, fixed, impulsive.

# Résumé

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Une étude quantitative pour des problèmes concernant les équations et inclusions différentielles impulsives fractionnaires

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Cette thèse porte sur l'étude quantitative pour les problèmes concernant les équations et inclusions différentielles impulsives fractionnaires. Le but de cette thèse est de fournir les conditions suffisantes pour l'existence des solutions pour les problème de Cauchy concernant les équations et inclusions différentielles impulsives fractionnaires avec la dérivée d'Hadamard et la dérivée de Caputo-Hadamard. Pour cela, on va utiliser des méthodes topologiques tel que, les théorèmes du points fixes (Banach, Schaefer et L'alternative non linéaires de Leray-Schauder) , théorème de Mönch combiné avec la mesure de non compacité de Kuratowski.

**Phrases et mots clés :** Equation différentielle fractionnaire, inclusion différentielle fractionnaire, dérivée fractionnaire, point fixe, impulsive.

## المخلص

### دراسة كمية للمسائل المتعلقة بالمعادلات و الاحتواءات التفاضلية النبضية الكسرية

تتناول هذه الأطروحة دراسة كمية لحل المسائل المتعلقة بالمعادلات و الاحتواءات التفاضلية النبضية الكسرية.

الغرض من هذه الأطروحة هو توفير الشروط الكافية لوجود حلول مسائل كوشي، فيما يتعلق بالمعادلات و الاحتواءات التفاضلية النبضية الكسرية مع اشتقاق هدمارد، و كذلك سوف نستخدم اشتقاق كابتو-هدمارد.

ومن أجل كل هذا نستخدم طرق طوبولوجية، نذكر منها نظريات النقاط الثابتة (بناخ و شيفر و المتناوبة غير الخطية للوراي شودر)، و ايضا نظرية منيخ مع نظرية القياس غير متراسة لكراتوسكي.

**جُمل و كلمات مفتاحية :** المعادلة التفاضلية الكسرية، الاحتواء التفاضلي الكسري، كسور، ثابتة، نبضية.

## PUBLICATIONS

1. S. Hamani, **A. Hammou** and J.Henderson, Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative, *Comm. on appl. nonl. anal.* **24** (2017), N 3, 48-58.
2. **A. Hammou**, S. Hamani and J. Henderson, Impulsive Hadamard Fractional Differential Equations In Banach Spaces, *Pan. Math. J.* **28** (2018), No. 2, 52-62.
3. **A. Hammou**, S. Hamani and J. Henderson, Initial Value Problem For Impulsive Caputo-Hadamard Fractional Differential Inclusions, *Comm. on appl. nonl. anal.* **26** (2019), N 2, 17-35.
4. **A. Hammou** and S. Hamani, Impulsive Fractional Differential Equations Involving The Caputo-Hadamard Fractional Derivative In Banach Space, ( Submitted).

# Introduction

The fractional calculus is a theory of integrals and derivatives of arbitrary real or even complex order. It is a generalization of the classical calculus and therefore preserves many of the basic properties as an intensively developing area of the calculus during the last couple decades it offers tremendous new features for research and thus becomes more and more in use in various applications.

The beginning of the fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695, where the notation for differentiation of non-integer order  $\frac{1}{2}$  is discussed. In addition, Leibniz writes :”Thus it follows that  $d^{\frac{1}{2}}$  will be equal to  $x\sqrt{\frac{dx}{x}}$ . This is an apparent paradox from which, one day, useful consequences will be drawn” (see Miller and Ross [83], p.1) Nowadays, not only fractions but is also arbitrary real and even complex numbers are considered as order of differentiation. Nevertheless, the name ”fractional calculus” is kept for the general theory.

A lot of contributions to the theory of fractional calculus up to the middle of the 20<sup>th</sup> century, of famous mathematicians are known: Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Heaviside (1892-1912), Weyl (1917), Erdlyi (1939-1965) and many other (see Gorenflo and Mainardi[55]). However, this topic is a matter of particular interest just the last thirty years. The first specialized conference on fractional calculus and its applications in 1974 at the University of New Haven, USA, initiates the up-to-date books of Oldham and Spanier ([88], 1974) (the first monograph in the field), Samko, Kilbas, and Marichev([93], 1987), Miller and Ross ([83], 1993), Podlubny ([90], 1999), etc.

Fractional calculus (see Podlubny[90]) is a name for the theory of integrals and derivatives of arbitrary order (called fractional integrals and derivatives), which unifies and generalizes the integer-order differentiation and n-fold integration. In other words fractional derivatives and integrals can be considered as an ”interpolation” of the infinite sequence

$$\dots, \int_a^t \int_a^{\tau_1} f(\tau_2)d\tau_2d\tau_1, \int_a^t f(\tau_1)d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2f(t)}{dt^2}, \dots$$

of the classical n-fold integrals and n-fold derivatives.

The first application of a semi-derivative (derivative of order  $\frac{1}{2}$ ) is done by Abel in 1823 (see Miller and Ross [83], Oldham and Spanier[88]). This application of fractional calculus is in relation with the solution of the integral equation for the tautochrone problem. That problem deals with the determination of the shape of the curve such that the time

of descent of a frictionless point mass sliding down along the curve under the action of gravity is independent of the starting point.

The last decades prove that derivatives and integrals of arbitrary order are very convenient for describing properties, of real materials, e.g., polymers (Podlubny [90]). The new fractional order models are more satisfying than former integer-order ones. Fractional derivatives are an excellent tool for describing the memory and hereditary properties of various materials and processes while in integer-order models such effects are neglected.

The fractional calculus finds also applications in different fields of science (see Gorenflo and Mainardi[55]), theory of fractals, numerical analysis, physics, engineering, biology, economics and finance. Some problems of viscoelasticity are formulated and solved by M. Caputo (see Podlubny[90]) with his own definition of fractional differentiation. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, where for the description of the controlled system and the controller fractional differential equations are used.

However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [12, 96]. The fractional derivative that Hadamard [62] introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. Detailed descriptions of the Hadamard fractional derivative and integral can be found in [39, 40, 41]. Recently, Hadamard fractional calculus is getting attention which is an important part of theory of fractional calculus [77]. The works in [12, 39, 40, 41, 75, 79, 96] made development in fundamental theorem of Hadamard fractional calculus. A Caputo-type modification of Hadamard fractional derivative which is called Caputo-Hadamard fractional derivative was given in [73], and its fundamental theorems were proved in [54, 5, 97].

The theory of impulsive differential equations has become an important area of investigation in recent years. It has been the subject of mathematical research for almost fifty years. The first papers in this theory are related to the names of V. D. Milman and A. D. Mishkis in 1960. An impulsive system is a system of special kind consisting of a differential system and a difference system that respectively describe continuous evolutions and discrete events occurring in a mathematical model of a physical system. Many evolutionary processes are characterized by the fact that at certain moments between intervals of continuous evolutions they undergo changes of state abruptly. The durations of these changes are often negligible when compared to the total duration of the process, so that these changes can be reasonably approximated as instantaneous changes of state, or impulse. These evolutionary processes are suitably modeled as impulsive differential systems, or simply impulsive systems. Generally, an impulsive system is characterized by a pair of equations, a system of ordinary evolution equations that describes a continuous evolutionary process and a difference equation defining discrete impulsive actions. Impulsive differential equations, meanwhile, are fundamental in most branches of applied mathematics. They have applications in various fields such as physical and engineering sciences, population dynamics, theoretical physics, radiophysics, mathematical economy, chemistry, metallurgy, ecology, industrial robotics and biotechnology. This thesis is ar-

ranged as follows:

In Chapter 1, we introduce notations and some preliminary notions. In the first section of this chapter, we give some notations from the theory of Banach spaces. The second section, we give definitions concerning fractional calculus. In the third section, we recall some fixed point theory, here we give the main theorem that will be used in the following chapters. The last section is devoted Impulsive systems.

In Chapter 2, we present existence and uniqueness results of the Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative problem. We give three results, with the first result (Theorem 2.3) based on the Banach fixed point theorem, the second result (Theorem 2.4) based on Schaefer's fixed point theorem and the third result (Theorem 2.5) based on the Leray-Schauder nonlinear alternative. The last section is devoted to an example illustrating the applicability of the imposed conditions. These results be can considered as a contribution in reneging field.

In Chapter 3, we shall be concerned by Impulsive Fractional Differential Equations In Banach Space. In section 3.1, we study the problem for Hadamard Fractional Derivative, for  $r \in (0, 1]$ . In section 3.2 , we study the problem for Caputo-Hadamard Fractional Derivative for  $r \in (1, 2]$ , our approach is based on the Mönch's fixed point theorem. An example will be presented in the last illustrating the abstract theory.

In the fourth chapter, we shall be concerned by Initial Value Problems For Impulsive Caputo-Hadamard Fractional Differential Inclusions. In Section 4.2 we present an existence result when the right hand side is convex valued using the nonlinear alternative of Leray-Schauder type. In Section 4.3, we give two results for nonconvex valued right hand side. The first is based on a fixed point theorem due to Covitz and Nadler [45], and the second result is based on the nonlinear alternative of Leray-Schauder [58] for single-valued maps, combined with a selection theorem due to Bressan and Colombo [37] for lower semicontinuous multivalued maps with decomposable values. In Section 4.4, we present a result on the topological structure of the set of solutions of (4.1)-(4.4). In the section 4.5, we give an example to illustrate our main results.

# Chapter 1

## Preliminaries

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this thesis.

### 1.1 Notations and definitions

Let  $C(J, \mathbb{R})$  be the Banach space of all continuous functions from  $J$  into  $\mathbb{R}$  with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 0 \leq t \leq T\},$$

Let  $L^1(J, \mathbb{R})$  denote the Banach space of functions  $y : J \rightarrow \mathbb{R}$  that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)| dt.$$

$AC([a, b]; \mathbb{R})$  is the space of functions  $y : [a, b] \rightarrow \mathbb{R}$ , which are absolutely continuous. For  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ , we denote by  $AC^n([a, b])$  the space of real-valued functions  $f(x)$ , which have continuous derivatives up to order  $(n - 1)$  on  $[a, b]$  such that  $f^{(n-1)}(x) \in AC[a, b]$ :

$$AC^n([a, b]) = \left\{ f : [a, b] \rightarrow \mathbb{R} \text{ and } (D^{(n-1)}f)(x) \in AC[a, b] \text{ where } (D = \frac{d}{dx}) \right\},$$

$\mathbb{R}$  being the set of real numbers. In particular,  $AC^1[a, b] = AC[a, b]$ .

We also use a weighted modification of the space  $AC^n[a, b]$  ( $n \in \mathbb{N}$ ), in which the usual derivative  $D = \frac{d}{dx}$  is replaced by the so-called  $\delta$ -derivative, defined by  $\delta = xD$  ( $D = \frac{d}{dx}$ ). Such a modification, which we denote by  $AC_{\delta, \mu}^n([a, b])$  ( $n \in \mathbb{N}; \mu \in \mathbb{R}$ ), involves the real-valued Lebesgue measurable functions  $g$  on  $(a, b)$  such that  $x^\mu g(x)$  has  $\delta$ -derivatives up to order  $(n - 1)$  on  $[a, b]$  and  $\delta^{(n-1)}[x^\mu g(x)]$  is absolutely continuous on  $[a, b]$ .

$$AC_{\delta, \mu}^n[a, b] = \left\{ g : [a, b] \rightarrow \mathbb{R} : \delta^{(n-1)}[g(x)] \in AC([a, b]) \text{ where } \delta = x \frac{d}{dx} \right\}.$$

If  $\mu = 0$  and  $n = 1$ , the space  $AC_\delta^1[a, b]$  coincides with  $AC[a, b]$ . When  $a > 0$ , the space  $AC_{\delta, \mu}^n[a, b]$  is characterized by the following result [see Kilbas([75], theorem 3.1)].

**Theorem 1.1** (Fubini's theorem) *If  $f(x, y)$  is continuous over the region  $\mathbb{R}$  defined by  $a \leq x \leq b$  and,  $c \leq y \leq d$ , then*

$$\int \int_{\mathbb{R}} f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

**Theorem 1.2** (Arzela-Ascoli theorem)[63] *Let  $A$  be a subset of  $C(J; E)$ ;  $A$  is relatively compact in  $C(J; E)$  if and only if the following conditions are met:*

- (a) *The set  $A$  is bounded ie :*  
 $\exists k > 0 : \|f(x)\| \leq k, \forall x \in J$  and  $\forall f \in A$ .
- (b) *Set  $A$  is equicontinuous ie :*  
 $\forall \epsilon > 0, \exists \delta > 0 : |t_1 - t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\| \leq \epsilon$  for all  $t_1, t_2 \in J$  and all  $f \in A$ .
- (c) *For all  $x \in J$  : set  $\{f(x), f \in A\} \subset E$  is relatively compact.*

**Theorem 1.3 (Mazur)** *Let  $\{x_n\}$  be a weakly convergent sequence to  $x$  in a Banach space  $E$ . Then, there is a sequence of convex combination of elements of  $\{x_n\}$  which converges strongly to  $x$ .*

We define

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Let  $(X, \|\cdot\|)$  be a Banach space and  $Y$  be a subset of  $X$ . we denote by:

$$P(X) = \{Y \subset X : Y \neq \emptyset\},$$

$$P_c(X) = \{Y \in P(X) : Y \text{ is closed}\},$$

$$P_b(X) = \{Y \in P(X) : Y \text{ is bounded}\},$$

$$P_{cp}(X) = \{Y \in P(X) : Y \text{ is compact}\},$$

$$P_{cp,c}(X) = \{Y \in P(X) : Y \text{ is compact and convex}\}.$$

**Definition 1.4** *A multivalued map  $G : X \rightarrow P(X)$  is convex (closed) valued if  $G(x)$  is convex (closed) for all  $x \in X$ .*

**Definition 1.5** *A multivalued map  $G : X \rightarrow P(X)$  is bounded on bounded sets if  $G(B) = \cup_{x \in B} G(x)$  is bounded in  $X$  for all  $B \in P_b(X)$  (i.e.  $\sup_{x \in B} \{\sup\{|y| : y \in G(x)\}\} < \infty$ ).*

**Definition 1.6** A multivalued map  $G : X \rightarrow P(X)$  is called upper semi-continuous (u.s.c.) on  $X$  if for each  $x_0 \in X$ , the set  $G(x_0)$  is a nonempty closed subset of  $X$ , and if for each open set  $N$  of  $X$  containing  $G(x_0)$ , there exists an open neighborhood  $N_0$  of  $x_0$  such that  $G(N_0) \subseteq N$ .

**Definition 1.7** A multivalued map  $G : X \rightarrow P(X)$  is said to be completely continuous if  $G(\mathcal{B})$  is relatively compact for every  $\mathcal{B} \in P_b(X)$ .

**Definition 1.8** A multivalued map  $G : X \rightarrow P(X)$  has a fixed point if there is  $x \in X$  such that  $x \in G(x)$ .

**Remark 1.9** The fixed point set of the multivalued operator  $G$  will be denoted by  $\text{Fix}G$ .

**Definition 1.10** A multivalued map  $G : J \rightarrow P_d(\mathbb{R})$  is said to be measurable if for every  $y \in \mathbb{R}$ , the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

**Lemma 1.11** [72] If the multivalued map  $G$  is completely continuous with nonempty compact values, then  $G$  is u.s.c. if and only if  $G$  has a closed graph (i.e.  $x_n \rightarrow x_*$ ,  $y_n \rightarrow y_*$ ,  $y_n \in G(x_n)$  imply  $y_* \in G(x_*)$ ).

**Definition 1.12** A function  $F : [a, b] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is said to be Caratheódory if

- (1)  $t \rightarrow F(t, u)$  is measurable for each  $u \in \mathbb{R}$ ;
- (2)  $u \rightarrow F(t, u)$  is upper semicontinuous for almost all  $t \in [a, b]$ .

For each  $y \in C([a, b], \mathbb{R})$ , define the set of selections of  $F$  by

$$S_{F,y} = \{v \in L^1([a, b], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [a, b]\}.$$

Let  $(X, d)$  be a metric space induced from the normed space  $(X, |\cdot|)$ . Consider  $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}$  given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where  $d(A, b) = \inf_{a \in A} d(a, b)$ ,  $d(a, B) = \inf_{b \in B} d(a, b)$ . Then  $(P_{b,cl}(X), H_d)$  is a metric space and  $(P_d(X), H_d)$  is a generalized metric space (see [78]).

**Definition 1.13** A multivalued operator  $N : X \rightarrow P_d(X)$  is called

- (1)  $\gamma$ -Lipschitz if and only if there exists  $\gamma > 0$  such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X,$$

(2) a contraction if and only if it is  $\gamma$ -Lipschitz with  $\gamma < 1$ .

For convenience, we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

**Definition 1.14** ([15, 22]) *Let  $E$  be a Banach space and let  $\Omega_E$  be the family of bounded subsets of  $E$ . The Kuratowski measure of noncompactness is the map  $\alpha : \Omega_E \rightarrow [0, \infty)$  defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{j=1}^m B_j \text{ and } \text{diam}(B_j) \leq \epsilon\}, \text{ for } B \in \Omega_E.$$

**Properties:** The Kuratowski measure of noncompactness satisfies the following properties (for details, see [22],[15]).

- (1)  $\alpha(B) = 0 \Leftrightarrow \overline{B}$  is compact ( $B$  is relatively compact).
- (2)  $\alpha(B) = \alpha(\overline{B})$ .
- (3)  $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$ .
- (4)  $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ .
- (5)  $\alpha(cB) = |c|\alpha(B), c \in \mathbb{R}$ .
- (6)  $\alpha(\text{con}B) = \alpha(B)$ .

Here  $\overline{B}$  and  $\text{con}B$  denote the closure and the convex hull of the bounded set  $B$ , respectively.

## 1.2 Some properties of fractional calculus

### 1.2.1 Gamma function and Beta function

The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals

**Definition 1.15** *Gamma function is denoted and defined by the integral*

$$\Gamma(m) = \int_0^{+\infty} e^{-x} x^{m-1} dx \quad m > 0, x > 0. \quad (1.1)$$

### Properties of Gamma function

- (a)  $\Gamma(z + 1) = z\Gamma(z)$ ,  $Re(z) > 0$ .
- (b)  $\Gamma(m + 1) = m!$ ,  $\forall m \in \mathbb{N}$ .
- (c)  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

**Example 1.16** Let  $a > 1, x > 0$ .

$$\int_0^{\infty} \frac{x^a}{a^x} dx$$

We pose

$$I = \int_0^{\infty} \frac{x^a}{a^x} dx$$

and  $a^x = e^t$  we have

$$\begin{aligned} (I) &= \int_0^{\infty} \left( \frac{t}{\log a} \right)^a e^{-t} \frac{dt}{\log a} \\ &= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-t} t^a dt \\ &= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} t^{(a+1)-1} e^{-t} dt \\ &= \frac{1}{(\log a)^{a+1}} \Gamma(a + 1) \end{aligned}$$

**Definition 1.17** Beta function denoted by  $B(m, n)$  or  $B(m, n)$  is defined as

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad (m > 0, n > 0, \text{ and } x > 0). \quad (1.2)$$

### Properties of Beta function

- (a)  $B(m, n) = B(n, m)$ ,  $n > 0, m > 0$
- (b)  $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$   $\forall n, m; n > 0, m > 0$ .

## 1.2.2 Fractional integral and derivative of Riemann-Liouville

Cauchy's formula for repeated integration (Oldham and Spanier [88], p. 38 and podlubny [90], p. 64)

$$\begin{aligned} I_a^n f(t) &= \int_a^t \int_a^{\tau_1} \dots \int_a^{\tau_{n-1}} f(\tau) d\tau \dots d\tau_2 d\tau_1 \\ &= \frac{1}{(n-1)!} \int_a^t f(\tau) (t - \tau)^{n-1} d\tau, \end{aligned}$$

hold for  $n \in \mathbb{N}$ ,  $a, t \in \mathbb{R}$ ,  $t > a$  if  $n$  is substituted by a positive real number  $\alpha$  and  $(n-1)!$  by its generalization  $\Gamma(\alpha)$ , a formula for fractional integration is obtained.

**Definition 1.18** Suppose that  $\alpha > 0$ ,  $t > a$ ,  $\alpha, a, t \in \mathbb{R}$  and  $f$  is continuous Then the fractional operator

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau) (t - \tau)^{\alpha-1} d\tau, \quad (1.3)$$

is referred to as Riemann-Liouville fractional integral of order  $\alpha$ .

**Example 1.19** Let  $x > 0$ ,  $a > 0$

$$f(x) = (x - a)^\alpha, \quad x > a$$

we have

$$\begin{aligned} I_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} (t - a)^\alpha dt \end{aligned}$$

by setting

$$t = a + (x - a)v$$

and

$$dt = (x - a)dv$$

we find

$$\begin{aligned} I_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} (1 - v)^{\alpha-1} (x - a)^\alpha v^\alpha (x - a) dv \\ &= \frac{(x-a)^{2\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - v)^{\alpha-1} v^\alpha dv \\ &= \frac{(x-a)^{2\alpha}}{\Gamma(\alpha)} B(\alpha, \alpha + 1) \end{aligned}$$

( $B$  is the Beta function)

Since

$$B(r_1, r_2) = \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1 + r_2)},$$

we find

$$\begin{aligned} I_a^\alpha f(x) &= \frac{(x-a)^{2\alpha}\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(2\alpha+1)} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} (x-a)^{2\alpha}. \end{aligned}$$

**Theorem 1.20** Let  $f \in C([a, b], \mathbb{R})$ . For all  $\alpha > 0$  and  $\beta > 0$ , we have

$$I^\alpha I^\beta f = I^{\alpha+\beta} f$$

**Proof:** Let  $x > 0, t > 0$

We have

$$\begin{aligned} I_a^\beta (I_a^\alpha f(x)) &= \frac{1}{\Gamma(\beta)} \int_0^{x-a} t^{\beta-1} (I_a^\alpha f)(x-t) dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{x-a} t^{\beta-1} dt \int_a^{x-t} (x-t-u)^{\alpha-1} f(u) du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(u) du \int_0^{x-u} t^{\beta-1} (x-u-t)^{\alpha-1} dt. \end{aligned}$$

by setting

$$t = (x-u)v$$

and

$$dt = (x-u)dv$$

we find

$$\begin{aligned} I_a^\beta (I_a^\alpha f(x)) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(u) du \int_0^1 (v(x-u))^{\beta-1} (x-u-v(x-u))^{\alpha-1} (x-u) dv \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-u)^{\alpha+\beta-1} f(u) du \int_0^1 v^{\beta-1} (1-v)^{\alpha-1} dv \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-u)^{\alpha+\beta-1} f(u) du B(\beta, \alpha) \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-u)^{\alpha+\beta-1} f(u) du \\ &= I_a^{\alpha+\beta} f(x). \end{aligned}$$

□

**Definition 1.21** ([77]) Suppose that  $\alpha > 0, t > a, \alpha, a, t \in \mathbb{R}$  and  $f$  is continuous. Then

$$(D^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases} \quad (1.4)$$

is called the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order  $\alpha$

### 1.2.3 Fractional derivative of Caputo

In this subsection an alternative operator to the Riemann-Liouville operator (1.4) is considered (see Caputo [82])

**Definition 1.22** ([77]) Suppose that  $\alpha > 0, t > a, \alpha, a, t \in \mathbb{R}$  and  $f$  is continuous. The fractional operator

$$(D^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}^*, \end{cases} \quad (1.5)$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order  $\alpha$ . This operator is introduced by the Italian mathematician Caputo in 1967 ([82])

#### Example 1.23

Let  $a = 0, \alpha = \frac{1}{2}, (n = 1)$ , and  $f(t) = t$ .

Then, applying formula (1.5) we get

$${}^C D^{\frac{1}{2}} t = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau$$

Taking into account the properties of the Gamma function (1.1) and using substitution  $u := (t-\tau)^{\frac{1}{2}}$  the final result for the Caputo fractional derivative of the function  $f(t) = t$  is obtained as

$$\begin{aligned} {}^C D^{\frac{1}{2}} t &= \frac{1}{-\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d(t-\tau) \\ &= \frac{1}{-\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} du^2 \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2u}{u} du \end{aligned}$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi}}$$

Thus, it holds

$${}^C D^{\frac{1}{2}} t = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

**Lemma 1.24** ([77]) *For the Caputo fractional derivative it holds*

$${}^C D^\alpha C = 0, \quad C = \text{const}$$

## 1.2.4 Fractional integral and derivative of Hadamard

The fractional derivative due to Hadamard, introduced in 1892 ([62]), differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in ([75]). We firstly, present the definitions and some properties of the Hadamard fractional integrals and derivatives

**Definition 1.25** ([77]) *The Hadamard fractional integral of order  $r$  for a function  $h : [1, +\infty) \rightarrow \mathbb{R}$  is defined as*

$${}^H(I^r h)(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds, \quad r > 0,$$

*provided the integral exists.*

**Example 1.26** *Let  $f(x) = \left(\log \frac{x}{a}\right)^{\beta-1}$ ,  $x > 0$ ,  $a > 1$  and  $\beta \in \mathbb{R}$ .*

*We have*

$$I^\alpha \left(\log \frac{t}{a}\right)^{\beta-1} (x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \left(\log \frac{t}{a}\right)^{\beta-1} \frac{dt}{t},$$

*We put  $\tau = \frac{\log \frac{t}{a}}{\log \frac{x}{a}}$ , we find*

$$\begin{aligned} I^\alpha \left(\log \frac{t}{a}\right)^{\beta-1} (x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\frac{x}{a}\right)^{\alpha-1} (1-\tau)^{\alpha-1} \left(\log \frac{x}{a}\right)^{\beta-1} \tau^{\beta-1} \left(\log \frac{x}{a}\right) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \left(\log \frac{x}{a}\right)^{\alpha+\beta-1} B(\alpha, \beta) \\ &= \frac{1}{\Gamma(\alpha)} \left(\log \frac{x}{a}\right)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\log \frac{x}{a}\right)^{\alpha+\beta-1} \end{aligned}$$

**Definition 1.27** ([77]) For a function  $h$  given on the interval  $[1, +\infty)$ , the  $r$  Hadamard fractional-order derivative of  $h$ , is defined by

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n-r)} \left( t \frac{d}{dt} \right)^n \int_1^t \left( \log \frac{t}{s} \right)^{n-r-1} \frac{h(s)}{s} ds, \quad n-1 < r \leq n, \quad n = [r] + 1,$$

where  $[r]$  denotes the integer part of  $r$  and  $\log(\cdot) = \log_e(\cdot)$ .

**Example 1.28** Let  $f(x) = \left( \log \frac{x}{a} \right)^{\beta-1}$ ,  $x > 0, a > 1$ ,  $\beta \in \mathbb{R}$   $\beta > 0$ , and  $\delta = t \frac{d}{dt}$ , we have

$$\begin{aligned} \left( D_{a+}^\alpha \left[ \log \left( \frac{t}{a} \right)^{\beta-1} \right] \right) (x) &= \delta^n \left( I_{a+}^{n-\alpha} \left[ \log \left( \frac{t}{a} \right)^{\beta-1} \right] \right) (x) \\ &= \delta^n \left( \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} \left[ \log \left( \frac{x}{a} \right) \right]^{n-\alpha+\beta-1} \right) \\ &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} \delta^{n-1} \left( x \frac{d}{dx} \left[ \log \left( \frac{x}{a} \right) \right]^{n-\alpha+\beta-1} \right) \\ &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} \delta^{n-1} \left( x(n-\alpha+\beta-1) \frac{1}{x} \left[ \log \left( \frac{x}{a} \right) \right]^{n-\alpha+\beta-2} \right) \\ &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta-1)} \delta^{n-1} \left( \log \left( \frac{x}{a} \right) \right)^{n-\alpha+\beta-2} \\ &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta-2)} \delta^{n-2} \left( \log \left( \frac{x}{a} \right) \right)^{n-\alpha+\beta-3} \end{aligned}$$

By recurrence we have

$$\begin{aligned} \left( D_{a+}^\alpha \left( \log \left( \frac{t}{a} \right)^{\beta-1} \right) \right) (x) &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta-n)} \left[ \left( \log \frac{x}{a} \right)^{n-\alpha+\beta-(n+1)} \right] \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left( \log \frac{x}{a} \right)^{\beta-\alpha-1} \end{aligned}$$

If  $\beta = 1$  and  $\alpha \geq 0$

$$\begin{aligned} (D_{a+}^\alpha 1)(x) &= \delta^n (I_{a+}^{n-\alpha} 1)(x) \\ &= \delta^n \left( \frac{1}{\Gamma(n-\alpha)} \int_a^x \left( \log \frac{x}{t} \right)^{n-\alpha-1} 1 \frac{d}{dt} \right) \end{aligned}$$

we pose  $v = \frac{x}{t}$ , we have

$$\begin{aligned} (D_{a+}^\alpha 1)(x) &= \delta^n \left( \frac{1}{\Gamma(n-\alpha)} \int_1^{\frac{x}{a}} (\log v)^{n-\alpha-1} \frac{dv}{v} \right) \\ &= \delta^n \left( \frac{1}{\Gamma(n-\alpha)} \left[ \frac{(\log v)^{n-\alpha}}{n-\alpha} \right]_1^{\frac{x}{a}} \right) \end{aligned}$$

$$\begin{aligned}
&= \delta^n \left( \frac{1}{\Gamma(n - \alpha + 1)} (\log \frac{x}{a})^{n-\alpha} \right) \\
&= \delta^{n-1} \left( x \frac{d}{dx} \frac{1}{\Gamma(n - \alpha + 1)} (\log \frac{x}{a})^{n-\alpha} \right) \\
&= \delta^{n-1} \left( x \frac{n - \alpha}{\Gamma(n - \alpha + 1)} \frac{1}{x} (\log \frac{x}{a})^{n-\alpha-1} \right) \\
&= \delta^{n-1} \left( \frac{1}{\Gamma(n - \alpha)} (\log \frac{x}{a})^{n-\alpha-1} \right) \\
&= \delta^{n-2} \left( \frac{1}{\Gamma(n - \alpha - 1)} (\log \frac{x}{a})^{n-\alpha-2} \right)
\end{aligned}$$

By recurrence , we get

$$(D_{a^+}^\alpha 1)(x) = \frac{1}{\Gamma(n - \alpha + 1 - n)} \left( \log \frac{x}{a} \right)^{n-\alpha-n}$$

for  $(0 < \alpha < 1)$  we have

$$(D_{a^+}^\alpha 1)(x) = \frac{1}{\Gamma(-\alpha + 1)} \left( \log \frac{x}{a} \right)^{-\alpha}$$

**Remark 1.29** [93] From the fractional integral of Hadamard, we can see the following relations:

- $({}^H I^\alpha h)(e^t) = \frac{1}{\Gamma(\alpha)} \int_{e^a}^{e^t} (t - s)^{\alpha-1} h(e^s) ds = (I_{e^a}^\alpha h)(e^t)$
- $x \frac{d}{dx} ({}^H I^{\alpha+1} h)(t) = ({}^H I^\alpha h)(t)$

## 1.2.5 Fractional integral and derivative of Caputo-Hadamard

**Definition 1.30** ([54]). Let  $AC_\delta^n[a, b] = \{g : [a, b] \rightarrow \mathbb{C}, \delta^{n-1}g \in AC[a, b]\}$  where  $\delta = t \frac{d}{dt}$ ,  $0 < a < b < \infty$  and let  $\alpha \in \mathbb{C}$ , such that  $Re(\alpha) \geq 0$ . For a function  $g \in AC_\delta^n[a, b]$  the Caputo-Hadamard derivative of fractional order  $\alpha$  is defined as follows

(i) If  $\alpha \notin \mathbb{N}$ , and  $n - 1 < \alpha < n$  such that  $n = [Re(\alpha)] + 1$ , then

$$({}^{CH} D_a^\alpha g)(t) = \frac{1}{\Gamma(n - \alpha)} \left( t \frac{d}{dt} \right)^n \int_a^t \left( \log \frac{t}{s} \right)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s},$$

(ii) If  $\alpha = n \in \mathbb{N}$ , then  $({}^{CH} D_a^\alpha g)(t) = \delta^n g(t)$ ,

where in both cases,  $[Re(\alpha)]$  denotes the integer part of the real number  $Re(\alpha)$  and  $\log(\cdot) = \log_e(\cdot)$ .

**Lemma 1.31** ([54]) Let  $y \in AC_\delta^n[a, b]$  or  $\delta = t \frac{d}{dt}$  and  $\alpha \in \mathbb{C}$ . Then

$$I_a^\alpha ({}^{CH}D_a^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left( \log \frac{t}{a} \right)^k. \quad (1.6)$$

### 1.3 Some fixed point theorems

Fixed point theorems concern maps  $f$  of a set  $x$  into itself that, under certain conditions, admit a fixed point, that is, a point  $x \in X$  such that  $f(x) = x$ . The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology

**Theorem 1.32** (Covitz-Nadler)[45] Let  $(X, d)$  be a complete metric space. If  $N : X \rightarrow P_d(X)$  is a contraction, then  $\text{Fix}N \neq \emptyset$ .

**Theorem 1.33** (Nonlinear alternative of Leray Schauder) [58] Let  $X$  be a Banach space and  $C$  a nonempty convex subset of  $X$ . Let  $U$  a nonempty open subset of  $C$  with  $0 \in U$  and  $T : \bar{U} \rightarrow C$  continuous and compact operator. Then either

(a)  $T$  has fixed points. Or

(b) There exist  $u \in \partial U$  and  $\lambda \in [0, 1]$  with  $x = \lambda T(x)$ .

**Theorem 1.34** (Schaefer's theorem)[68] Let  $X$  be a Banach space and  $N : X \rightarrow X$  completely continuous operator. If the set

$$E(N) = \{x \in X : x = \lambda Nx \text{ for } \lambda \in [0, 1]\},$$

is bounded, then  $N$  has fixed points

Let us now recall the Mönch's fixed point theorem and an important lemma.

**Theorem 1.35** ([86],[8]) Let  $D$  be a bounded, closed and convex subset of a Banach space  $E$  such that  $0 \in D$ , and let  $N$  be a continuous mapping of  $D$  into itself. If the implications

$$V = \overline{c\bar{o}N(V)} \text{ or } V = N(V) \cup \{0\} \implies \alpha(V) = 0, \quad (1.7)$$

hold for every subset  $V$  of  $D$ , then  $N$  has a fixed point.

**Lemma 1.36** ([60]) If  $V \subset C(J, E)$  is a bounded and equicontinuous set, then

1. The function  $t \rightarrow \alpha(V(t))$  is continuous on  $J$ .

2.

$$\alpha \left( \left\{ \int_J x(t) dt, x \in V \right\} \right) \leq \int_J \alpha(v(t)) dt.$$

## 1.4 Impulsive systems

A particular class of hybrid systems are the impulsive systems, which present a combination of a continuous process described by an ordinary differential equation (EDO) and instantaneous jumps of the state or pulses. Impulsive differential systems have been observed naturally in several models and phenomena. For example, in the Bautin shock model, a clock mechanism, in the study of the distribution of medicines in the human body, in the control of Lotka-Volterra models, etc. The theory of impulsive differential equations was initiated by A. Mishkis and V. D. Mil'man in 1960 [84]. The development of this theory was relatively slow because of the difficulty of handling such equations, V. Lakshmikantham, L. Byszewski, D. Bainov and S. Simeonov and other researchers participated in the enrichment and popularization of the theory of impulsive differential equations from 1991 ([21],[20],[18],[19])

### 1.4.1 Description of impulsive systems

An impulsive system is defined by

**Definition 1.37** (i) a differential equation

$$y'(t) = f(t, y(t)) \quad (1.8)$$

Where  $f : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^n$ ,  $\Omega \subset \mathbb{R}^n$  is an open and  $n \in \mathbb{N}^*$

(ii)  $M, N : \mathbb{R}^+ \times \Omega \rightarrow P(\Omega)$ , are applications.

(iii)  $A : M \rightarrow N$  is a family of applications.

We denote  $y(t, t_0, y_0)$  is the solution of the system  $y'(t) = f(t, y(t))$  with of the initial condition  $(t_0, y_0)$ .

**Definition 1.38** A function  $y : \mathbb{R}^+ \rightarrow \Omega$  is said to be a solution of impulsive system if there exists a function  $y_I(t, t_0, y_0)$  such that  $y_I(t_0, t_0, y_0) = y_0$  and  $y_I(t, t_0, y_0) = f(t, y_I(t, t_0, y_0))$  as long as the point  $y(t, t_0, y_0)$  does not belong to the set  $M(t, (t, t_0, y_0))$ .

If the trajectory  $y(t, t_0, y_0)$  intersects the set  $M(t, x)$  at time  $t = t_1$ , the application  $A(t_1, P_{t_1})$  transfers the point  $P_{t_1} = y(t_1, t_0, y_0)$  (belonging to the set  $M(t_1, P_{t_1})$ ) at a point  $P_{t_1^+} = A(t_1, P_{t_1})P_{t_1}$  of the set  $N(t_1, P_{t_1})$ .

For the time  $t > t_1$ ,  $y_I(t, t_0, y_0)$  is solution of  $y'(t) = f(t, y(t))$  with the initial condition  $(t_1, P_{t_1^+})$ . We define  $y_I(t, t_0, y_0) = y(t, t_1, P_{t_1^+})$  for  $t > t_1$  and this as long as  $y_I(t, t_0, y_0)$  does not belong to the set  $M(t, y_I(t, t_0, y_0))$ . If there is a time  $t_2 > t_1$ , such that  $P_{t_2} \in M(t_2, P_{t_2})$ , the point  $P_{t_2}$  will be transferred to  $P_{t_2^+} \in N(t_2, P_{t_2})$  by the application  $A$ . Thus, the evolution process continues to advance as long as the system solution (1.8) exists.

**Remark 1.39** Let  $t_1, t_2, \dots, t_k, \dots$  be the moments during which  $P_t$  meets the set  $M(t, x)$ . These moments are called instants of pulses. Moreover  $\lim_{k/t_0 \rightarrow \infty} t_k = T < +\infty$

we will describe below two particular types of impulsive systems.

### 1.4.2 Systems with fixed pulse times

For this first type of system, we give ourselves a strictly increasing sequence  $t_1 < t_2 < \dots < t_k < \dots$  tending towards infinity, instants of impulses. The sets  $M(t, y)$  and  $N(t, y)$  are

$$M(t, y) = N(t, y) = \emptyset, \quad t \neq t_k, \quad k = 1, 2, \dots$$

$$M(t, y) = N(t, y) = \Omega, \quad t = t_k, \quad k = 1, 2, \dots$$

The applications  $A(t, y)$  are only defined for instants  $t = t_k$  by the suite of applications  $(A_k)_{k \geq 0}$  defined by

$$A_k; \Omega \rightarrow \Omega$$

$$y \rightarrow A_k(y) = y + I_k(y),$$

Where  $I_k; \Omega \rightarrow \Omega$  So with this choice of  $M, N$  and  $A$ , we can describe an impulsive differential system whose pulses occur at times fixed by the next mathematical model

$$y'(t) = f(t, y(t)), \quad t \neq t_k, \quad k = 1, 2, \dots$$

$$\Delta y(t) = I_k(y(t)), \quad t = t_k, \quad k = 1, 2, \dots$$

$$y(t_0^+) = y_0$$

Where  $\Delta y(t_k) = y(t_k^+) - y(t_k)$

### 1.4.3 Systems with variable pulse times

These systems constitute a generalization systems with instants of fixed pulses. For this type of impulsive systems, We gives a countable infinity of functions  $\tau_k : \Omega \rightarrow \mathbb{R}^+$  such that  $\tau_k(y) < \tau_{k+1}(y)$  and  $\lim_{k \rightarrow +\infty} \tau_k(y) = +\infty$  for  $y \in \Omega$  The sets  $M(t, y)$  and  $N(t, y)$  are

$$M(t, y) = N(t, y) = \emptyset, \quad t \neq \tau_k(y), \quad k = 1, 2, \dots$$

$$M(t, y) = N(t, y) = \Omega, \quad t = \tau_k(y), \quad k = 1, 2, \dots$$

and the applications  $A(t, y)$  are only defined for the instants  $\tau = \tau_k(y)$ . So such a system is defined by

$$y'(t) = f(t, y(t)), \quad t \neq \tau_k(y), \quad k = 1, 2, \dots$$

$$\Delta y(t) = I_k(y(t)), \quad t = \tau_k(y), \quad k = 1, 2, \dots$$

Since the instants of pulses depend on the state  $y$  of the solution, this type of systems presents more difficulties than fixed pulse time systems.

### 1.4.4 Application of impulsive systems

Nowadays, impulsive systems have become more and more important in some real processes and phenomena studied in physics, in pathology [42], in technology [26], in population dynamics [63, 64], in biotechnology, especially in networks of biological neurons [36] and economics [32]. In recent years, there has been an important development in the theory of impulsive differential equations with fixed moments, see works [9], [62] and [70]. In what follows, we will present an example of application of impulsive systems in medicine. The following system represents a mathematical model describing the evolution of a heterogeneous tumor, under periodic chemotherapeutic treatment with drugs instantaneous effects represented by pulses

$$x' = r_1(x, y)x \quad (1.9)$$

$$y' = r_2(x, y)y \quad (1.10)$$

$$x(t_n^+) = \eta_n(D, x(t_n), y(t_n)) \quad (1.11)$$

$$y(t_n^+) = \theta_n(D, x(t_n), y(t_n)) \quad (1.12)$$

Where

$D$  is the dose of the drug administered.

$x, y$  are respectively the biomass of the sensitive cells and the resistant cells.

$r_1(x, y)$  and  $r_2(x, y)$  are respectively the growth rates of the sensitive cells and the resistant cells.

The values  $\eta_n(D, x(t_n), y(t_n))$  and  $\theta_n(D, x(t_n), y(t_n))$  are respectively the biomass of sensitive cells and resistant cells that survive after the  $n$ th  $D$  dose of the drug administered at the moment  $t_n$ .

The sequence  $(t_n)$  is strictly increasing.

The system (1.9 -1.12) represents the case where several drugs are administered one after another, in order, with a certain period  $T$ : drug 1 is administered at time  $t_1$ , the drug 2 at time  $t_2$  and so on until the last drug  $n$ .

If we consider the case of two drugs then the period is  $T = t_2 - t_1$ .

The tumor consists of sensitive cells and resistant cells, the evolution of their biomass is equal to the biomass of the sensitive cells  $x(t_1)$  and the biomass of the resistant cells  $y(t_1)$ .

When a dose  $D$  of drug  $A$  is administered at time  $t$  the biomass of the tumor becomes

$$x(t_1^+) - y(t_1^+) = \eta_1(D, x(t_1), y(t_1)) - \theta_1(D, x(t_1), y(t_1))$$

The drug removes only a small fraction of the resistant cell biomass. To reduce a significant biomass of resistant cells, a  $D$  dose of drug  $B$  at time  $t = t_2$ ;  $t_2 > t_1$  and periodically take the same process by drug  $A$  until eradication of the tumor.

### 1.4.5 The space of continuous functions by pieces

Solutions of an impulsive differential equation are usually continuous functions by pieces. So the appropriate functional space for such solutions is space continuous functions by

pieces. Let

$$J_0 = [0, t_1]; J_k = (t_k, t_{k+1}], k = 1, \dots, m, m \in \mathbb{N}^*.$$

Let  $y$  be a continuous function. We note by  $y(t_k^-)$  and  $y(t_k^+)$  the limits on the left and right of  $y$  at  $t = t_k$ . Then for all  $T \geq 1 + [t_1]$ .

We introduce the space

$$PC([0, T], X) = \left\{ \begin{array}{l} y : [0, T] \rightarrow X, y \in C(J_k \cap [0, T], X), k = 1, \dots, m, \text{ and there exist} \\ y(t_k^+) \text{ and, } y(t_k^-), k = 1, \dots, m, \text{ with } y(t_k^-) = y(t_k) \end{array} \right\}.$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup\{\|y(t)\| : 0 \leq t \leq T\}.$$

# Chapter 2

## Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative

### 2.1 Introduction

For  $0 < r \leq 1$ , this chapter deals with the existence of solutions of initial value problems (IVP for short), for the impulsive fractional order differential equation,

$${}^H D^r y(t) = f(t, y(t)), \text{ for almost each } t \in J = [1, T], \quad (2.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (2.2)$$

$$y(1) = 0, \quad (2.3)$$

where  ${}^H D^r$  is the Hadamard fractional derivative,  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function,  $I_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k = 1, \dots, m$ ,  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ ,  $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ , or  $y(t_k^+) = \lim_{h \rightarrow 0^+} y(t_k + h)$  and  $y(t_k^-) = \lim_{h \rightarrow 0^-} y(t_k + h)$  represent the right and left limits of  $y(t)$  at  $t = t_k$ ,  $k = 1, \dots, m$ .

The results obtained in this chapter are presented in <sup>1</sup>.

### 2.2 Existence of solutions

Consider the set of functions

$$PC(J, \mathbb{R}) = \left\{ \begin{array}{l} y : J \rightarrow \mathbb{R}, y \in C((t_k, t_{k+1}], \mathbb{R}), k = 1, \dots, m, \text{ and there exist} \\ y(t_k^+) \text{ and } y(t_k^-), k = 1, \dots, m, \text{ with } y(t_k^-) = y(t_k) \end{array} \right\}.$$

This set is a Banach space with the norm

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<sup>1</sup>S. Hamani, **A. Hammou** and J.Henderson, Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative, *Commu on appl nonl anal.* **24** (2017), no 3, 48-58.

$$\|y\|_{PC} = \sup\{|y(t)| : 1 \leq t \leq T\}.$$

And let

$$PC'(J, \mathbb{R}) = \left\{ \begin{array}{l} y : J \rightarrow \mathbb{R}, y \in AC((t_k, t_{k+1}], \mathbb{R}), k = 1, \dots, m, \text{ and there exist} \\ y(t_k^+) \text{ and, } y(t_k^-), k = 1, \dots, m, \text{ with } y(t_k^-) = y(t_k) \end{array} \right\}.$$

This set is a Banach space with the norm

$$\|y\|_{PC'} = \sup\{|y(t)| : 1 \leq t \leq T\}.$$

Set

$$J' := J \setminus \{t_1, \dots, t_m\}.$$

**Definition 2.1** A function  $y \in PC(J, \mathbb{R}) \cap PC'(J, \mathbb{R})$  is said to be a solution of (2.1)-(2.3), if  $y$  satisfies  ${}^H D^r y(t) = f(t, y(t))$  on  $J'$  and satisfies conditions (2.2)-(2.3).

To prove the existence of solutions to (2.1)-(2.3), we need the following auxiliary lemma.

**Lemma 2.2** Let  $0 < r \leq 1$  and let  $\rho \in C(J', \mathbb{R}) \cap AC(J', \mathbb{R})$ . A function  $y$  is a solution of the fractional integral equation

$$y(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}, \quad \text{if } t \in [1, t_1] \\ \left(\frac{\log(t-t_k)}{\log(t_k)}\right)^{r-1} \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} \rho(s) \frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k)) \right] \\ + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}, \quad \text{if } t \in (t_k, t_{k+1}], k=1, \dots, m \end{array} \right\}, \quad (2.4)$$

if and only if  $y$  is a solution of the impulsive fractional IVP

$${}^H D^r y(t) = \rho(t), \quad \text{for each } t \in J', \quad (2.5)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (2.6)$$

$$y(1) = 0. \quad (2.7)$$

**Proof:** Applying the Hadamard fractional integral of order  $r$  to both sides of  ${}^H D^r y(t) = \rho(t)$ , for a.e.  $t \in J = [1, T]$ ,  $0 < r \leq 1$ , we have

$$y(t) = c(\log t)^{r-1} + {}^H I^r \rho(t). \quad (2.8)$$

If  $t \in [1, t_1]$ , then from  $y(1) = 0$ , we have  $c = 0$  and

$$y(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.$$

If  $t \in (t_1, t_2]$ , then from (2.6) and (2.8), we have

$$\begin{aligned}
y(t) &= c(\log(t - t_1))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s} \\
&= \frac{y(t_1^+)}{(\log t_1)^{r-1}} \times (\log(t - t_1))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s} \\
&= \frac{\Delta y|_{t=t_1} + y(t_1^-)}{(\log t_1)^{r-1}} \times (\log(t - t_1))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s} \\
&= \frac{1}{(\log t_1)^{r-1}} \left[ I_1(y(t_1^-)) + \frac{1}{\Gamma(r)} \int_1^{t_1} (\log \frac{t_1}{s})^{r-1} \rho(s) \frac{ds}{s} \right] \times (\log(t - t_1))^{r-1} \\
&+ \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned}$$

Finally

$$\begin{aligned}
y(t) &= \left( \frac{\log(t - t_1)}{\log t_1} \right)^{r-1} \left[ \frac{1}{\Gamma(r)} \int_1^{t_1} (\log \frac{t_1}{s})^{r-1} \rho(s) \frac{ds}{s} + I_1(y(t_1^-)) \right] \\
&+ \frac{1}{\Gamma(r)} \int_{t_1}^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned} \tag{2.9}$$

If  $t \in (t_2, t_3]$ , then from (2.6) and (2.8), we have

$$\begin{aligned}
y(t) &= c(\log(t - t_2))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s} \\
&= \frac{y(t_2^+)}{(\log t_2)^{r-1}} \times (\log(t - t_2))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s} \\
&= \left[ \frac{y(t_2^-) + I_2(y(t_2^-))}{(\log t_2)^{r-1}} \right] \times (\log(t - t_2))^{r-1} + \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned}$$

Finally

$$\begin{aligned}
y(t) &= \left( \frac{\log(t - t_2)}{\log t_2} \right)^{r-1} \left[ \frac{1}{\Gamma(r)} \left( \int_1^{t_1} (\log \frac{t_1}{s})^{r-1} \rho(s) \frac{ds}{s} \right. \right. \\
&+ \left. \int_{t_1}^{t_2} (\log \frac{t_2}{s})^{r-1} \rho(s) \frac{ds}{s} \right) + I_1(y(t_1^-)) + I_2(y(t_2^-)) \left. \right] \\
&+ \frac{1}{\Gamma(r)} \int_{t_2}^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned} \tag{2.10}$$

If  $t \in (t_k, t_{k+1}]$ , then again from (2.8), we get (2.4).

Conversely, assume that  $y$  satisfies the impulsive fractional integral equation (2.4). If  $t \in [1, t_1]$ , then  $y(1) = 0$  and using that  ${}^H D^r$  is the left inverse of  $I^r$ , we get  ${}^H D^r y(t) = \rho(t)$ , for a.e.  $t \in [1, t_1]$ , while if  $t \in (t_k, t_{k+1}]$ ,  $k = 1, \dots, m$ , we get  ${}^H D^r y(t) = \rho(t)$ , for a.e.  $t \in [t_k, t_{k+1}]$ . Also, we can easily show that  $\Delta y|_{t=t_k} = I_k(y(t_k^-))$ ,  $k = 1, \dots, m$ .  $\square$

Our first result is based on the Banach fixed point theorem.

**Theorem 2.3** *Assume the following conditions hold:*

(H1) *There exists a constant  $l > 0$  such that*

$$|f(t, u) - f(t, \bar{u})| \leq l|u - \bar{u}| \text{ for each } t \in J \text{ and } u, \bar{u} \in \mathbb{R}.$$

(H2) *There exists a constant  $l^* > 0$  such that*

$$|I_k(u) - I_k(\bar{u})| \leq l^*|u - \bar{u}| \text{ for each } u, \bar{u} \in \mathbb{R} \text{ and } k = 1, \dots, m.$$

If

$$\left[ \frac{l(m+1)(\log T)^r}{\Gamma(r+1)} + ml^* \right] < 1, \quad (2.11)$$

then (2.1)-(2.3) has a unique solution on  $J$ .

**Proof:** We transform the problem (2.1)-(2.3) into a fixed point problem. Consider the operator  $F : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$  defined by

$$\begin{aligned} F(y)(t) &= \left( \frac{\log(t - t_k)}{\log(t_k)} \right)^{r-1} \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} f(s, y(s)) \frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k^-)) \right] \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} f(s, y(s)) \frac{ds}{s}. \end{aligned}$$

Clearly, the fixed points of the operator  $F$  are solutions of the problem (2.1)-(2.3). We shall use the Banach contraction principle to prove that  $F$  has a fixed point. That is, we shall show  $F$  is a contraction.

Let  $x, y \in PC(J, \mathbb{R})$ . Then, for each  $t \in J$  we have

$$\begin{aligned}
|F(x)(t) - F(y)(t)| &\leq \left( \frac{\log(t - t_k)}{\log t_k} \right)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{\alpha-1} |f(s, x(s)) \right. \\
&\quad \left. - f(s, y(s)) \right| \frac{ds}{s} \\
&\quad + \sum_{k=1}^m |I_k(x(t_k^-) - I_k(y(t_k^-)))| + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (\log \frac{t}{s})^{\alpha-1} |f(s, x(s)) \\
&\quad \left. - f(s, y(s)) \right| \frac{ds}{s} \\
&\leq \left( \frac{\log(t - t_k)}{\log t_k} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{\alpha-1} l |x(s) - y(s)| \frac{ds}{s} \\
&\quad + \sum_{k=1}^m l^* |x(t_k^-) - y(t_k^-)| + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (\log \frac{t}{s})^{\alpha-1} l |x(s) - y(s)| \frac{ds}{s} \\
&\leq \frac{l \|x - y\|_\infty}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{\alpha-1} \frac{ds}{s} + \sum_{k=1}^m l^* \|x - y\|_\infty \\
&\quad + \frac{l \|x - y\|_\infty}{\Gamma(\alpha)} \int_{t_k}^t (\log \frac{t}{s})^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{lm(\log T)^\alpha}{\Gamma(\alpha + 1)} \|x - y\|_\infty + ml^* \|x - y\|_\infty + \frac{l(\log T)^\alpha}{\Gamma(\alpha + 1)} \|x - y\|_\infty \\
&\leq \left[ \frac{l(m + 1)(\log T)^\alpha}{\Gamma(\alpha + 1)} + ml^* \right] \|x - y\|_\infty.
\end{aligned}$$

Thus

$$\|F(x) - F(y)\|_\infty \leq \left[ \frac{l(m + 1)(\log T)^\alpha}{\Gamma(\alpha + 1)} + ml^* \right] \|x - y\|_\infty.$$

Consequently by (2.11),  $F$  is a contraction. As a consequence of the Banach fixed point theorem, we deduce that  $F$  has a unique fixed point which is the solution of the problem (2.1)-(2.3).  $\square$

Our second result is based on Schaefer's fixed point theorem.

**Theorem 2.4** *Assume the following conditions hold:*

(H3) *The function  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.*

(H4) *There exists a constant  $M > 0$  such that  $|f(t, u)| \leq M$  for each  $t \in J$  and each  $u \in \mathbb{R}$ .*

(H5) *The functions  $I_k : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and there exists a constant  $M^* > 0$  such that  $|I_k(u)| \leq M^*$  for each  $u \in \mathbb{R}$ ,  $k = 1, \dots, m$ .*

*Then (2.1)-(2.3) has at least one solution on  $J$ .*

**Proof:** We shall use Schaefer's fixed point theorem to prove that  $F$  has a fixed point. The proof will be given in several steps.

**Step 1:**  $F$  is continuous.

Let  $y_n$  be a sequence such that  $y_n \rightarrow y$  in  $PC(J, \mathbb{R})$ . Then for each  $t \in J$ ,

$$\begin{aligned} |F(y_n)(t) - F(y)(t)| &\leq \left( \frac{\log(t - t_k)}{\log(t_k)} \right)^{r-1} \times \\ &\quad \left( \frac{1}{\Gamma(r)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} |f(s, y_n(s)) - f(s, y(s))| \frac{ds}{s} \right. \\ &\quad \left. + \sum_{k=1}^n |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \right) \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} |f(s, y_n(s)) - f(s, y(s))| \frac{ds}{s}. \end{aligned}$$

Since  $f$  and  $I_k$ ,  $k = 1, \dots, m$ , are continuous functions, we have

$$\|F(y_n) - F(y)\|_{\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 2:**  $F$  maps bounded sets into bounded sets in  $PC(J, \mathbb{R})$ .

Indeed, it is enough to show that for any  $\eta^*$ , there exists a positive constant  $l$  such that for each  $y \in B_{\eta^*} = \{y \in PC(J, \mathbb{R}) : \|y\|_{\infty} \leq \eta^*\}$ , we have  $\|F(y)\|_{\infty} \leq l$ . By (H4) and (H5), we have for each  $t \in J$ ,

$$\begin{aligned} |F(y)(t)| &\leq \left| \left( \frac{\log(t - t_k)}{\log(t_k)} \right)^{r-1} \right| \times \\ &\quad \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} |f(s, y(s))| \frac{ds}{s} + \sum_{k=1}^n |I_k(y(t_k^-))| \right] \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} |f(s, y(s))| \frac{ds}{s} \\ &\leq \left[ \frac{Mm(\log T)^r}{\Gamma(r+1)} + \frac{M(\log T)^r}{\Gamma(r+1)} + mM^* \right] \\ &\leq \left[ \frac{M(m+1)(\log T)^r}{\Gamma(r+1)} + mM^* \right] := l. \end{aligned}$$

**Step 3:**  $F$  maps bounded sets into equicontinuous sets of  $PC(J, \mathbb{R})$ .

Let  $\lambda_1, \lambda_2 \in J$ ,  $\lambda_1 < \lambda_2$ ,  $B_{\eta^*}$  be a bounded set of  $PC(J, \mathbb{R})$  as in Step 2, and let  $y \in B_{\eta^*}$ . Then

$$|F(y)(\lambda_2) - F(y)(\lambda_1)| = \frac{1}{\Gamma(r)} \int_0^{\lambda_1} \left| (\log \frac{\lambda_2}{s})^{r-1} - (\log \frac{\lambda_1}{s})^{r-1} \right| |f(s, y(s))| \frac{ds}{s}$$

$$\begin{aligned}
& + \sum_{0 < t_k < \lambda_2 - \lambda_1} |I_k(y(t_k^-))| \\
& + \frac{1}{\Gamma(r)} \int_{\lambda_1}^{\lambda_2} \left(\log \frac{\lambda_2}{s}\right)^{\alpha-1} |f(s, y(s))| \frac{ds}{s} \\
\leq & \frac{M}{\Gamma(r+1)} \left[ 2\left(\log \frac{\lambda_2}{\lambda_1}\right)^r + (\log \lambda_2)^r - (\log \lambda_1)^r \right] \\
& + \sum_{0 < t_k < \lambda_2 - \lambda_1} |I_k(y(t_k^-))|.
\end{aligned}$$

As  $\lambda_1 \rightarrow \lambda_2$ . The right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that  $F : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$  is completely continuous.

**Step 4: A priori bounds.**

Now it remains to show that the set

$$\epsilon = \{y \in PC(J, \mathbb{R}) : y = \lambda F(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let  $y \in \epsilon$ , then  $y = \lambda F(y)$  for some  $0 < \lambda < 1$ . Thus, for each  $t \in J$  we have

$$\begin{aligned}
y(t) & = \left(\frac{\log(t - t_k)}{\log(t_k)}\right)^{r-1} \left[ \frac{\lambda}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s} + \sum_{k=1}^m |I_k(y(t_k^-))| \right. \\
& \left. + \frac{\lambda}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s} \right].
\end{aligned}$$

Then, by (H4) and (H5) (as in Step 2), it follows that, for each  $t \in J$ , we have

$$|y(t)| \leq \left[ \frac{Mm(\log T)^r}{\Gamma(r+1)} + \frac{M(\log T)^r}{\Gamma(r+1)} + mM^* \right].$$

This for every  $t \in J$ , we have

$$\|y\|_\infty \leq \left[ \frac{Mm(\log T)^r}{\Gamma(r+1)} + \frac{M(\log T)^r}{\Gamma(r+1)} + mM^* \right] := R.$$

This shows that the set  $\epsilon$  is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that  $F$  has a fixed point which is a solution of the problem (2.1)-(2.3).  $\square$

In the following theorem we give an existence result for the problem (2.1)-(2.3) by applying the nonlinear alternative of Leray-Schauder type and for which the conditions (H4) and (H5) are weakened.

**Theorem 2.5** Assume that (H2) and the following conditions hold:

(H6) There exist  $\phi_f \in C(J, \mathbb{R}^+)$  and  $\psi : [0, \infty) \rightarrow (0, \infty)$  continuous and non-decreasing such that

$$|f(t, u)| \leq \phi_f(t)\psi(|u|) \text{ for all } t \in J, u \in \mathbb{R}.$$

(H7) There exists  $\psi^* : [0, \infty) \rightarrow (0, \infty)$  continuous and nondecreasing such that

$$|I_k(u)| \leq \psi^*(|u|) \text{ for all } u \in \mathbb{R}, k = 1, \dots, m.$$

(H8) There exists a number  $\bar{M} > 0$  such that

$$\frac{\bar{M}}{\psi(\bar{M}) \frac{m\phi_f^0(\log T)^\alpha}{\Gamma(\alpha+1)} + \psi(\bar{M}) \frac{\phi_f^0(\log T)^\alpha}{\Gamma(\alpha+1)} + m\psi^*(\bar{M})} > 1,$$

where  $\phi_f^0 = \sup\{\phi_f(t) : t \in J\}$ .

Then(2.1)-(2.3) has at least one solution on  $J$ .

**Proof:** Consider the operator  $F$  defined in Theorems 2.3 and 2.4. It can be easily shown that  $F$  is continuous and completely continuous. For  $\lambda \in [0, 1]$ , let  $y$  be such that, for each  $t \in J$ , we have  $y(t) = \lambda(Fy)(t)$ . Then from(H6)-(H7), we have for each  $t \in J$ ,

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} \phi_f(s) \psi(|y(s)|) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} \phi_f(s) \psi(|y(s)|) \frac{ds}{s} + \sum_{0 < t_k < t} \psi^*(|y(s)|) \\ &\leq \psi(\|y\|_\infty) \frac{m\phi_f^0(\log T)^r}{\Gamma(r+1)} + \psi(\|y\|_\infty) \frac{\phi_f^0(\log T)^r}{\Gamma(r+1)} + m\psi^*(\|y\|_\infty). \end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty) \frac{m\phi_f^0(\log T)^r}{\Gamma(r+1)} + \psi(\|y\|_\infty) \frac{\phi_f^0(\log T)^r}{\Gamma(r+1)} + m\psi^*(\|y\|_\infty)} \leq 1.$$

Then by condition (H8), there exists  $\bar{M}$  such that  $\|y\|_\infty \neq \bar{M}$ . Let

$$U = \{y \in PC(J, \mathbb{R}) : \|y\|_\infty \langle \bar{M} \rangle\}.$$

The operator  $F : \bar{U} \rightarrow PC(J, \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y = \lambda F(y)$  for some  $\lambda \in (0, 1)$ . As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that  $F$  has a fixed point  $y \in \bar{U}$  which is a solution of the problem (2.1)-(2.3). This completes the proof.  $\square$

## 2.3 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the impulsive fractional initial-value problem,

$${}^H D^r y(t) = \frac{e^{-t}|y(t)|}{(9 + e^t)(1 + |y(t)|)}, \text{ for a.e. } t \in J = [1, e], t \neq \frac{3}{2}, \quad 0 < r \leq 1, \quad (2.12)$$

$$\Delta y \Big|_{t=\frac{3}{2}} = \frac{\left| y\left(\frac{3}{2}\right)^- \right|}{3 + \left| y\left(\frac{3}{2}\right)^- \right|}, \quad (2.13)$$

$$y(1) = 0. \quad (2.14)$$

Set  $f(t, x) = \frac{e^{-t}x}{(9 + e^t)(1 + x)}$ ,  $(t, x) \in J \times [0, \infty)$ , and  $I_k(x) = \frac{x}{3 + x}$ ,  $x \in [0, \infty)$ . Let  $x, y \in [0, \infty)$  and  $t \in J$ . Then we have

$$\begin{aligned} |f(t, x) - f(t, y)| &= \frac{e^{-t}}{(9 + e^t)} \left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \\ &= \frac{e^{-t}|x - y|}{(9 + e^t)(1 + x)} \\ &\leq \frac{e^{-t}|x - y|}{(9 + e^t)} \\ &\leq \frac{1}{10}|x - y|. \end{aligned}$$

Hence the condition (H1) holds with  $l = \frac{1}{10}$ .

Let  $x, y \in [0, \infty)$ . Then we have

$$|I_k(x) - I_k(y)| = \left| \frac{x}{3 + x} - \frac{y}{3 + y} \right| = \frac{3|x - y|}{(3 + x)(3 + y)} \leq \frac{1}{3}|x - y|.$$

Hence the condition (H2) holds with  $l^* = \frac{1}{3}$ . We shall check that condition (11) is satisfied with  $T = e$  and  $m = 1$ . Indeed,

$$\left[ \frac{l(m + 1)(\log T)^\alpha}{\Gamma(r + 1)} + ml^* \right] < 1 \text{ iff } \Gamma(r + 1) > \frac{3}{10}, \quad (2.15)$$

which is satisfied for some  $r \in (0, 1]$ . Then by Theorem 2.3, the problem (2.12)-(2.14) has a unique solution on  $[1, e]$  for values of  $r$  satisfying (2.15).

# Chapter 3

## Impulsive Fractional Differential Equations In Banach Spaces

### 3.1 Impulsive Fractional Differential Equations Involving Hadamard Fractional Derivative

In this section, the obtained results are presented in <sup>1</sup>

#### 3.1.1 Introduction

This section with the existence of solutions of initial value problems (IVP for short), for the impulsive fractional order differential equation,

$${}^H D^r y(t) = f(t, y(t)), \text{ for almost each } t \in J := [1, T], \quad 0 < r \leq 1, \quad (3.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.2)$$

$$y(1) = 0, \quad (3.3)$$

where  ${}^H D^r$  is the Hadamard fractional derivative,  $f : J \times E \rightarrow E$  is a continuous function,  $(E, \|\cdot\|_E)$  is a Banach space,  $I_k : E \rightarrow E$ ,  $k = 1, \dots, m$ ,  $1 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$ ,  $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ ,  $y(t_k^+) = \lim_{h \rightarrow 0^+} y(t_k + h)$  and  $y(t_k^-) = \lim_{h \rightarrow 0^-} y(t_k + h)$  represent the right and left limits of  $y(t)$  at  $t = t_k$ ,  $k = 1, \dots, m$ .

#### 3.1.2 Existence of solutions

Consider the set of functions

$$PC(J, E) = \left\{ \begin{array}{l} y : J \rightarrow E, y \in C((t_k, t_{k+1}], E), k = 1, \dots, m, \text{ and there exist} \\ y(t_k^+) \text{ and, } y(t_k^-), k = 1, \dots, m, \text{ with } y(t_k^-) = y(t_k). \end{array} \right\}$$

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<sup>1</sup>**A. Hammou**, S. Hamani and J. Henderson, Impulsive Hadamard fractional differential equations in Banach spaces, *PanAmer. Math. J.* **28** (2018), No. 2, 52-62.

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup\{\|y(t)\|_E : 1 \leq t \leq T\}.$$

And let

$$PC'(J, E) = \left\{ \begin{array}{l} y : J \rightarrow E, y \in AC((t_k, t_{k+1}], E), k = 1, \dots, m, \text{ and there exist} \\ y(t_k^+) \text{ and } y(t_k^-), k = 1, \dots, m, \text{ with, } y(t_k^-) = y(t_k). \end{array} \right\}$$

This set is a Banach space with the norm

$$\|y\|_{PC'} = \sup\{\|y(t)\|_E : 1 \leq t \leq T\}$$

Set

$$J' := J \setminus \{t_1, \dots, t_m\}.$$

**Definition 3.1** A function  $y \in PC(J, E) \cap PC'(J, E)$  is said to be a solution of (3.1)-(3.3), if  $y$  satisfies  ${}^H D^r y(t) = f(t, y(t))$  on  $J'$  and satisfies conditions (3.2)-(3.3).

To prove the existence of a solution to (3.1)-(3.3), we need the following auxiliary lemma.

**Lemma 3.2** Let  $0 < r \leq 1$  and let  $\rho \in C(J', E) \cap AC(J', E)$ . A function  $y$  is a solution of the fractional integral equation

$$y(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}, \quad \text{if } t \in [1, t_1], \\ \left( \frac{\log(t-t_k)}{\log(t_k)} \right)^{r-1} \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} \rho(s) \frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k)) \right] \\ + \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} \rho(s) \frac{ds}{s}, \quad \text{if } t \in (t_k, t_{k+1}], k = 1, \dots, m \end{array} \right\}. \quad (3.4)$$

if and only if  $y$  is a solution of the impulsive fractional IVP

$${}^H D^r y(t) = \rho(t), \quad \text{for each } t \in J', \quad (3.5)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.6)$$

$$y(1) = 0. \quad (3.7)$$

The result of this chapter is based on the theorem of fixed point of Monch combined with the measure of non-compactness of Kuratowski

**Theorem 3.3** Assume the following hypotheses hold.

(H'1) The function  $f : J \times E \rightarrow E$  satisfies the Carathéodory conditions.

(H'2) There exists  $p \in L^\infty(J, \mathbb{R}_+)$  such that

$$\|f(t, y)\| \leq p(t)\|y\| \quad \text{for a.e. } t \in J \text{ and each } y \in E.$$

(H'3) There exists a constant  $k^* > 0$  such that

$$\|I_k(y)\| \leq k^* \|y\| \quad \text{for each } y \in E.$$

(H'4) For each bounded set  $B \subset E$ , we have

$$\alpha(f(t, B)) \leq p(t)\alpha(B).$$

(H'5) For each bounded set  $B \subset E$ , we have

$$\alpha(I_k(B)) \leq k^* \alpha(B), \quad k = 1, \dots, m.$$

Then the IVP (3.1)-(3.3) has at least one solution in  $C(J, E)$ , provided that

$$\left[ \frac{(m+1)(\log T)^r \|p\|_{L^\infty}}{\Gamma(r+1)} + mk^* \right] < 1. \quad (3.8)$$

where  $p^* = \sup_{t \in J} p(t)$ .

**Proof:** Transform the problem (3.1)-(3.3) into a fixed point problem. Consider the operator

$$Ny(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} f(s, y(s)) \frac{ds}{s}, \quad \text{if } t \in [1, t_1] \\ \left( \frac{\log(t-t_k)}{\log(t_k)} \right)^{r-1} \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} f(s, y(s)) \frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k)) \right] \\ + \frac{1}{\Gamma(r)} \int_{t_k}^t (\log \frac{t}{s})^{r-1} f(s, y(s)) \frac{ds}{s}, \quad \text{if } t \in (t_k, t_{k+1}], k=1, \dots, m \end{array} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of  $N$  are solutions to (3.1)-(3.3). Let  $R > 0$  and consider the set

$$D_R = \{y \in C(J, E) : \|y\|_\infty \leq R\}.$$

We shall show that  $N$  satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

**Step 1:**  $N$  is continuous.

Let  $\{y_n\}$  be a sequence such that  $y_n \rightarrow y$  in  $C(J, E)$ . Then, for each  $t \in J$ ,

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \left| \frac{\log(t-t_k)}{\log(t_k)} \right|^{r-1} \left| \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (\log \frac{t_k}{s})^{r-1} \|f(s, y_n(s)) \right. \\ &\quad \left. - f(s, y(s))\| \frac{ds}{s} \right. \\ &\quad + \sum_{k=1}^m \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\| + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| (\log \frac{t}{s})^{r-1} \right| \times \\ &\quad \|f(s, y_n(s)) - f(s, y(s))\| \frac{ds}{s}. \end{aligned}$$

Let  $\rho > 0$  be such that

$$\|y_n\|_\infty \leq \rho \text{ and } \|y\|_\infty \leq \rho.$$

By (H'2)-(H'3) we have

$$\|f(s, y_n(s)) - f(s, y(s))\| \leq 2\rho p(s) := \sigma(s); \sigma \in L^1(J, \mathbb{R}_+).$$

Since  $f$  and  $I_k, k = 1, \dots, m$ , are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 2:**  $N$  maps  $D_R$  into itself.

For each  $y \in D_R$ , by (H'2), (H'3) and (3.6), we have for each  $t \in J$ ,

$$\begin{aligned} \|(Ny)(t)\| &\leq \left| \left( \frac{\log(t - t_k)}{\log(t_k)} \right)^{r-1} \right| \left[ \frac{1}{\Gamma(r)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right)^{r-1} \right| \|f(s, y(s))\| \frac{ds}{s} \right. \\ &\quad \left. + \sum_{k=1}^m \|I_k(y(t_k^-))\| \right] \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right)^{r-1} \right| \|f(s, y(s))\| \frac{ds}{s} \\ &\leq \left[ \frac{(m+1)(\log T)^r \|p\|_{L^\infty}}{\Gamma(r+1)} + mk^* \right] \leq R. \end{aligned}$$

**Step 3:**  $N(D_R)$  is bounded and equicontinuous.

By Step 2, it is obvious that  $N(D_R) \subset C(J, E)$  is bounded. For the equicontinuity of  $N(D_R)$ , let  $\lambda_1, \lambda_2 \in J$ ,  $\lambda_1 < \lambda_2$ , and  $y \in D_R$ . We have

$$\begin{aligned} \|(Ny)(\lambda_2) - (Ny)(\lambda_1)\| &= \left\| \frac{1}{\Gamma(r)} \int_0^{\lambda_1} \left[ \left( \log \frac{\lambda_2}{s} \right)^{r-1} - \left( \log \frac{\lambda_1}{s} \right)^{r-1} \right] f(s, y(s)) \frac{ds}{s} \right. \\ &\quad \left. + \sum_{0 < t_k < (\lambda_2 - \lambda_1)} I_k(y(t_k^-)) + \frac{1}{\Gamma(r)} \int_{\lambda_1}^{\lambda_2} \left( \log \frac{\lambda_2}{s} \right)^{r-1} f(s, y(s)) \frac{ds}{s} \right\| \\ &\leq \frac{\|p\|_{L^\infty} \|y\|}{\Gamma(r+1)} \left[ 2 \left( \log \frac{\lambda_2}{\lambda_1} \right)^r + \left( \log \lambda_2 \right)^r - \left( \log \lambda_1 \right)^r \right] \\ &\quad + \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \|I_k(y(t_k^-))\| \end{aligned}$$

As  $\lambda_1 \rightarrow \lambda_2$ , the right-hand side of the above inequality tends to zero.

Now let  $V$  be a subset of  $D_R$  such that  $V \subset \overline{\text{co}}(N(V) \cup \{0\})$ .  $V$  is bounded and equicontinuous, and therefore the function  $t \rightarrow \vartheta(t) := \alpha(V(t))$  is continuous on  $J$ . By

(H'2)-(H'3), Lemma 1.36, and the properties of the measure  $\alpha$  we have for each  $t \in J$

$$\begin{aligned} \vartheta(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \int_1^t \frac{(m+1)(\log T)^r p(s) \alpha(V(s)) ds}{\Gamma(r+1)} + mk^* \alpha(V(t)) \\ &\leq \|\vartheta\|_\infty \left[ \frac{(m+1)(\log T)^r \|p\|_{L^\infty}}{\Gamma(r+1)} + mk^* \right]. \end{aligned}$$

This means that

$$\|\vartheta\|_\infty \left[ 1 - \left( \frac{(m+1)(\log T)^r \|p\|_{L^\infty}}{\Gamma(r+1)} + mk^* \right) \right] \leq 0.$$

By (3.8) it follows that  $\|\vartheta\|_\infty = 0$ , that is,  $\vartheta = 0$  for each  $t \in J$ , and then  $V(t)$  is relatively compact in  $E$ . In view of the Ascoli-Arzelà theorem,  $V$  is relatively compact in  $D_R$ . Applying now Theorem 1.35, we conclude that  $N$  has a fixed point which is a solution of the problem (3.1)-(3.3).  $\square$

### 3.1.3 An Example

let

$$E = l^1 = \{(y_1, y_2, \dots, y_n, \dots), \sum_{i=1}^{+\infty} |y_n| < \infty\}$$

be our Banach space with the norm

$$\|y\|_E = \sum_{i=1}^{+\infty} |y_n|$$

We apply the main result of the chapter (Theorem 3.3) to the following system of fractional differential equations

$${}^H D^r y(t) = \frac{1}{9 + e^t} |y_n(t)|, \text{ for a.e. } t \in J = [1, e], t \neq 3/2, 0 < r \leq 1, \quad (3.9)$$

$$\Delta y|_{t=3/2} = \frac{1}{3 + |y_n(3/2)^-|}, \quad (3.10)$$

$$y(1) = 0. \quad (3.11)$$

Where

$$f_n(t, x) = \frac{x_n}{9 + e^t}, \quad (t, x) \in J \times E,$$

and

$$I_k(x) = \frac{1}{3 + x_n}, \quad x \in E.$$

$$y = (y_1, y_2, \dots, y_n, \dots)$$

Set

$$f = (f_1, f_2, \dots, f_n, \dots)$$

Clearly, condition (H'1) holds, and conditions (H'2) and (H'4) hold with

$$p(t) = \frac{1}{9 + e^t}.$$

And hence the conditions (H'3) and (H'5) hold with  $k^* = \frac{1}{3}$ . We shall check that condition (3.8) is satisfied with  $T = e$  and  $m = 1$ . Indeed,

$$\left[ \frac{(m+1)(\log T)^r \|p\|_{L^1}}{\Gamma(r+1)} + mk^* \right] < 1, \quad \text{if } \Gamma(r+1) > \frac{1}{3}, \quad (3.12)$$

which is satisfied for some  $r \in (0, 1]$ . Then by Theorem 3.3, the problem (3.9)-(3.11) has a solution on  $[1, e]$  for values of  $r$  satisfying (3.12).

## 3.2 Impulsive Fractional Differential Equations Involving The Caputo-Hadamard Fractional Derivative

In this section, the most results are taken from <sup>2</sup>

### 3.2.1 Introduction

This section with the existence of solutions for the initial value problems (IVP for short), for fractional order differential equations

$${}^{CH}D^r y(t) = f(t, y(t)), \text{ for a.e., } t \in J = [a, T], a > 0, t \neq t_k, k = 1, \dots, m, 1 < r \leq 2, \quad (3.13)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.14)$$

$$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.15)$$

$$y(a) = y_1, y'(a) = y_2 \quad (3.16)$$

where  ${}^{CH}D^r$  is the Caputo-Hadamard fractional derivative,  $F : J \times E \rightarrow E$  is a function,  $I_k$  and  $\bar{I}_k : E \rightarrow E$ ,  $k = 1, \dots, m$  are functions,  $a = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ ,  $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ ,  $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$ ,  $y(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} y(t_k + \varepsilon)$  and  $y(t_k^-) = \lim_{\varepsilon \rightarrow 0^-} y(t_k + \varepsilon)$  represent the right and left limits of  $y$  at  $t = t_k$ ,  $k = 1, \dots, m$  and  $E$  is a Banach space.

### 3.2.2 Existence of solutions

**Definition 3.4** A function  $y \in PC(J, E) \cap PC'(J, E)$  with its  $\alpha$ - derivative exists on  $J'$  is said to be a solution of (3.13)-(3.16) if satisfies the differential equation  ${}^{CH}D^r y(t) = f(t, y(t))$  on  $J'$ , and conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$

$$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m,$$

$$y(a) = y_1, y'(a) = y_2$$

are satisfied.

To prove the existence of a solution to (3.13)-(3.16), we need the following auxiliary lemma.

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<sup>2</sup>**A. Hammou** and S. Hamani, Impulsive Fractional Differential Equations Involving The Caputo-Hadamard Fractional Derivative In Banach Space,(submitted)

**Lemma 3.5** Let  $1 < r \leq 2$  and let  $\rho \in AC(J', \mathbb{R})$ . A function  $y$  is a solution of the fractional integral equation

$$y(t) = \begin{cases} y_1 + ay_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}, & \text{if } t \in [a, t_1] \\ y_1 + ay_2 \log\left(\frac{t}{a}\right) + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\ + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\ + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)), & \text{if } t \in (t_k, t_{k+1}], k = 1, \dots, m, \end{cases} \quad (3.17)$$

if and only if  $y$  is a solution of the fractional IVP

$${}^{cH}D_a^r y(t) = \rho(t), \quad \text{for each, } t \in J', \quad (3.18)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.19)$$

$$\Delta' y|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.20)$$

$$y(a) = y_1, y'(a) = y_2 \quad (3.21)$$

**Proof** Let  $y$  be a solution of (3.18)-(3.21). Applying the Hadamard fractional integral of order  $r$  to both sides of (3.18), using conditions (3.19)- (3.21) and Lemma(1.31) we get  
If  $t \in [a, t_1]$

$$y(t) = c_1 + c_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.$$

Hence  $c_1 = y(a) = y_1$  and  $c_2 = y'(a) = y_2$

$$y(t) = y_1 + ay_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.$$

If  $t \in (t_1, t_2]$

$$y(t) = c_1 + c_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}. \quad (3.22)$$

We have

$$\Delta y|_{t=t_1} = y(t_1^+) - y(t_1^-)$$

and

$$\begin{aligned} I_1(y(t_1^-)) &= c_1 + c_2 \log\left(\frac{t_1}{a}\right) \\ &- \left( y_1 + ay_2 \log\left(\frac{t_1}{a}\right) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \right) \end{aligned}$$

Hence

$$c_1 + c_2 \log\left(\frac{t_1}{a}\right) = y_1 + ay_2 \log\left(\frac{t_1}{a}\right) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s} + I_1(y(t_1^-))$$

We have

$$\Delta y'|_{t=t_1} = y'(t_1^+) - y'(t_1^-)$$

and

$$\begin{aligned} \bar{I}_1(y(t_1^-)) &= \frac{c_2}{t_1} - \left(\frac{a}{t_1}\right) y_2 \\ &+ \frac{1}{t_1 \Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \end{aligned}$$

Hence

$$c_2 = ay_2 + \frac{1}{t_1 \Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \rho(s) \frac{ds}{s} + \bar{I}_1(y(t_1^-)) \quad (3.23)$$

and

$$\begin{aligned} c_1 &= y_1 - \frac{\log\left(\frac{t_1}{a}\right)}{t_1 \Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s} + I_1(y(t_1^-)) - t_1 \log\left(\frac{t_1}{a}\right) \bar{I}_1(y(t_1^-)) \end{aligned} \quad (3.24)$$

Then by (3.23)-(3.24) and (3.22), we have

$$\begin{aligned} y(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) + \frac{\log\left(\frac{t}{t_1}\right)}{t_1 \Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} + I_1(y(t_1^-)) + t_1 \log\left(\frac{t}{t_1}\right) \bar{I}_1(y(t_1^-)). \end{aligned}$$

If  $t \in (t_k, t_{k+1}]$ , then again from Lemma (1.31) we obtain(3.17). Conversely, assume that  $y$  satisfies the impulsive fractional integral equation (3.17). If  $t \in [a, t_1]$ , then  $y(a) = y_1, y'(a) = y_2$  and using that  ${}^{CH}D_a^r$  is the left inverse of  $I_a^r$ , we get

$${}^{CH}D_a^r y(t) = \rho(t), \text{ for all } t \in [a, t_1].$$

Let  $t \in (t_k, t_{k+1}]$ ,  $k = 1, \dots, m$ . We have  ${}^{CH}D_a^r C = 0$ , for any constant  $C$ , then

$${}^{CH}D_a^r y(t) = \rho(t), \text{ for all } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

$$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m.$$

□

**Theorem 3.6** *Suppose that (H'1)-(H'5) and the following hypotheses are verified.*

(H'6) *There exists a constant  $k_1^* > 0$  such that*

$$\|\bar{I}_k(y)\| \leq k_1^* \|y\| \text{ for each } y \in E.$$

(H'7) *For each bounded set  $B \subset E$ , we have*

$$\alpha(\bar{I}_k(B)) \leq k_1^* \alpha(B), \quad k = 1, \dots, m.$$

*Then the IVP (3.13)-(3.16) has at least one solution in  $C(J, E)$ , provided that*

$$\frac{pm(\log T)^r}{\Gamma(r)} + \frac{p(\log T)^r}{\Gamma(r+1)} + mk^* + mTk_1^* \log T < \frac{1}{2}. \quad (3.25)$$

where

$$p = \operatorname{ess\,supp}_{t \in J}(t).$$

**Proof:** Transform the problem (3.13)-(3.16) into a fixed point problem. Consider the operator

$$Ny(t) = \begin{cases} y_1 + ay_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s}, & \text{if } t \in [a, t_1] \\ y_1 + ay_2 \log\left(\frac{t}{a}\right) + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} f(s, y(s)) \frac{ds}{s} \\ + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s} \\ + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)), & \text{if } t \in (t_k, t_{k+1}], k = 1, \dots, m, \end{cases} \quad (3.26)$$

Clearly, from Lemma (3.5), the fixed points of  $N$  are solutions to (3.13)-(3.16).

Let  $R > 0$  with  $|y_1| + |ay_2| \log T < \frac{R}{2}$  and consider the set

$$D_R = \{y \in C(J, E) : \|y\|_\infty \leq R\}.$$

We shall show that  $N$  satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

**Step 1:**  $N$  is continuous.

Let  $\{y_n\}$  be a sequence such that  $y_n \rightarrow y$  in  $C(J, E)$ . Then, for each  $t \in J$ ,

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \left| \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \right| \int_{t_{k-1}}^{t_k} \left| \left(\log \frac{t_k}{s}\right) \right|^{r-2} \\ &\quad \|f(s, y_n(s)) - f(s, y(s))\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left(\log \frac{t}{s}\right) \right|^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| \frac{ds}{s} \\ &+ \sum_{k=1}^m \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\| \\ &+ \sum_{k=1}^m \left| t_k \log\left(\frac{t}{t_k}\right) \right| \|\bar{I}_k(y_n(t_k^-)) - \bar{I}_k(y(t_k^-))\| \end{aligned}$$

Let  $\rho > 0$  be such that

$$\|y_n\|_\infty \leq \rho \text{ and } \|y\|_\infty \leq \rho.$$

By (H'2)-(H'3) we have

$$\|f(s, y_n(s)) - f(s, y(s))\| \leq 2\rho p(s) := \sigma(s); \sigma \in L^1(J, \mathbb{R}_+).$$

Since  $f$ ,  $I_k$  and  $\bar{I}_k$ ,  $k = 1, \dots, m$ , are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 2:**  $N$  maps  $D_R$  into itself.

For each  $y \in D_R$ , by (H'2), (H'3) and (3.21), we have for each  $t \in J$ ,

$$\|(Ny)(t)\| \leq |y_1| + |ay_2| \left| \log\left(\frac{t}{a}\right) \right| + \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \left| \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \right| \int_{t_{k-1}}^{t_k} \left| \left(\log \frac{t_k}{s}\right) \right|^{r-2} \|f(s, y(s))\| \frac{ds}{s}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} \|f(s, y(s))\| \frac{ds}{s} + \sum_{k=1}^m \|I_k(y(t_k^-))\| \\
& + \sum_{k=1}^m \left| t_k \log \left( \frac{t}{t_k} \right) \right| \|\bar{I}_k(y(t_k^-))\| \\
\leq & |y_1| + |ay_2| \log T + \frac{m \log T}{\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} p \|y\| \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} p \|y\| \frac{ds}{s} + mk^* \|y\| + mT \log Tk_1^* \|y\| \\
\leq & |y_1| + |ay_2| \log T + \frac{pm(\log T)^r}{\Gamma(r)} \|y\| + \frac{p(\log T)^r}{r\Gamma(r)} \|y\| + mk^* \|y\| + mT \log Tk_1^* \|y\| \\
\leq & R.
\end{aligned}$$

**Step 3:**  $N(D_R)$  is bounded and equicontinuous.

By Step 2, it is obvious that  $N(D_R) \subset C(J, E)$  is bounded.

For the equicontinuity of  $N(D_R)$ , let  $\lambda_1, \lambda_2 \in J$ ,  $\lambda_1 < \lambda_2$ , and  $y \in D_R$ . We have

$$\begin{aligned}
\|(Ny)(\lambda_2) - (Ny)(\lambda_1)\| &= \left| ay_2 \log \left( \frac{\lambda_2}{\lambda_1} \right) \right| \\
&+ \frac{1}{\Gamma(r-1)} \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \left| \frac{\log \left( \frac{\lambda_2}{t_k} \right)}{t_k} \right| \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \|f(s, y(s))\| \frac{ds}{s} \\
&+ \frac{\left| \log \left( \frac{\lambda_2}{\lambda_1} \right) \right|}{\Gamma(r-1)} \sum_{0 < t_k < \lambda_1} \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \|f(s, y(s))\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} \|f(s, y(s))\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{\lambda_1} \left[ \left( \log \frac{\lambda_2}{s} \right)^{r-1} - \left( \log \frac{\lambda_1}{s} \right)^{r-1} \right] \|f(s, y(s))\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{\lambda_1}^{\lambda_2} \left( \log \frac{\lambda_2}{s} \right)^{r-1} \|f(s, y(s))\| \frac{ds}{s} \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \|I_k(y(t_k^-))\| \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \left| t_k \log \left( \frac{\lambda_2}{t_k} \right) \right| \|\bar{I}_k(y(t_k^-))\| \\
&+ \log \left( \frac{\lambda_2}{\lambda_1} \right) \sum_{0 < t_k < \lambda_1} |t_k| \|\bar{I}_k(y(t_k^-))\|
\end{aligned}$$

then,

$$\begin{aligned}
\|(Ny)(\lambda_2) - (Ny)(\lambda_1)\| &\leq \left| ay_2 \log \left( \frac{\lambda_2}{\lambda_1} \right) \right| \\
&+ \frac{1}{\Gamma(r-1)} \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \left| \frac{p \log \left( \frac{\lambda_2}{t_k} \right)}{t_k} \right| \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \frac{ds}{s} \\
&+ \frac{p \left| \log \left( \frac{\lambda_2}{\lambda_1} \right) \right|}{\Gamma(r-1)} \sum_{0 < t_k < \lambda_1} \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left| p \left( \log \frac{t}{s} \right) \right|^{r-1} \frac{ds}{s} \\
&+ \frac{p}{\Gamma(r)} \int_{t_k}^{\lambda_1} \left[ \left( \log \frac{\lambda_2}{s} \right)^{r-1} - \left( \log \frac{\lambda_1}{s} \right)^{r-1} \right] \frac{ds}{s} \\
&+ \frac{p}{\Gamma(r)} \int_{\lambda_1}^{\lambda_2} \left( \log \frac{\lambda_2}{s} \right)^{r-1} \frac{ds}{s} \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \|I_k(y(t_k^-))\| \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} |t_k \log \left( \frac{\lambda_2}{t_k} \right)| \|\bar{I}_k(y(t_k^-))\| \\
&+ \log \left( \frac{\lambda_2}{\lambda_1} \right) \sum_{0 < t_k < \lambda_1} |t_k| \|\bar{I}_k(y(t_k^-))\|
\end{aligned}$$

As  $\lambda_1 \rightarrow \lambda_2$ , the right-hand side of the above inequality tends to zero.

Now let  $V$  be a subset of  $D_R$  such that  $V \subset \bar{c}o(N(V) \cup \{0\})$ .  $V$  is bounded and equicontinuous, and therefore the function  $t \rightarrow \vartheta(t) := \alpha(V(t))$  is continuous on  $J$ . By (H'2)-(H'6), Lemma 1.36, and the properties of the measure  $\alpha$  we have for each  $t \in J$

$$\begin{aligned}
\vartheta(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \int_a^t \left( \frac{p(s)m(\log T)^r \alpha(V(s)) ds}{\Gamma(r)} + \frac{p(s)(\log T)^r \alpha(V(s)) ds}{r\Gamma(r)} \right. \\
&\quad \left. + mk^* \alpha(V(t)) + mT \log T k_1^* \alpha(V(t)) \right) \\
&\leq \|\vartheta\|_\infty \left[ \frac{\|p\|_{L^\infty} m (\log T)^r}{\Gamma(r)} + \frac{\|p\|_{L^\infty} (\log T)^r}{r\Gamma(r)} + mk^* + mT k_1^* \log T \right].
\end{aligned}$$

This means that

$$\|v\|_\infty \left[ 1 - \left( \frac{\|p\|_{L^\infty} m (\log T)^r}{\Gamma(r)} + \frac{\|p\|_{L^\infty} (\log T)^r}{r\Gamma(r)} + mk^* + mT k_1^* \log T \right) \right] \leq 0.$$

By (3.25) it follows that  $\|\vartheta\|_\infty = 0$ , that is,  $\vartheta = 0$  for each  $t \in J$ , and then  $V(t)$  is relatively compact in  $E$ . In view of the Ascoli-Arzelà theorem,  $V$  is relatively compact in  $D_R$ . Applying now Theorem 3.3 and 3.6, we conclude that  $N$  has a fixed point which is a solution of the problem (3.13)-(3.16). □

### 3.2.3 An Example

Let  $E = l^1 = \{(y_1, y_2, \dots, y_n, \dots)\}$ ,  $\sum_{i=1}^{+\infty} |y_i| < +\infty$  be our Banach space with the norm

$$\|y\|_E = \sum_{i=1}^{+\infty} |y_i|$$

We apply the main result of the chapter Theorem 3.3 and 3.6 to the following system of fractional differential equations

$${}^{CH}D^r y(t) = \frac{1}{9 + e^t} |y_n(t)|, \text{ for a.e., } t \in J = [1, e], a > 0, t \neq \frac{3}{2}, 1 < r \leq 2, \quad (3.27)$$

$$\Delta y|_{t=\frac{3}{2}} = \frac{1}{3 + |y_n(\frac{3}{2}^-)|}, \quad (3.28)$$

$$\Delta y'|_{t=\frac{3}{2}} = \frac{1}{5 + |y_n(\frac{3}{2}^-)|}, \quad (3.29)$$

$$y(a) = 0, y'(a) = 0 \quad (3.30)$$

Where

$$f_n(t, x) = \frac{x_n}{9 + e^t}(t, x) \in J \times E$$

and

$$I_k(x) = \frac{1}{3 + x_n}$$

$$\bar{I}_k(x) = \frac{1}{5 + x_n}$$

$$y = (y_1, \dots, y_n, \dots)$$

set  $f = (f_1, \dots, f_n, \dots)$

clearly conditions  $(H'_1)$  hold, and conditions  $(H'_2) - (H'_4)$  hold with

$$p(t) = \frac{1}{9 + e^t}$$

and where the conditions  $(H'_3) - (H'_4)$  and  $(H'_5) - (H'_7)$  hold with  $k^* = \frac{1}{3}$  and  $k_1^* = \frac{1}{5}$ . We shall that condition(3.25)is satisfied for some  $r \in (1, 2]$ . Then by Theorem 3.3 and 3.6,

the problem (3.27)-(3.30) has a solution on  $[a, e]$  for where values for  $r$  satisfying (3.25) is satisfied with  $T = e$  and  $m = 2$ . Indeed.

$$\frac{pm(\log T)^r}{\Gamma(r)} + \frac{p(\log T)^r}{r\Gamma(r)} + mk^* + mT \log T k_1^* < \frac{1}{2} \text{ if } \frac{\Gamma(r+1)}{2r+1} > \frac{15}{5-4e}$$

which is satisfied for some  $r \in (1, 2]$ . Then by Theorem 3.3 and 3.6, the problem(3.27)-(3.30) has a solution on  $[1, e]$  for values of  $r$  satisfying.

# Chapter 4

## Initial Value Problems for Impulsive Caputo-Hadamard Fractional Differential Inclusions

The article <sup>1</sup> contains the most results presented in this chapter.

### 4.1 Introduction

This chapter deals with the existence of solutions of the initial value problem (IVP for short), for the fractional order differential inclusion,

$${}^{CH}D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [a, T], a > 0, t \neq t_k, k = 1, \dots, m, 1 < r \leq 2, \quad (4.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (4.2)$$

$$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (4.3)$$

$$y(a) = y_1, y'(a) = y_2, \quad (4.4)$$

where  ${}^{CH}D^r$  is the Caputo-Hadamard fractional derivative,  $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is a multi-valued map,  $\mathcal{P}(\mathbb{R})$  is the family of all nonempty subsets of  $\mathbb{R}$ ,  $I_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k = 1, \dots, m$  are continuous functions,  $a = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ ,  $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ ,  $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$ ,  $y(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} y(t_k + \varepsilon)$  and  $y(t_k^-) = \lim_{\varepsilon \rightarrow 0^-} y(t_k + h)$  represent the right and left limits of  $y(t)$  at  $t = t_k$ ,  $k = 1, \dots, m$ , and  $y_1, y_2 \in \mathbb{R}$ .

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<sup>1</sup>**A. Hammou**, S. Hamani and J. Henderson, Initial Value Problems for Impulsive Caputo-Hadamard Fractional Differential Inclusions, *Communication on applied nonlinear analysis*, **26** (2019), N 2, P 17-35.

## 4.2 The convex case

In this section, we assume that  $F$  is a compact and convex valued multivalued map. Consider the set of functions

$$PC(J, \mathbb{R}) = \{y : J \rightarrow \mathbb{R}, y \in AC_{\delta}^2((t_k, t_{k+1}], \mathbb{R}), k = 0, \dots, m,$$

and there exist  $y(t_k^+)$  and  $y(t_k^-)$ ,  $k = 1, \dots, m$ , with  $y(t_k^-) = y(t_k)$ \}.

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

Set  $J' = J \setminus \{t_1, \dots, t_m\}$ .

**Definition 4.1** *A function  $y \in PC(J, \mathbb{R})$  is said to be a solution of (4.1)-(4.4) if there exists a function  $\rho \in L^1([a, T], \mathbb{R})$  such that  $\rho(t) \in F(t, y(t))$  a.e.  $t \in J$ , satisfies  ${}^{CH}D^r y(t) = \rho(t)$  on  $J'$ , and satisfies the conditions (4.2) -(4.4).*

To prove the existence of a solution to (4.1)-(4.4), we need the previous auxiliary lemma.

**Theorem 4.2** *Assume the following hypotheses hold:*

**(H"1)**  $F : J \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(\mathbb{R})$  is a Carathéodory multi-valued map.

**(H"2)** *There exist  $p \in C(J, \mathbb{R}^+)$  and  $\psi : [0, \infty) \rightarrow (0, \infty)$  continuous and nondecreasing such that*

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v|, v \in F(t, u)\} \leq p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

**(H"3)** *There exists  $\psi^* : [0, \infty) \rightarrow (0, \infty)$  continuous and nondecreasing such that*

$$\|I_k(u)\| \leq \psi^*(|u|) \text{ for each } u \in \mathbb{R}.$$

(H"4) There exists  $\psi_1^* : [0, \infty) \rightarrow (0, \infty)$  continuous and nondecreasing such that

$$\|\bar{I}_k(u)\| \leq \psi_1^*(|u|) \text{ for each } u \in \mathbb{R}.$$

(H"5) There exists a number  $M > 0$  such that

$$\begin{aligned} M > & |y_1| + |ay_2| \log T + \left(\frac{m}{\Gamma(r)} + \frac{1}{r\Gamma(r)}\right)(\log T)^r p^* \psi(M) \\ & + m\psi^*(M) + T\psi_1^*(M) \log T \end{aligned} \quad (4.5)$$

where  $p^* = \sup_{t \in J} \{p(t)\}$ .

(H"6) There exists  $l \in L^\infty(J, \mathbb{R}^+)$ , such that

$$H_d(F(t, u), F(t, \bar{u})) \leq l(t)|u - \bar{u}| \quad \text{for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t, 0)) \leq l(t) \quad \text{a.e. } t \in J.$$

Then the IVP (4.1)-(4.4) has at least one solution on  $J$ .

**Proof:** Transform the problem (4.1)-(4.4) into a fixed point problem. Consider the multivalued operator,  $N : PC(J, \mathbb{R}) \rightarrow \mathcal{P}(PC(J, \mathbb{R}))$  defined by

$$\begin{aligned} N(y) = & \{h \in PC(J, \mathbb{R}), h(t) = y_1 + ay_2 \log \left(\frac{t}{a}\right) \\ & + \sum_{k=1}^m \frac{\log \left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\ & + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\ & + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log \left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)), \rho \in S_{F,y}\}. \end{aligned}$$

From Lemma (3.5), the fixed points of  $N$  are solutions to (4.1)-(4.4). We shall show that  $N$  satisfies the assumptions of the nonlinear alternative of Leray-Schauder. The proof will be given in several steps.

**Step 1:**  $N(y)$  is convex for each  $y \in PC(J, E)$ .

Let  $h_1, h_2$  belong to  $N(y)$ , then there exist  $\rho_1, \rho_2 \in S_{F,y}$  such that for each  $t \in J$  and for each  $i = 1, 2$ ,

$$h_i(t) = y_1 + ay_2 \log \left(\frac{t}{a}\right)$$

$$\begin{aligned}
& + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_i(s) \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho_i(s) \frac{ds}{s} \\
& + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

Let  $0 \leq \lambda \leq 1$ . Then, for each  $t \in J$ , we have

$$\begin{aligned}
(\lambda h_1 + (1-\lambda)h_2)(t) & = y_1 + ay_2 \log\left(\frac{t}{a}\right) \\
& + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} (\lambda \rho_1 + (1-\lambda)\rho_2)(s) \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} (\lambda \rho_1 + (1-\lambda)\rho_2)(s) \frac{ds}{s} \\
& + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

Since  $S_{F,y}$  is convex (because  $F$  has convex values), we have

$$\lambda h_1 + (1-\lambda)h_2 \in N(y).$$

**Step 2:**  $N$  maps bounded sets into bounded sets in  $PC(J, \mathbb{R})$ .

Let  $B_{\mu_*} = \{y \in PC(J, \mathbb{R}) : \|y\|_\infty \leq \mu_*\}$  be a bounded set in  $PC(J, \mathbb{R})$  and  $y \in B_{\mu_*}$ . Then for each  $h \in N(y)$ , there exists  $\rho \in S_{F,y}$  such that

$$\begin{aligned}
h(t) & = y_1 + ay_2 \log\left(\frac{t}{a}\right) \\
& + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
& + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

By (H"2)-(H"4), we have, for each  $t \in J$ ,

$$\begin{aligned}
|h(t)| &\leq |y_1| + |ay_2| \left| \log \left( \frac{t}{a} \right) \right| + \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \left| \frac{\log \left( \frac{t}{t_k} \right)}{t_k} \right| \\
&\quad \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} |\rho(s)| \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} |\rho(s)| \frac{ds}{s} \\
&\quad + \sum_{k=1}^m |I_k(y(t_k^-))| \\
&\quad + \sum_{k=1}^m \left| t_k \log \left( \frac{t}{t_k} \right) \right| |\bar{I}_k(y(t_k^-))| \\
&\leq |y_1| + |ay_2| \log T + \frac{m \log T}{\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} p(s) \psi(\|y\|_\infty) \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} p(s) \psi(\|y\|_\infty) \frac{ds}{s} \\
&\quad + m\psi^*(\|y\|_\infty) + mT \log T \psi_1^*(\|y\|_\infty) \\
&\leq |y_1| + |ay_2| \log T + \frac{p^* m (\log T)^r}{\Gamma(r)} \psi(\mu_*) + \frac{p^* (\log T)^r}{r\Gamma(r)} \psi(\mu_*) \\
&\quad + m\psi^*(\mu_*) + mT \log T \psi_1^*(\mu_*)
\end{aligned}$$

Thus

$$\begin{aligned}
\|h\|_\infty &\leq |y_1| + |ay_2| \log T + \frac{p^* m (\log T)^r}{\Gamma(r)} \psi(\mu_*) + \frac{p^* (\log T)^r}{r\Gamma(r)} \psi(\mu_*) \\
&\quad + m\psi^*(\mu_*) + mT \log T \psi_1^*(\mu_*) \\
&:= \ell.
\end{aligned}$$

**Step 3:**  $N$  maps bounded sets into equicontinuous sets of  $PC(J, \mathbb{R})$ .

Let  $\lambda_1, \lambda_2 \in J$ ,  $\lambda_1 < \lambda_2$ , and let  $B_{\mu^*}$  be a bounded set of  $PC(J, \mathbb{R})$  as in Step 2. Let  $y \in B_{\mu^*}$  and  $h \in N(y)$ .

Then

$$\begin{aligned}
|h(\lambda_2) - h(\lambda_1)| &\leq \left| ay_2 \log \left( \frac{\lambda_2}{\lambda_1} \right) \right| \\
&+ \frac{1}{\Gamma(r-1)} \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \left| \frac{\log \left( \frac{\lambda_2}{t_k} \right)}{t_k} \right| \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \rho(s) \frac{ds}{s} \\
&+ \frac{\left| \log \left( \frac{\lambda_2}{\lambda_1} \right) \right|}{\Gamma(r-1)} \sum_{0 < t_k < \lambda_1} \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \rho(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} \rho(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{\lambda_1} \left[ \left( \log \frac{\lambda_2}{s} \right)^{r-1} - \left( \log \frac{\lambda_1}{s} \right)^{r-1} \right] \rho(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{\lambda_1}^{\lambda_2} \left( \log \frac{\lambda_2}{s} \right)^{r-1} \rho(s) \frac{ds}{s} \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} \|I_k(y(t_k^-))\| \\
&+ \sum_{0 < t_k < (\lambda_2 - \lambda_1)} |t_k \log \left( \frac{\lambda_2}{t_k} \right)| \| \bar{I}_k(y(t_k^-)) \| \\
&+ \log \left( \frac{\lambda_2}{\lambda_1} \right) \sum_{0 < t_k < \lambda_1} |t_k| \| \bar{I}_k(y(t_k^-)) \|.
\end{aligned}$$

As  $\lambda_1 \rightarrow \lambda_2$ , the right hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we conclude that  $N : PC(J, \mathbb{R}) \rightarrow \mathcal{P}(PC(J, \mathbb{R}))$  is completely continuous.

**Step 4:**  $N$  has a closed graph.

Let  $y_n \rightarrow y_*$ ,  $h_n \in N(y_n)$  and  $h_n \rightarrow h_*$ . We need to show that  $h_* \in N(y_*)$ .  $h_n \in N(y_n)$

means that there exists  $\rho_n \in S_{F, y_n}$ , such that, for each  $t \in J$

$$\begin{aligned}
h_n(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_n(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho_n(s) \frac{ds}{s} \\
&+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

We must show that there exists  $\rho_* \in S_{F, y_*}$  such that, for each  $t \in J$ ,

$$\begin{aligned}
h_*(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_*(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho_*(s) \frac{ds}{s} \\
&+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

Since  $F(t, \cdot)$  is upper semi-continuous, then for every  $\epsilon > 0$ , there exists a natural number  $n_0(\epsilon)$  such that, for every  $n \geq n_0$ , we have

$$\rho_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0, 1), \quad \text{a.e. } t \in J.$$

Since  $F(\cdot, \cdot)$  has compact values, then there exists a subsequence  $\rho_{n_m}(\cdot)$  such that

$$\rho_{n_m}(\cdot) \rightarrow \rho_*(\cdot) \quad \text{as } m \rightarrow \infty,$$

and

$$\rho_*(t) \in F(t, y_*(t)), \quad \text{a.e. } t \in J.$$

For every  $w \in F(t, y_*(t))$ , we have

$$|\rho_{n_m}(t) - v_*(t)| \leq |\rho_{n_m}(t) - w| + |w - \rho_*(t)|.$$

Then

$$|\rho_{n_m}(t) - \rho_*(t)| \leq d(\rho_{n_m}(t), F(t, y_*(t))).$$

We obtain an analogous relation by interchanging the roles of  $\rho_{n_m}$  and  $\rho_*$ , and it follows that

$$|\rho_{n_m}(t) - \rho_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t) \|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned} |h_{n_m}(t) - h_*(t)| &\leq \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} |\rho_{n_m}(s) - \rho_*(s)| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} |\rho_{n_m}(s) - \rho_*(s)| \frac{ds}{s} \\ &+ \sum_{k=1}^m |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))| \\ &+ \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) |\bar{I}_k(y_{n_m}(t_k^-)) - \bar{I}_k(y_*(t_k^-))|. \\ &\leq \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} l(s) \|y_{n_m} - y_*\|_\infty \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} l(s) \|y_{n_m} - y_*\|_\infty \frac{ds}{s} \\ &+ \sum_{k=1}^m |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))| \\ &+ \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) |\bar{I}_k(y_{n_m}(t_k^-)) - \bar{I}_k(y_*(t_k^-))|. \\ &\leq \frac{m(\log T)^r}{\Gamma(r)} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty \\ &+ \frac{(\log T)^r}{r\Gamma(r)} \int_a^{T^a} l(s) ds \|y_{n_m} - y_*\|_\infty \\ &+ \sum_{k=1}^m |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))| \\ &+ \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) |\bar{I}_k(y_{n_m}(t_k^-)) - \bar{I}_k(y_*(t_k^-))|. \end{aligned}$$

So, as  $m \rightarrow \infty$ ,

$$\begin{aligned}
\|h_{n_m}(t) - h_*(t)\|_\infty &\leq \frac{m(\log T)^r}{\Gamma(r)} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty \\
&+ \frac{(\log T)^r}{r\Gamma(r)} \int_a^T l(s) ds \|y_{n_m} - y_*\|_\infty \\
&+ \sum_{k=1}^m |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))| \\
&+ \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) |\bar{I}_k(y_{n_m}(t_k^-)) - \bar{I}_k(y_*(t_k^-))| \\
&\rightarrow 0.
\end{aligned}$$

**Step 5:** *A priori bounds on solutions.*

Let  $y \in PC(J, \mathbb{R})$  be such that  $y \in \lambda N(y)$  with  $\lambda \in (0, 1]$ . Then there exists  $\rho \in S_{F,y}$  such that, for each  $t \in J$

$$\begin{aligned}
|y(t)| &\leq |y_1| + |ay_2| \log(T) + \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \left| \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \right| \\
&\times \int_{t_{k-1}}^{t_k} \left| \left( \log \frac{t_k}{s} \right) \right|^{r-2} \rho(s) \psi |y(s)| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left| \left( \log \frac{t}{s} \right) \right|^{r-1} \rho(s) \psi |y(s)| \frac{ds}{s} \\
&+ \sum_{k=1}^m \psi^* |y(s)| \\
&+ \sum_{k=1}^m \left| t_k \log\left(\frac{t}{t_k}\right) \right| \psi_1^* |y(s)| \\
&\leq |y_1| + |ay_2| \log T + \left( \frac{m}{\Gamma(r)} + \frac{1}{r\Gamma(r)} \right) (\log T)^r p^* \psi (\|y\|_\infty) \\
&+ m\psi^* (\|y\|_\infty) + T \log T \psi_1^* (\|y\|_\infty).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|y\|_\infty &\leq |y_1| + |ay_2| \log T + \left( \frac{m}{\Gamma(r)} + \frac{1}{r\Gamma(r)} \right) (\log T)^r p^* \psi \|y\|_\infty \\
&+ m\psi^* (\|y\|_\infty) + T \log T \psi_1^* (\|y\|_\infty).
\end{aligned}$$

Then by condition (H'5), we have that  $\|y\|_\infty \neq M$ .

Let  $U = \{y \in PC(J, \mathbb{R}) : \|y\|_\infty < M\}$ . The operator  $N : \bar{U} \rightarrow \mathcal{P}(PC(J, \mathbb{R}))$  is upper semi-continuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y \in \lambda N(y)$  for some  $\lambda \in (0, 1]$ . As a consequence of the nonlinear alternative

of Leray-Schauder, we deduce that  $N$  has a fixed point  $y \in \bar{U}$  which is a solution of the problem (4.1)-(4.4). This completes the proof.  $\square$

### 4.3 The nonconvex case

We present in this section a result for the problem (4.1)-(4.4) with a nonconvex valued right hand side. Our result is based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [45].

**Theorem 4.3** *Assume (H"6) and the following hypotheses hold:*

(H"7)  $F : J \times \mathbb{R} \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  has the property that  $F(\cdot, u) : J \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  is measurable, convex valued and integrably bounded for each  $u \in \mathbb{R}$ .

(H"8) There exist constants  $l^* > 0$  and  $l_1^* > 0$  such that

$$|I_k(u) - I_k(\bar{u})| \leq l^*|u - \bar{u}|, \text{ for each } u, \bar{u} \in \mathbb{R}, \text{ and } k = 1, 2, \dots, m,$$

$$|\bar{I}_k(u) - \bar{I}_k(\bar{u})| \leq l_1^*|u - \bar{u}|, \text{ for each } u, \bar{u} \in \mathbb{R}, \text{ and } k = 1, 2, \dots, m.$$

Let  $l = \sup_{t \in J} \{l(t)\}$ . If

$$\left[ \frac{m(\log T)^r}{\Gamma(r)} + \frac{(\log T)^r}{r\Gamma(r)} + ml^* + ml_1^*T \log T \right] < 1 \quad (4.6)$$

then the IVP (4.1)-(4.4) has at least one solution on  $J$ .

**Proof:** We shall show that  $N$  satisfies the assumptions of theorem 1.32. The proof will be given in two steps.

**Step 1:**  $N(y) \in \mathcal{P}_{cl}(PC(J, \mathbb{R}))$  for each  $y \in PC(J, \mathbb{R})$ .

Indeed, let  $\{y_n\}_{n \geq 1} \subset N(y)$  be such that  $y_n \rightarrow \bar{y}$  in  $PC(J, \mathbb{R})$ . Then,  $\bar{y} \in PC(J, \mathbb{R})$  and there exists  $\rho_n \in S_{F, y}$  such that, for each  $t \in J$ ,

$$\begin{aligned} y_n(t) &= y_1 + ay_2 \log \left( \frac{t}{a} \right) \\ &+ \sum_{k=1}^m \frac{\log \left( \frac{t}{t_k} \right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left( \log \frac{t_k}{s} \right)^{r-2} \rho_n(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left( \log \frac{t}{s} \right)^{r-1} \rho_n(s) \frac{ds}{s} \\ &+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log \left( \frac{t}{t_k} \right) \bar{I}_k(y(t_k^-)). \end{aligned}$$

Using the fact that  $F$  has compact values and from (H'6), we may pass to a subsequence if necessary to get that  $\rho_n$  converges weakly to  $\rho$  in  $L_w^1(J, \mathbb{R})$  (the space endowed with the weak topology). An application of Mazur's theorem implies that  $\{\rho_n\}$  converges strongly to  $\rho$  and hence  $\rho \in S_{F,y}$ . Then for each  $t \in J$ ,

$$\begin{aligned} y_n(t) &\rightarrow \bar{y}(t) = y_1 + ay_2 \log\left(\frac{t}{a}\right) \\ &+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\ &+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)). \end{aligned}$$

So,  $\bar{y} \in N(y)$ .

**Step 2:** *There exists  $\gamma < 1$  such that*

$$H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty \text{ for each } y, \bar{y} \in PC(J, \mathbb{R}).$$

Let  $y, \bar{y} \in PC(J, \mathbb{R})$  and  $h_1 \in N(y)$ . Then, there exists  $\rho_1 \in F(t, y(t))$  such that for each  $t \in J$

$$\begin{aligned} h_1(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) \\ &+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_1(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho_1(s) \frac{ds}{s} \\ &+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)). \end{aligned}$$

From (H'6) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y}(t))) \leq l(t) |y(t) - \bar{y}(t)|.$$

Hence, there exists  $w \in F(t, \bar{y}(t))$  such that

$$|\rho_1(t) - w| \leq l(t) |y(t) - \bar{y}(t)|, \quad t \in J.$$

Consider  $U : J \rightarrow \mathcal{P}(\mathbb{R})$  given by

$$U(t) = \{w \in \mathbb{R} : |\rho_1(t) - w| \leq l(t) |y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator  $V(t) = U(t) \cap F(t, \bar{y}(t))$  is measurable, there exists a function  $\rho_2(t)$  which is a measurable selection for  $V$ . So,  $\rho_2(t) \in F(t, \bar{y}(t))$ , and for each  $t \in J$

$$|\rho_1(t) - \rho_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, \quad t \in J.$$

Let us define for each  $t \in J$ ,

$$\begin{aligned} h_2(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) \\ &+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_2(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho_2(s) \frac{ds}{s} \\ &+ \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)). \end{aligned}$$

Then for each  $t \in J$ ,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} |\rho_1(s) - \rho_2(s)| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} |\rho_1(s) - \rho_2(s)| \frac{ds}{s} \\ &+ \sum_{k=1}^m |I_k(y(t_k^-)) - I_k(\bar{y}(t_k^-))| \\ &+ \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) |\bar{I}_k(y(t_k^-)) - \bar{I}_k(\bar{y}(t_k^-))|. \end{aligned}$$

Thus

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(r-1)} \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} |y(s) - \bar{y}(s)| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)| \frac{ds}{s} \\ &+ \sum_{k=1}^m l^* |y(t_k^-) - (\bar{y}(t_k^-))| + \sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right) l_1^* |(y_1(t_k^-)) - (\bar{y}_1(t_k^-))| \\ &\leq \frac{m(\log T)^r}{\Gamma(r)} \|y - \bar{y}\|_\infty + \frac{(\log T)^r}{r\Gamma(r)} \|y - \bar{y}\|_\infty \\ &+ ml^* \|y - \bar{y}\|_\infty + ml_1^* T \log T \|y - \bar{y}\|_\infty. \end{aligned}$$

Therefore,

$$\|h_1 - h_2\|_\infty \leq \left[ \frac{m(\log T)^r}{\Gamma(r)} + \frac{(\log T)^r}{r\Gamma(r)} + ml^* + ml_1^* T \log T \right] \|y - \bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of  $y$  and  $\bar{y}$ , it follows that

$$H_d(N(y), N(\bar{y})) \leq \left[ \frac{m(\log T)^r}{\Gamma(r)} + \frac{(\log T)^r}{r\Gamma(r)} + ml^* + ml_1^* T \log T \right] \|y - \bar{y}\|_\infty.$$

So by (4.6),  $N$  is a contraction and thus, by theorem 1.32,  $N$  has a fixed point  $y$  which is solution to (4.1)-(4.4). The proof is complete.  $\square$

## 4.4 Topological structure of the set of solutions

In this section we give a result on the topological structure of the set of solutions of (4.1)-(4.4).

**Theorem 4.4** *Assume that (H"1), (H"5), (H"6) and the following hypotheses hold:*

**(H"9)** *There exists  $f_1 \in C(J, \mathbb{R})$  such that*

$$\|F(t, u)\|_{\mathcal{P}} \leq f_1(t), \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

**(H"10)** *There exists  $\zeta \in \mathbb{R}_+^*$  such that*

$$|I_k(u)| \leq \zeta \text{ for each } u \in \mathbb{R}.$$

**(H"11)** *There exists  $\zeta_1 \in \mathbb{R}_+^*$  such that*

$$|\bar{I}_k(u)| \leq \zeta_1 \text{ for each } u \in \mathbb{R}.$$

*Then the set of solutions of problem (4.1)-(4.4) is nonempty and compact in  $PC(J, \mathbb{R})$ .*

**Proof:** Let

$$S = \{y \in PC(J, \mathbb{R}) \mid y \text{ is a solution of (4.1)-(4.4)}\}.$$

From Theorem 4.2,  $S \neq \emptyset$ . Let  $\{y_n\}_{n \in \mathbb{N}} \subset S$ . Then there exists  $\rho_n \in S_{F, y_n}$  such that, for each  $t \in J$ ,

$$\begin{aligned} y_n(t) &= y_1 + ay_2 \log\left(\frac{t}{a}\right) \\ &+ \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \rho_n(s) \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left( \log \frac{t}{s} \right)^{r-1} \rho_n(s) \frac{ds}{s} \\
& + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log \left( \frac{t}{t_k} \right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

From (H"1), (H"9), (H"10) and (H"11), it follows that,

There exists  $M_1 > 0$  such that  $\|y_n\| \leq M_1$ , for all  $n \geq 1$ .

Using (H"1), (H"9), (H"10) and (H"11), it also follows that  $\{y_n\}_{n \geq 1}$  is equicontinuous in  $PC(J, \mathbb{R})$ , and then there exists a subsequence, relabeled  $\{y_n\}$ , such that  $\{y_n\}$  converges to  $y$  in  $PC(J, \mathbb{R})$ .

Applying (H"1) we obtain that, for each  $\varepsilon > 0$ , there exists  $n_0(\varepsilon) \in \mathbb{N}$  such that,  $\rho_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0, 1)$ , for each  $n \geq n_0$  and for a.e.  $t \in J$ . But  $F(\cdot, \cdot)$  has compact values, and so there exists a subsequence  $\{\rho_{n_m}(\cdot)\}$  such that

$$\rho_{n_m} \rightarrow \rho(\cdot) \text{ as } m \rightarrow \infty,$$

and

$$\rho(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

We have also

$$|\rho_{n_m}(t)| \leq f_1(t) \text{ a.e. } t \in J.$$

By applying the Lebesgue dominated convergence theorem, we obtain  $\rho \in L^1(J, \mathbb{R})$ , and then  $\rho \in S_{F,y}$ . So,

$$\begin{aligned}
y(t) & = y_1 + ay_2 \log \left( \frac{t}{a} \right) \\
& + \sum_{k=1}^m \frac{\log \left( \frac{t}{t_k} \right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left( \log \frac{t_k}{s} \right)^{r-2} \rho(s) \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left( \log \frac{t}{s} \right)^{r-1} \rho(s) \frac{ds}{s} \\
& + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m t_k \log \left( \frac{t}{t_k} \right) \bar{I}_k(y(t_k^-)).
\end{aligned}$$

Hence  $S$  is compact. □

## 4.5 An example

We apply our result of the chapter in the convex case, Theorem 4.2, to the following fractional differential inclusion

$${}^{CH}D^r y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J = [1, e], \quad t \neq \frac{3}{2}, \quad 1 < r \leq 2, \quad (4.7)$$

$$\Delta y|_{t=\frac{3}{2}} = \frac{1}{3 + |y(\frac{3}{2}^-)|}, \quad (4.8)$$

$$\Delta y'|_{t=\frac{3}{2}} = \frac{1}{5 + |y(\frac{3}{2}^-)|}, \quad (4.9)$$

$$y(1) = 0, \quad y'(1) = 0, \quad (4.10)$$

where  ${}^{CH}D^r$  is the Caputo-Hadamard fractional derivative, and  $F : [1, e] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is a multivalued map satisfying

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

and where  $f_1, f_2 : [1, e] \times \mathbb{R} \mapsto \mathbb{R}$  are such that, for each  $t \in [1, e]$ ,  $f_1(t, \cdot)$  is lower semi-continuous (i.e., the set  $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$  is open for each  $\mu \in \mathbb{R}$ ), and such that, for each  $t \in [1, e]$ ,  $f_2(t, \cdot)$  is upper semi-continuous (i.e., the set  $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$  is open for each  $\mu \in \mathbb{R}$ ). Assume that there are  $p \in C([J, \mathbb{R}^+)$  and  $\psi : [0, \infty) \mapsto (0, \infty)$  continuous and nondecreasing such that

$$\max(|f_1(t, y)|, |f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in J \text{ and } y \in \mathbb{R},$$

and

$$I_k(y) = \frac{1}{3 + y},$$

$$\bar{I}_k(x) = \frac{1}{5 + y}.$$

It is clear that  $F$  is compact and convex valued, and it is upper semi-continuous.

Assume that there are  $\psi^*, \psi_1^* : [0, \infty) \mapsto (0, \infty)$  with  $\psi^*(t) = \frac{1}{3}t$  and  $\psi_1^*(t) = \frac{1}{5}t$ . We have  $m = 2$ , and assume there exists a number  $M > 0$  such that

$$\frac{M}{\left(\frac{2}{\Gamma(r)} + \frac{1}{r\Gamma(r)}\right) (p^*\psi M + \frac{2}{5}M + \frac{\epsilon}{3}M)} > 1,$$

where  $p^* = \sup_{t \in J} \{p(t)\}$ .

Since all the conditions of Theorem 4.2 are satisfied, problem (4.7)-(4.10) has at least one solution  $y$  on  $[1, e]$ .

# Conclusion and Perspectives

In this thesis, we have considered the existence of solutions of the initial value problems (IVP for short), for the impulsive fractional order differential equations and inclusions ( $r \in (0, 1]$  or  $r \in (1, 2]$ ). For this, we based on three well-known fixed point theorems (Banach, Schaefer and Leray-Schauder). Then, in the fourth chapter, we used the Mönch's fixed point theorem combined with the technic of measure of noncompactness of Kuratowski. And, in the fifth chapter, we used Leray-Schauder for the convex case and Covitz-Nadler for the non-convex case. In the futur, we will study the existence of solution of the initial value problems (IVP for short) for the impulsive fractional functional differential equations and inclusions with State-Dependent Delay and Caputo-hadamard derivative.

# Bibliography

- [1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.
- [2] S. Abbas , M. Benchohra , S. Hamani and J. Henderson, Upper and lower solutions method for Caputo- hadamard fractional inclusions, *Math. Moravica* 23 (2019), N 1,P 107-118.
- [3] S. Abbas, M. Benchohra and G. M. N'Gu'er'ekata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
- [4] S. Abbas, M. Benchohra and G. M. N'Gu'er'ekata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [5] Y. Adjabi, F. Jarad, D. Baleanu and T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, *J. Comput. Anal. Appl.* **21** (4) (2016), 661-681.
- [6] R. P Agarwal, M. Benchohra and S. Hamani, A survey on existence results for Boundary value problems for nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* Vol. 109 **3** (2010), 973-1033.
- [7] R. P. Agarwal, M. Benchohra and D. Seba, An the application of measure of non-compactness to the existence of solutions for fractional differential equations, *Results Math.***55** , 3-4 (2009), 221-230.
- [8] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, *Cambridge Tracts in Mathematics* **141** Cambridge University Press, Cambridge, UK, (2001).
- [9] B. Ahmad, Existence of solutions for fractional equations of order  $q \in (2, 3]$  with anti-periodic conditions, *J. Appl. Math. Compt.* **24** (2011), 822-825.
- [10] B. Ahmad and J. J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Larray Schauder degree theory, *Topol. Meth. Nonl Anal.* **35** (2010), 295-304.

- [11] B. Ahmad, S. K. Ntouyas and A. Alsaedi, New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions, *Bound Val Problem.* **275**(2013), 1-14.
- [12] B. Ahmad and S. K. Ntouyas, Initial value problems for hybrid Hadamard fractional equations, *Elect. J. Diff.* **2014** (2014), No. 161, p. 1-8.
- [13] B. Ahmad and V. Otero Espiner, Existence of solutions for fractional inclusions with anti-periodic boundary conditions, *Bound. Val Probl* **11** (2009): Art ID 625347.
- [14] E. Ait Dads , M. Benchohra and S. Hamani , Impulsive fractional differential inclusions involving the Capito fractional derivative, *Frac. Cal. and Appl. Math* 12 No. 1(2009), 15-38.
- [15] R. R. Akhmerov, M. I. Kamenski, A. S. Patapov, A. E. Rodkina and B. N. Sadovski, Measure of noncompactness and condensing operators: Translated from the 1986 Russian original by A. Iacop. Operator theory : Advances and Applications, **55**, Birkhauser Verlag, Bassel, 1992.
- [16] J. P. Aubin and A. Cellina, Differential Inclusions, Springer-Verlag, Berlin-Heidelberg, New York, 1984.
- [17] J.P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.
- [18] D.D.Bainov and V. Covachev, Impulsive differential equations with a small parameter, vol.24 of Series on Advances in Mathematics for Applied Sciences, World Scientific, Singapore(1994).
- [19] D.D.Bainov, V. Lakshmikantham, and P, S. Simeonov, Theory of impulsive differential equations, vol. 6 of Series in Modern Applied Mathematics, World Scientific, Singapore, (1989)
- [20] D.D.Bainov and P,S. Simeonov, Impulsive differential equations: Asymptotic properties of the solutions, vol. 28 of Series on Advan in Mathematics for Applied Sciences, World Scientific, Singapore, (1995)
- [21] D.D.Bainov and P,S. Simeonov, Oscillation theory of impulsive differential equations, international publications, orlando, Fla, USA, (1998).
- [22] J. Banas and K. Goebel, Measure of noncompactness in Banach spaces, In lecture Notes in Pure and Applied Mathematics, Vol **60**, Marcel Dekker, New York.
- [23] J. Banas and K. Sadarangani, On some measures of noncompactness in the space of continous functions, *Nonli Anal.* **60**, No. 2 (2008) 377-383.
- [24] L. C. Becker, T. A. Burton and I. K. Purnaras, An Inversion of a Fractional Differential Equation and Fixed point, *Nonli. Dyna. and Syst The*, **15**(3) (2015) 242.

- [25] F. Belhannache, S. Hamani and J .Hendrson, Impulsive Fractional Differential Inclusions for the Liouville-Caputo-Hadamard Fractional Derivative, *Commu. in appl nonl anal*, 25(2018), N 4,P 52-67.
- [26] F. Belhannache, S. Hamani, and J. Henderson; Upper and lower solutions methods for impulsive Caputo-Hadamard fractional differential inclusions, *Electr. J. Diff Equat*, (2019), No. 22, pp. 1-13.
- [27] M. Benchohra, S. Djebali and S. Hamani, Boundary value problems of differential inclusions with Riemann-Liouville fractional derivative, *Nonlinear Oscillation* . 14 **1** (2011), 7-20.
- [28] M. Benchohra and S. Hamani, Boundary value problems for differential inclusions with fractional order, *Diss. Math. Diff. Inclusions. Control and Optimization.*,**28** (2008), 147-164.
- [29] M .Benchohra, S. Hamani, J.J Nieto and B. A. Slimani , Existence of solutions to differential inclusions with fractional order and impulses, *Electron. J. Diff. Equ.*, Vol. 2010(2010), No. 80, pp. 1-18.
- [30] M. Benchohra and S. Hamani, Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative, *Topol. Meth. Nonlin Anal.*, **32**(1) (2008), 115-130.
- [31] M. Benchohra, J. Henderson and D. Seba, Boundary value problems for fractional differential inclusions in Banach Space, *Frac. Diff. Cal* **2** (2012), 99-108.
- [32] M. Benchohra, J. Henderson and D. Seba, Measure of noncompactenes and fractional differential equations in Banach Space, *Commun. Appl. Anal* **12** 4(2008), 419-428.
- [33] M. Benchohra, J. J. Nieto and D. Seba, Measure of noncompactenes and fractional and hyperbolic partial fracational differential equations in Banach Space, *Panamer. Math. J* **20** 3(2010), 27-37.
- [34] W. Benhamida, S. Hamani and J .Hendrson, Boundary value problems for Caputo-Hadamard fractional differential equations, *Adv in the The of Nonl Anal and Applic*, **2**, N 3 (2018), 138-145.
- [35] W. Benhamida and S. Hamani, Measure Noncompacteness and Caputo-Hadamart Fractional Differential Equations in Banach Spaces, *Inter. Balkan. J. of Math*, **1**(2018), N 3,P 98-106.
- [36] C. Berge, Espaces topologiques: Fonctions multivoques, *Dunod*, Paris, 1959.
- [37] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

- [38] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type, *Math. Nachr.* **189** (1998), 23-31.
- [39] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Composition of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.
- [40] P. L. Butzer, A. A. Kilbas and J. J. Trujillo; Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.
- [41] P. L. Butzer, A. A. Kilbas and J. J. Trujillo; Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.
- [42] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem. *Selected problems of mathematics*, 25-30, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, 6, Cracow Univ. Technol, Krakw, 1995.
- [43] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
- [44] T. Chen and W. Liu, An anti-periodic boundary value problem for the fractional differential equations with a  $p$ -Laplacian operator, *Appl. Math. Lett.* **25** (11) (2012), 1671-1675.
- [45] H. Covitz and S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.
- [46] K. Deimling, Multivalued Differential Equations, Walter De Gruyter, Berlin-New York, 1992.
- [47] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609-625.
- [48] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.
- [49] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity, in "Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.
- [50] K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms* **16** (1997), 231-253.

- [51] M. Frigon, Théorèmes d'existence de solution d'inclusions différentielles, Topological methods in differential equations and inclusions, NATO ASI Series, Ser. C: Math. and Phys. Sci. , Kluwer, Dordrecht,
- [52] A. Fryszkowski, Continuous selections for a class of nonconvex multivalued maps, *Studia. Math.* **76** (1983), 163-174.
- [53] A. Fryszkowski, Fixed Point Theory for Decomposable Sets. Topological Fixed Point Theory and Its Applications, **2**. Kluwer Academic Publishers, Dordrecht, 2004.
- [54] Y. Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Difference Equ.* **2014** (2014), No. 10, 12 pp.
- [55] Gorenflo R. and Mainardi F., Essentials of fractional calculus, Preprint submitted to MaphySto center, january 28, 2000.
- [56] J R. Graef, N. Guerraiche and S. Hamani, Boundary value problems for fractional differential inclusions with Hadamrd type derivatives in Banach spaces, *Stud. Univ. Babeş-Bolyai. Math*, **62** (2017), No. 4, 427-438.
- [57] J R. Graef, N. Guerraiche and S. Hamani, Initial value problem of fractional functional differential inclusions with Hadamard type derivative, *Sur. in math. and its appl*, **13**, 2018, 27-40.
- [58] A. Granas and J. Dugundji, Fixed Point Theory. Springer-Verlag, New York, 2003.
- [59] N. Guerraiche, S. Hamani and J. Henderson, Nonlinear boundary value Problems for Hadamard fractional differential inclusions with integral boundary conditions, *Adv. in Dyn. Sys. and Appl*, **12** (2017), No. 2, 107-121.
- [60] D. Guo, V. Lakshmikantham and X. Liu, Nonlinear integral equations in abstract spaces, Mathematics and its Applications, **373**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
- [61] V. Gupta and J. Dabas, Existence Results for a Fractional Integro-Differential Equation with Nonlocal Boundary Conditions and Fractional Impulsive Conditions *Nonl. Dyn. and Sys. The.* **15**(4) (2015) 370382
- [62] J. Hadamard, Essai sur l'étude des fonctions donnees par leur developement de Taylor, *J. Mat. Pure Appl.* Ser. 8 (1892) 101-186.
- [63] J. K. Hale and S. V. Lunel, Introduction to functional differential equations, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.
- [64] S. Hamani, A. Hammou and J. Henderson, Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative, *Comm. on appl. nonl. analy.*, 24(2017), N 3, 48-58.

- [65] A. Hammou, S. Hamani and J. Henderson, Impulsive Hadamard Fractional Differential Equations in Banach Spaces, *Pan. Math. J.*, **28** N 2 (2018), 52-62.
- [66] A. Hammou, S. Hamani and J. Henderson, Initial value problem for Impulsive Caputo Hadamard Fractional Differential Inclusions, *Comm. on appl. nonl. anal.*, **26** (2019), N 2, 17-35.
- [67] H. P. Heinz, On the behaviour of measure of noncompactness with respect of differentiation and integration of vector-valued function, *Nonlinear. Anal* **7** (1983), 1351-1371.
- [68] J. Henderson and C. Tisdell, Topological transversality and boundary value problems on time scales, *J. Math. Anal. Appl.* **289** (2004), 110-125.
- [69] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, *Rheologica Acta*, **45**(2006), no. 5, 765-772.
- [70] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
- [71] R. Hilfer, Threefold Introduction to fractional derivatives. R. Klages, G. Radons and M. Sokolov, Anomalous Transport: Foundations and Applications, Wiley-VCH.
- [72] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht, 1997.
- [73] F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Difference Equ.* **2012** (2012), No. 142, 8 pp.
- [74] E. R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* (2007), No. 3, p. 11.
- [75] Kilbas, A.A. (2001). Hadamard-type fractional calculus, *J. Korean Math. Soc*, **38**, 1191-1204.
- [76] A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84-89.
- [77] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.
- [78] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer, Dordrecht, The Netherlands, 1994.

- [79] M. Klimek, Sequential fractional differential equations with Hadamard derivative, *Commun. Nonlinear Sci. Numer. Simul.* **16** (12) (2011), 4689-4697.
- [80] V. Lakshmikantham and S. Leela, Nonlinear differential equations in abstract spaces, International Series in Mathematics: Theory, Methods and Applications, **2**, Pergamon Press, Oxford, UK, 1981.
- [81] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equation, *Bull. Accd. Pol. Sci., Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.
- [82] C. M., Linear model of dissipation whose Q is almost frequency independent-II. The Geophysical Journal of the Royal Astronomical Society. Vol. 13, 1967, 529-539
- [83] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
- [84] V.D. Mil'man and A.D. Myshkis, on the stability of motion in the presence of impulsives, *Sib. Math. J.*, 1, (1960), 233-237.
- [85] S. M. Momani, S. B. Hadid and Z. M. Alawneh, Some analytical properties of solutions of differential equations of noninteger order, *Int. J. Math. Math. Sci.* **2004**(2004), 697-701.
- [86] H. Mönch, Boundary value problem for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **75**, No. 5 (1980), 985-999.
- [87] U. Mosco and J. l. Joly, A propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles, *J. of Funct. Analy.* **34**, 107-137 (1979).
- [88] Oldham K. and Spanier J., The fractional calculus, Academic Press, New York. London, 1974
- [89] D. O'Regan and R. Precup, Fixed point theorems for set-valued maps and existence principles for integral inclusions, *J. Math. Anal. Appl.* 245 (2000), 594-612.
- [90] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
- [91] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, *Fract. Calculus Appl. Anal.* **5** (2002), 367-386.
- [92] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary and L. Dorčák, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281-296.
- [93] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives: Theory and applications. Gordon and Breach (1993).

- [94] M. Sowmya and A.S. Vatsala, Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence, *Nonlinear Dynamics and Systems Theory*, **15** (2) (2015) 198208.
- [95] S. Szufła, On the application of measure of noncompactness to existence theorems, *Rendiconti del Seminario Matematico Della Università di Padova* **75** (1986), 1-14.
- [96] P. Thiramanus, S. K. Ntouyas and J. Tariboon, Existence and uniqueness results for Hadamard- type fractional differential equations with nonlocal fractional integral boundary conditions, *Abstr. Appl. Anal.* (2014), Art. ID 902054, 9 pp.
- [97] A. Zahed and S. Hamani, Boundary value problems for Caputo-Hadamard fractional differential inclusions with non local conditions, *Pen. Math. J.*, **29** N 4 (2019),15-28.