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## Course Support Document

## Title :

# Concepts of Transfer Phenomena

**Level : L2/ Process Engineering, Petrochemical Industries**

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# *Foreword*

This course, « *Concepts of Transfer Phenomena* » is intended for students enrolled for the L2 in Process Engineering and Petrochemical Engineering. To understand this subject, students must have acquired sufficient knowledge of thermodynamics and chemical kinetics.

The phenomena of heat, mass, and momentum transfer are of paramount importance in science and technology, as they play a role in numerous industrial sectors.

This course aims to provide students with fundamental knowledge of the various phenomena of heat, mass, and momentum transfer.

- To introduce the basic concepts of heat transfer and then guide students through calculations.

- To present the fundamental laws that describes mass transfer processes.

- To demonstrate the balance equations for equilibrium and fluid flow.

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*ملخص*

يُسلط مقرر «مفاهيم ظواهر الانتقال» الضوء على الأسس الجوهرية اللازمة لاكتساب الكفاءات في مجالات انتقال الحرارة، وانتقال المادة، وانتقال كمية الحركة. وهو موجه لطلاب السنة الثانية (ليسانس) في تخصصات هندسة الطرائق والصناعات البتروكيميائية. يستعرض الفصل الأول المفاهيم الأولية المتعلقة بأنماط الانتقال في الأنظمة؛ حيث يبدأ بتقديم تعريفات أساسية حول: النظام، وقيود النظام، وظواهر الانتقال، والقوانين الخطية للنقل الانتشاري (الجزيئي) للكميات ، وهي: قانون فوريه، وقانون فيك، وقانون نيوتن. ويختتم الفصل بصياغة معادلات الموازنة (معادلات الحفظ) الشاملة للكميات الشاملة، سواء في حالات التوازن أو في حالات جريان الموائع. أما الفصل التالي، المخصص لانتقال الحرارة، فيقدم نظرة عامة حول هذا الموضوع، ويشرح كيفية صياغة موازنة الطاقة، مع دراسة أنماط انتقال الحرارة الثلاثة: الانتقال، والحمل، والإشعاع. ويتناول الفصل الثالث انتقال المادة، موضحاً مبادئ الانتقال الانتشاري للمادة والانتقال بالحمل. وأخيراً، يتناول الفصل الرابع انتقال كمية الحركة، مستعرضاً خصائص الموائع، واستاتيكا الموائع، والمعادلات العامة للحفظ .

وقد أُرفق كل فصل بمجموعة من الأمثلة الحسابية المحلولة، والتي صُممت لتغطي كافة المعارف النظرية التي يتناولها المقرر.

*كلمات مفتاحية*: ظواهر الانتقال، انتقال المادة، انتقال الحرارة، انتقال كمية الحركة.

***Abstract***

This course, "Concepts of Transport Phenomena," introduces the fundamental principles required to develop expertise in the areas of heat, mass, and momentum transfer. It is designed for second-year undergraduate students in the fields of process engineering and the petrochemical industry. Chapter 1 reviews the elementary concepts underlying transport phenomena. It begins with essential definitions: such as those for a system, system constraints, and transport phenomena and introduces the linear laws governing diffusive (molecular) transport: the laws of Fourier, Fick, and Newton. It concludes by establishing balance (conservation) equations for extensive properties, applicable to both static systems and fluid flow. The subsequent chapter on heat transfer provides an overview of thermal transport, the formulation of energy balances, and an examination of the three modes of heat transfer: conduction, convection, and radiation. The third chapter, focused on mass transfer, explains the principles of diffusive mass transfer and convective mass transfer. The final chapter, covering momentum transfer, addresses fluid properties, fluid statics, and general conservation equations. Each chapter includes a selection of solved calculation examples, designed to encompass the full range of theoretical knowledge presented throughout the course.

***Key words:*** Transfer phenomena, matter transfer, heat transfer, momentum transfer.

***Résumé***

Ce cours « Notions de phénomènes de transfert » met en avant les bases fondamentales nécessaires à l'acquisition de compétences dans le domaine des transferts de la chaleur, de matière et de quantité de mouvement. Il s'adresse aux étudiants 2ème année licence filières génie des procédés et industries pétrochimies.

Le chapitre 1 rappelle les notions élémentaires aux modes de transfert des systèmes. On commençant d'abord de donner des définitions : d'un système, contraintes d'un système, les phénomènes de transfert, lois linéaires de transport diffusif (moléculaire) des extensités : loi de Fourier, Fick et Newton. En fin établir les équations de bilan (équation de conservation) d'une extensité relatives à l'équilibre et à l'écoulement des fluides. Le chapitre suivant de transfert de chaleur présente des généralités sur le transfert thermique, formulation de bilan énergétique, étudier les trois modes de transfert thermique : conduction, convection, rayonnement. Le troisième chapitre sur le transfert de matière, explique le principe de transfert de masse diffusif (transfert de matière par diffusion) et transfert de masse par convection (Transfert de matière par convection).

Le dernier chapitre de transfert de quantité de mouvement présente les propriétés des fluides, statique des fluides, et les équations générales de conservation.

Chaque chapitre est accompagné de quelques exemples de calcul corrigés. Ces exemples sont censés couvrir l'ensemble des connaissances théoriques développées dans le cours.

***Mots clés :*** Phénomènes de transfert, transfert de matière, transfert de chaleur, transfert de quantité de mouvement.

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# Chapter I :

# Introduction to modes of transfer

## I.1. General information and definitions

### I.1.1 Definition of a system

A system is defined as all the matter contained within a volume V bounded by a closed envelope with an external surface area S. The matter outside this surface constitutes the external environment.

- If, during the observation period, the system does not exchange matter with the external environment, it is said to be closed. Otherwise, it is said to be open. Depending on the circumstances, it may also be semi-open or semi-closed. Only an open system can function in a steady-state or permanent state.

- Similarly, a system is considered isolated if it does not exchange heat with its surroundings. It is considered non-isolated if there is heat exchange.

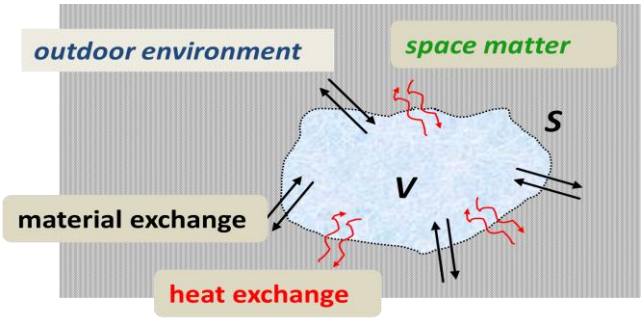



**Figure I. 1** Definition of a system

- A system is characterized by two categories of parameters:

**Extensive** quantities, which are functions of the chosen system: mass, number of moles, etc. The variation of the matter located inside the volume V causes the variation of these quantities. These extensive quantities are conserved: a certain quantity of any of these three extensive quantities cannot spontaneously appear or disappear from the volume V. If it appears or disappears, it is because it has been exchanged with the external environment.

**Intensive** quantities, independent of the choice of system: density, concentration, etc. These are independent of the variation of the amount of matter located inside V.

## I.1.2 Constraints of a System

The operating regime of any physicochemical device results from the combination of a very large number of factors characterizing either the chemical transformation itself or the processes of mass, heat, and momentum transport, etc. All the mathematical relationships between these different factors are called constraints of the system.

When mathematically describing the operation of such a system, it is sometimes difficult to be certain that all the constraints have been taken into account, and conversely, it is difficult to ensure that the numerous mathematical relationships one is tempted to write are truly independent of each other. By enumerating all these relationships, it is generally accepted that they belong to the following seven groups:

### I.1.2.a Principles of Conservation of Extensive Quantities

- Conservation of Mass
- Conservation of Energy
- Conservation of Momentum

### I.1.2.b Laws of Static Equilibrium

These laws express mechanical, physical, or chemical equilibrium. They constitute applications, in various forms, of the first law of thermodynamics. For example, to express the mechanical equilibrium of a system, we state that the resultant force and the resultant moment of the external forces applied to the system are zero. The physicochemical equilibrium of a system is expressed by the equality of the chemical potentials of each of the constituents in each of the phases present.

### I.1.2.c Kinetic Laws

These express the direction and rate of evolution of a system. In fact, the direction of evolution of any isolated system is determined by a single law: the second law of thermodynamics, which states that the entropy of this system can only increase. On the other hand, the rate of evolution refers to the numerous laws that govern the rates of various elementary processes: rates of chemical reactions, rates of mass transport (Fick’s law), heat transport (Fourier’s law), and momentum transport (Newton’s law).

#### I.1.2.d Capacitive Relations

These are the relations that express the quantity, per unit volume, of each of the extensive quantities (matter, heat, momentum).

#### I.1.2.e Geometric Texture Relations

These are all the relations that give the shape and dimensions of a device or a phase domain: for example, the distribution of pore diameters in a solid catalyst, the distribution of droplet diameters in an emulsion, etc. These relationships also define the residence time distribution of a fluid flowing through the system, and so on. If, in addition, one wishes to study the system's operation from an economic perspective, two further sets of relationships must be introduced.

#### I.1.2.f Cost formulas

#### I.1.2.g Optimization equations

### I.1.3 Definition of the transfer phenomenon

A transfer phenomenon or transport phenomenon is an irreversible process in which a physical quantity is transported by molecules, originating from the inhomogeneity of an intensive quantity. It is the spontaneous tendency of physicochemical systems to standardize these quantities that causes transport.

The study of each transport phenomenon refers to a specific transferred entity, such as heat, mass, or momentum.

### I.1.4 Different transfer phenomena

The best-known transfer phenomena are:

#### I.1.4.1 Heat transfer

In this process, the quantity transferred is heat (temperature). This transfer occurs between two areas with different temperatures: it always happens from the higher temperature to the lower temperature. The temperature difference is called the driving force of heat transfer.

#### I.1.4.2 Mass transfer

In this process, the transferred quantity is matter (mass concentration). This transfer occurs between two areas with different mass concentrations, always from the higher concentration

to the lower concentration. The difference in concentration is called the driving force of mass transfer.

### I.1.4.3 Momentum transfer

In this case, the transferred quantity is momentum (velocity). This transfer occurs between two entities with different velocities; it always happens from the entity with the higher velocity to the one with the lower velocity. The difference in velocity is called the driving force of the momentum transfer.

## I.2 Linear laws of diffusive (molecular) transport of extensities

In a medium that is both still (no overall flow) and tranquil (no eddies or vortices), the only possible movements occur at the molecular level. The thermal agitation of molecules, ions, or free electrons within matter leads to macroscopic evolutions of systems, observable at our scale. We will describe below the phenomenological laws governing these evolutions for each of the three scales.

### I.2.1 Diffusive transport of matter – Fick's law

Consider a stationary and undisturbed phase, a mixture of two components, A (solvent) and B (solute). Under the influence of thermal agitation, the molecules of B tend to migrate from areas of higher concentration to areas of lower concentration.

This is a microscopic phenomenon of molecular transport, which takes place within a **macroscopically stationary phase,** and is called molecular diffusion **or** diffusive transport of matter.

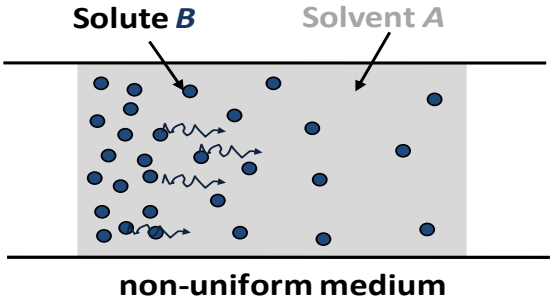



**Figure I. 2** Diffusive transport of matter

After a longer or shorter period of time, the environment becomes uniform, quantifiable on our scale. This is the macroscopic manifestation of a microscopic phenomenon.

This kinetic process, leading the phase towards equilibrium, is represented by a phenomenological model which is translated by a simple linear law called **Fick's law** :

$$\boldsymbol{\phi_M(x) = -\mathcal{D}\,\frac{dC}{dx}(x)} \qquad \dots (\mathbf{I.1})$$

With

$\boldsymbol{\phi_M(x)}$ : molar flux density of $B$ at position $x$ $\quad [\boldsymbol{mol \cdot s^{-1} \cdot m^{-2}}]$

Molar flux (or flow rate) of $B$ passing through, by diffusion, the unit cross-sectional area Ω (cross-section normal to the diffusion axis), at position $x$.

$\frac{\boldsymbol{dC}}{\boldsymbol{dx}}(\boldsymbol{x})$: concentration gradient of $B$, along the diffusion axis, at position $x$.

$\boldsymbol{\mathcal{D}}$: proportionality coefficient, which we will call diffusivity.

Note that the linearity in this law is a linear relationship between the flux density and the concentration gradient. **[m² · s⁻¹]**

The sign **[–]** is introduced into the relation when the gradient $\frac{\boldsymbol{dC}}{\boldsymbol{dx}}$ is negative (the chosen x-axis coincides with the diffusion axis).

Writing Fick's law for three-way transport

$$\boldsymbol{\phi_M(x) = -\mathcal{D}\ \frac{\partial C}{\partial x}(x)}$$

$$\boldsymbol{\phi_M(y) = -\mathcal{D}\ \frac{\partial C}{\partial y}(y)}$$

$$\boldsymbol{\phi_M(z) = -\mathcal{D}\ \frac{\partial C}{\partial z}(z)}$$

$$\overrightarrow{\boldsymbol{\phi_M}} = -\boldsymbol{\mathcal{D}} \,.\, \overrightarrow{\boldsymbol{grad}\ C} \qquad \ldots (\mathbf{I.2}) \qquad \textbf{Three-way Transport}$$

The diffusivity $\boldsymbol{\mathcal{D}}$ is considered to be constant in all three directions.

### I.2.2 Diffuse heat transport – Fourier's law

Let's heat a metal rod, steel for example, at one end for a short time. As a result of this momentary heating, a temperature difference develops between the two ends.

After a certain time, the temperature of the steel rod becomes uniform again.

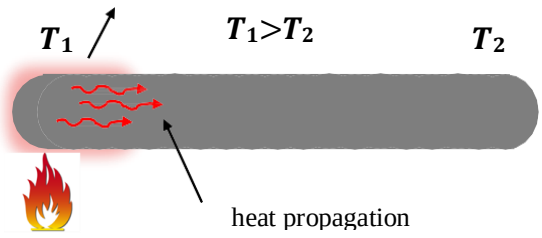



**Figure I. 3** Diffuse heat transport

The heat has propagated along this solid body, from the hot end to the cold end. This is a transfer of kinetic energy of thermal agitation (vibrational energy) from the molecules at the hot end, which have greater kinetic energy, to the molecules at the cold end, which have less kinetic energy. This is called diffusive heat transport along the rod, commonly known as thermal conduction. This transfer of kinetic energy generally occurs through collisions between particles (molecules, atoms, ions, free electrons).

The same phenomenon will occur within a still and tranquil fluid phase when a temperature difference is created along a given direction.

Thermal conduction is governed by a phenomenological law known as **Fourier's law** :

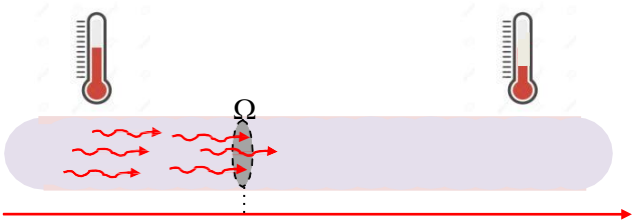



**Figure I. 4** Unidirectional conduction

$$\boldsymbol{\phi_T(x) = -\lambda \frac{dT}{dx}(x)} \qquad \text{... (I. 3)}$$

With

$\boldsymbol{\phi_T(x)}$ : heat flux density at position $x$ **[J · s$^{-1}$ · m$^{-2}$]** or **[W · m$^{-2}$]**

heat flux (or flow rate) passing by conduction through a unit cross-sectional area Ω (cross-section normal to the conduction axis) at position $x$.

$\frac{dT}{dx}$ : temperature gradient along the conduction axis, at position $x$

λ : thermal conductivity of the medium. **[J·s$^{-1}$·°C$^{-1}$·m$^{-1}$]** or **[W·°C$^{-1}$·m$^{-1}$]**

For the case of three-way conduction, Fourier's law is written:

$$\boldsymbol{\phi_T(x) = -\lambda \frac{\partial T}{\partial x}(x)}$$

$$\boldsymbol{\phi_T(y) = -\lambda \frac{\partial T}{\partial y}(y)}$$

$$\phi_T(z) = -\lambda \frac{\partial T}{\partial z}(z)$$

$$\overrightarrow{\phi_T} = -\lambda \cdot \overrightarrow{grad}\, T \qquad \text{... (I. 4)} \qquad \textbf{Three-way Transport}$$

The thermal conductivity of the medium λ is considered constant in all three directions.

### I.2.3 Diffusive transport of momentum – Newton's law

Imagine a fluid phase at rest between two parallel planes (P) and (P'). Animate plane (P) with a translational motion at speed V, and keep plane (P') fixed. We will observe a state of motion within the fluid (macroscopic phenomenon).

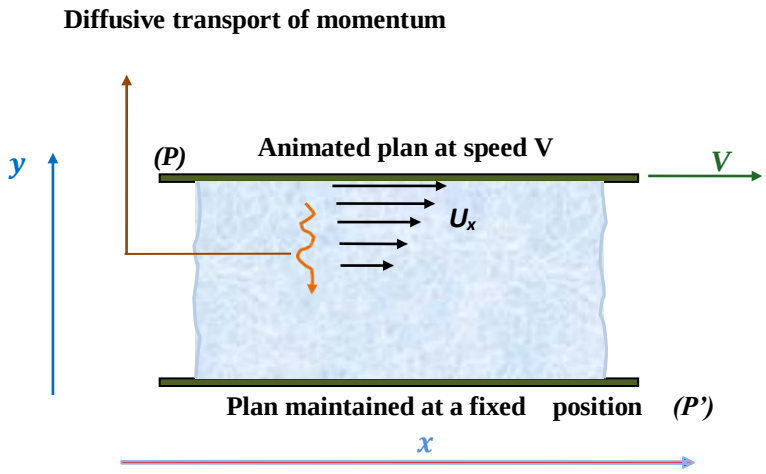



**Figure I. 5** Flow caused between two parallel planes (by thought)

This flow is established progressively from the fluid layer directly in contact with (P), which will acquire the same velocity (principle of adhesion), to the layer in contact with (P'), which will remain stationary (same principle). The intermediate layers will move at decreasing velocities $U_x(y)$ in the direction opposite to the y-axis. Motion is transmitted from one layer to the other through what is called viscous friction: the molecules of the faster-moving layer are held by the lower-velocity layer in contact with it. These molecules thus transport their momentum from one layer to the other, transversely to the overall flow (a microscopic phenomenon), this is **diffusive momentum transport**.

The diffusive transport of momentum (viscous friction) is governed by a phenomenological law known as **Newton's law**:

$$\boldsymbol{\tau}(\boldsymbol{y}) = -\boldsymbol{\mu}\,\frac{\boldsymbol{\partial u_x}}{\boldsymbol{\partial y}}(\boldsymbol{y}) \qquad \ldots (\mathbf{I.5})$$

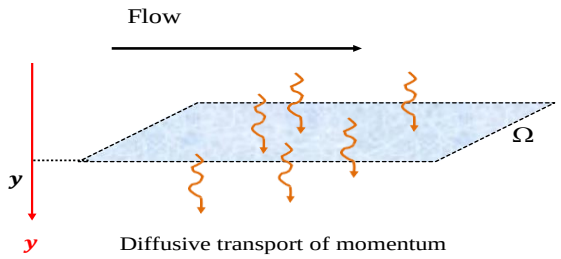



**Figure I. 6** Diffusive transport of momentum

With

$\boldsymbol{\tau}(\boldsymbol{y})$: tangential viscous friction stress (momentum flux density) at position $y$

momentum flux (flow rate) passing through, by diffusion, the unit cross-section Ω (cross-section normal to the diffusion axis), at position $y$.

$[\boldsymbol{kg{\cdot}m^{-1}{\cdot}s^{-2}}]$ or $[\boldsymbol{Pa}]$.

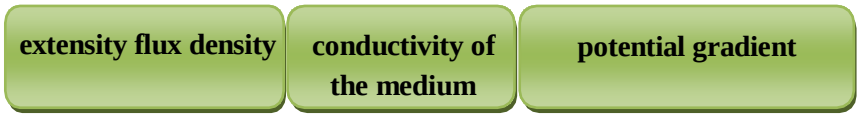
$\frac{\boldsymbol{\partial u_x}}{\boldsymbol{\partial y}}(\boldsymbol{y})$: longitudinal velocity gradient along the diffusion axis (lateral longitudinal velocity gradient), at position $y$.

$\boldsymbol{\mu}$: dynamic viscosity coefficient of the fluid $[\boldsymbol{Pa{\cdot}\,s}]$.

### I.2.4 Analogy of Linear Diffusive Transport Laws

The three phenomenological laws (I.1), (I.3), and (I.5) exhibit an analogy in their expressions. Indeed, the flux density of the extensity is the product of a proportionality coefficient, reflecting the ease of extensity transport in the medium under consideration (conductivity), and the gradient of the quantity whose difference gave rise to the transport phenomenon of the extensity in question, and which therefore corresponds to a potential. Thus, the same mathematical formalism can be adopted for all three laws:

$$\phi = -\lambda \cdot \overrightarrow{grad}\, V \qquad \text{... (I. 6)}$$




### I.2.5 Linear capacitance laws

The diffusive transport of an intensity in a given medium under the effect of the potential difference prevailing within the medium suggests that at each point, in steady state, there exists a volume concentration $g$ of the intensity, which is proportional (to a first approximation) to the potential $V$ at that point:

$$g = c \cdot V \qquad \text{... (I.7)}$$

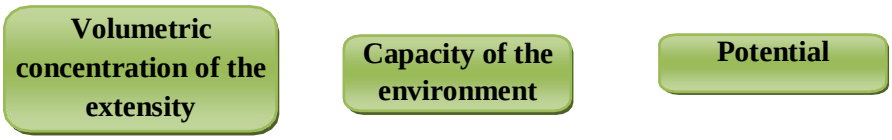



**Table I. 1** The proportionality factor c is the capacity of the medium at the point considered

| *Extensity* | *Volumetric concentration of the extensibility* $g$ | *Potential* $V$ | Capacity $c$ |
|---|---|---|---|
| **Number of moles** | $C\ [mol \cdot m^{-3}]$ | $C\ [mol \cdot m^{-3}]$ | 1 |
| **Quantity of heat** | $\rho C_p T\ [J \cdot m^{-3}]$ | $T$ **[°C]** | $\rho C_p\ [J \cdot °C^{-1} \cdot m^{-3}]$ |
| **Momentum** | $\rho u\ [kg \cdot m \cdot s^{-1} \cdot m^{-3}]$ | $u\ [m \cdot s^{-1}]$ | $\rho\ [kg \cdot m^{-3}]$ |

### I.2.6 Expressions of the diffusive transport laws using double linearity

By combining the two expressions (I.6) and (I.7), we obtain the form:

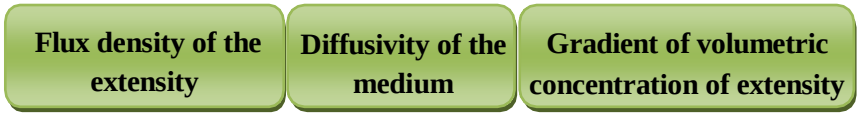
$$\boldsymbol{\phi = -\lambda \, . \overrightarrow{grad} \left(\frac{g}{c}\right)} \qquad \ldots (\mathbf{I.8})$$

If we consider that the capacity remains constant in the space of the medium, equation (I.8) becomes:

$$\boldsymbol{\phi = -\frac{\lambda}{c} . \overrightarrow{grad}\, g} \qquad \ldots (\mathbf{I.9})$$

The ratio $\frac{\lambda}{c}$ is called the diffusivity $\boldsymbol{D}$ of the medium. We obtain the doubly linear form

$$\boldsymbol{\phi} \quad = \quad - \quad \boldsymbol{D} \quad . \quad \boldsymbol{\overrightarrow{grad}\, g} \qquad \ldots (\mathbf{I.10})$$




With the correspondence summarized in the table below.

**Table I. 2** Diffusivity quantities of different extensities

| *Extensity* | *Diffusivity* | *Dimension* |
|---|---|---|
| **Number of moles** | $D = \mathcal{D}$ **Molecular diffusivity** | $[\boldsymbol{L^2 \cdot T^{-1}}]$ |
| **Quantity of heat** | $D = \alpha = \frac{\lambda}{\rho C_p}$ **Thermal diffusivity** | $[\boldsymbol{L^2 \cdot T^{-1}}]$ |
| **Momentum** | $D = \nu = \frac{\mu}{\rho}$ **Kinematic viscosity** | $[\boldsymbol{L^2 \cdot T^{-1}}]$ |

### I.3 General form of the balance equation (conservation equation) of an extensity

A balance is generally written within a volume $V$ delimited by a closed contour of area $S$, within the space where the diffusive transport of the considered extent occurs. This is called the global balance, and the resulting equation is an integral equation (global equation).

However, it can also be written within an element $dV$ inside $V$. This is the local balance. A differential equation (local equation) is obtained. Solving this equation allows us to determine the evolution of the corresponding potential along the diffusion axis. Very often in process engineering, the contour delimiting the control volume $V$ (or $dV$) consists of an inlet section and an outlet section. This is referred to as a single-inlet, single-outlet system. This is the case that interests us here. The balance equation reflects the principle of conservation of extensity in the system under consideration:

**Flow rate of extensity entering the system** + **(+) Source or (-) well** = **Flow rate exiting the system** + **Accumulation flow rate in the system**

Consider a stationary and quiet phase, in a cylindrical geometry for example, the site of diffusive transport of the extent $G$ (quantity of matter, heat). Momentum is not considered here. Let's write the balance of this extent in the element $dV = \Omega\, dx$, located between $\boldsymbol{x}$ and $\boldsymbol{x+dx}$:

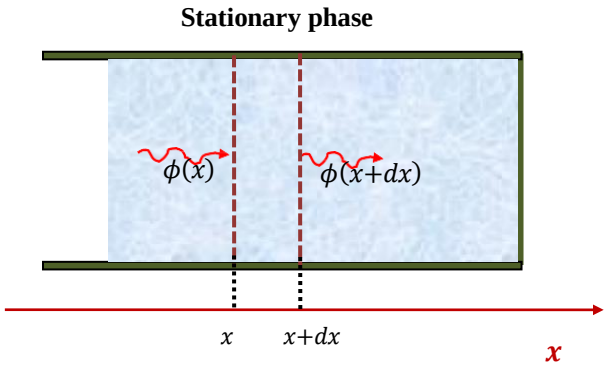



**Figure I. 7** The conservation equation of an extensity for a cylindrical geometry

$$\boldsymbol{\phi(x).\Omega + 0 = \phi(x+dx).\Omega + \frac{\partial}{\partial t}(g.\Omega dx)} \qquad \ldots(\mathbf{I.11})$$

↓ neither spring nor well

Who gives :

$$\boldsymbol{\frac{\partial \phi}{\partial x} + \frac{\partial g}{\partial t} = 0} \qquad \ldots(\mathbf{I.12})$$

Substituting $\phi$ with its expression in terms of $g$: $\phi = -D \cdot \frac{\partial g}{\partial x}$, we obtain another form of the equation (I.12) :

$$\frac{\partial g}{\partial t} - D\,\frac{\partial^2 g}{\partial x^2} = 0 \qquad \ldots (\mathbf{I.13})$$

Integrating equation (I.13) requires one initial condition and two boundary conditions with respect to $g$ to obtain the volume density distribution of the extent $G$ : $\boldsymbol{g(x,t)}$. The other form of equation (I.12) is obtained by replacing $\phi$ and $g$ with their respective expressions in terms of $V$: $\phi = -\lambda \cdot \frac{\partial V}{\partial x}$, and $g = c \cdot V$ :

$$\frac{\partial V}{\partial t} - D\,\frac{\partial^2 V}{\partial x^2} = 0 \qquad \ldots (\mathbf{I.14})$$

Integrating equation (I.14) gives the potential distribution: $\boldsymbol{V(x,t)}$. In the case of diffusive transport in steady state, the accumulation rate is zero and equation (I.12) becomes:

$$\frac{d\phi}{dx} = 0 \qquad \ldots (\mathbf{I.15})$$

Equation (I.15) indicates that the flux density $\phi$ of the extensity $G$ remains constant, independent of x in steady state, and that $\boldsymbol{g}$ and $\boldsymbol{V}$ are linear with x.

In the case of three-directional diffusive transport, the local balance is written on a parallelepiped-shaped element $dV$ of dimensions $dx$, $dy$, $dz$, conceptually cut out of the medium. After rearrangement, we obtain:

$$\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} + \frac{\partial g}{\partial t} = 0 \qquad \ldots (\mathbf{I.16})$$

Or in vector form :

$$div\,\vec{\phi} + \frac{\partial g}{\partial t} = 0 \qquad \ldots (\mathbf{I.17})$$

In steady state, equation (I.17) takes the form

$$div\,\vec{\phi} = 0 \qquad \ldots (\mathbf{I.18})$$

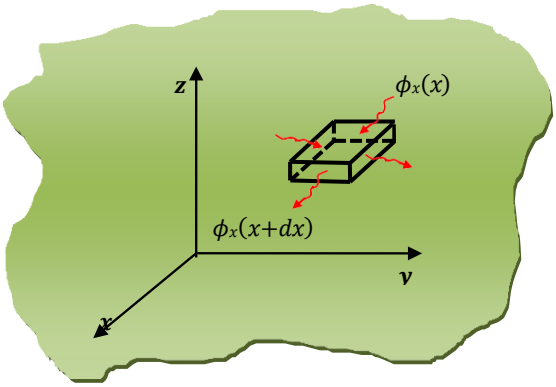



**Figure I. 8** Three-directional diffusive transport

**Example :** Unidirectional diffusive transport (along the x-axis)

**Material:** $-\mathcal{D}\ \frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial t} = 0$ ⟵ **Fick's 2nd Law**

**Heat:** $-\alpha\ \frac{\partial^2 T}{\partial x^2} + \frac{\partial T}{\partial t} = 0$ ⟵ **Heat equation**

# Chapter II :
# Heat Transfer

## II.1 Review of Heat Transfer

### II.1.1 Definition heat transfer

Heat transfer, or thermal transfer, between two bodies is an energy interaction resulting from a temperature difference between them.

### II.1.2 Temperature

Temperature is a state variable that characterizes the degree of particle agitation. The temperature of a body is a measure of its molecular agitation. At the macroscopic level, this agitation is simply thermal energy, more commonly known as heat.

### II.1.3 Heat

Heat is a form of energy that flows under the effect of a temperature difference from high to low temperatures.

- **Notes on the concept of heat**

- Heat is a form of energy; it is a gain of thermal energy, which results in an increase in thermal agitation, or a loss of energy, which results in a decrease in this agitation.

- The amplitude of an atom's oscillations around its average position increases with temperature. Therefore, the greater the thermal energy (heat), the greater its measurement (temperature).

### II.1.4 Difference Between Temperature and Heat

In everyday language, the words heat and temperature are often used interchangeably: "What heat !" is an expression meaning that the temperature is high.

However, in physics, heat and temperature are two very different concepts, which are nevertheless linked by the fact that it is the temperature difference between two points in a system that is responsible for the movement of heat.

We can also express the relationship between heat and temperature by saying that a set of two bodies is in thermal equilibrium when these two bodies are at the same temperature, that is, they no longer exchange heat.
We will also see that the units are not the same.

- Temperature T in °C or K.
- Heat Q, in J or calories (cal) (1 cal = 4,18 J).

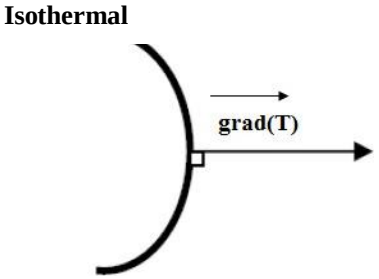
### II.1.5 Temperature field

Energy transfers are determined from the evolution of temperature in space and time:
T = f(x, y, z, t). The instantaneous value of the temperature at any point in space is a scalar called the temperature field. We will distinguish two cases:

- Temperature field independent of time: the regime is said to be steady-state or stationary.
- Temperature field changes with time: the regime is said to be variable or transient.

### II.1.6 Temperature Gradient

If we connect all points in space that have the same temperature, we obtain a surface called an isothermal surface. The temperature variation per unit length is greatest along the normal to the isothermal surface. This variation is characterized by the temperature gradient:




$$\overrightarrow{grad}(T) = \vec{n}\frac{\partial T}{\partial n} \qquad \dots (\mathbf{II.1})$$

$\vec{n}$ : unit vector of the normal

$\frac{\partial T}{\partial n}$ : derivative of the temperature along the normal

### II.1.7 Heat flux

Heat flows under the influence of a temperature gradient from high to low temperatures. The amount of heat transferred per unit of time and per unit area of the isothermal surface is called **the heat flux density**:

$$\phi = \frac{1}{S} \cdot \frac{dQ}{dt} \qquad \dots (\mathbf{II.2})$$

Where

S is the area of the surface (m²).

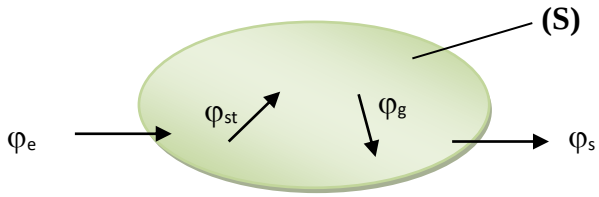
The **heat flux** is defined as the amount of heat transferred across the surface S per unit of time.

$$\boldsymbol{\varphi = \frac{dQ}{dt}} \qquad \text{... (II. 3)}$$

## II.2 Formulation of a heat transfer problem

### II.2.1 Energy balance

First, a system (S) must be defined by its spatial boundaries, and then an inventory of the different heat fluxes that influence the state of the system must be established, which can be:




$$\boldsymbol{\varphi_e + \varphi_g = \varphi_s + \varphi_{st}} \qquad \text{... (II. 4)}$$

**Energy system and balance**

$\varphi_{st}$: Stored heat flux
$\varphi_g$: Heat flux generated
$\varphi_e$: Incoming heat flux
$\varphi_s$: Outgoing heat flux
} in the system (S)

### II.2.2 Expression of energy fluxes

Next, the expressions for the different energy fluxes must be established. Substituting these expressions into the energy balance, we obtain the differential equation, the solution of which allows us to determine the temperature evolution at each point in the system.

#### II.2.2.1 Energy storage

Energy storage in a body corresponds to an increase in its internal energy over time, (at constant pressure and in the absence of a change of state):

$$\boldsymbol{\varphi_{st} = \rho V c \frac{\partial T}{\partial t}} \qquad \text{... (II. 5)}$$

With

$\varphi_{st}$: Stored heat flux (W)

$\rho$: Density (kg $m^{-3}$), V: Volume ($m^3$); c: Specific heat (J $kg^{-1}$ °$C^{-1}$)

T: Temperature (°C); t: Time (s)

The product $\rho Vc$ is called the thermal capacitance of the body.

### II.2.2.2 Energy Generation

This occurs when another form of energy (chemical, electrical, mechanical, nuclear) is converted into thermal energy. It can be written in the form:

$$\boldsymbol{\varphi_g = \dot{q}\, V} \qquad \text{... (II. 6)}$$

With

$\varphi_g$ : Generated thermal energy flux (W)

$\dot{q}$ : Generated energy density (W $m^{-3}$); V: Volume ($m^3$)

## II.3 Modes of heat transfer

Three modes of heat transfer are generally distinguished:

**1.** Thermal conduction

**2.** Convection

**3.** Thermal radiation

### II.3.1 Conduction

The high molecular agitation of the hot zone transfers kinetic energy to the cooler zones through a phenomenon called heat conduction. Conduction is a diffusion process that allows heat to propagate within a solid body without any appreciable displacement of matter.

In most cases, conduction is studied in solid media, since in fluid media, it is often coupled with a movement of matter and therefore a convection mechanism.

Conduction occurs progressively:

If the end of a solid is heated, there is a gradual transfer of heat.

**Example:** Metal bar heated at one of its ends.

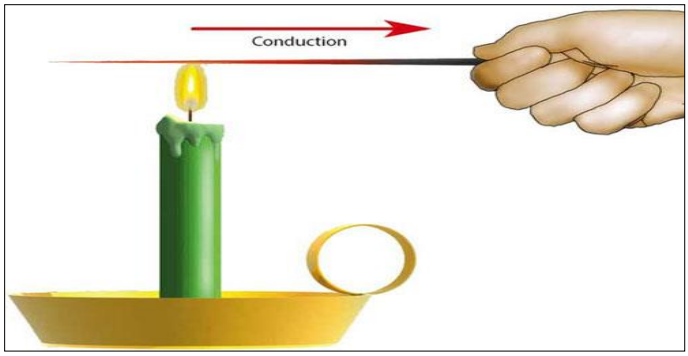
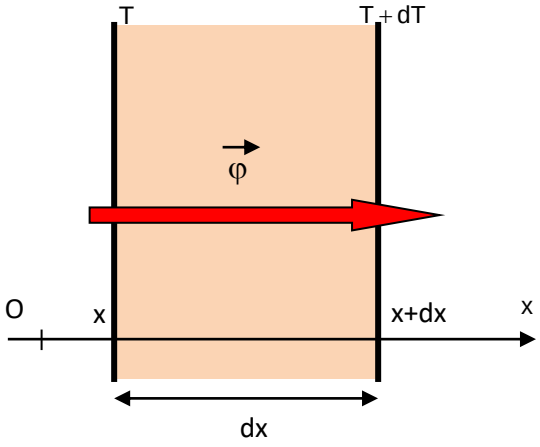



**Figure II. 1** Example of heat transfer by conduction

Heat exchange by conduction is defined by **Fourier's law** (1822); the instantaneous value of the quantity of heat transferred per unit of time is proportional to:

- The cross-sectional area perpendicularly traversed by the heat flow.
- The thermal conductivity of the material traversed, which depends on the physical properties of the material.
- The temperature gradient, which is the ratio of the temperature change to the distance traveled by the heat flow.



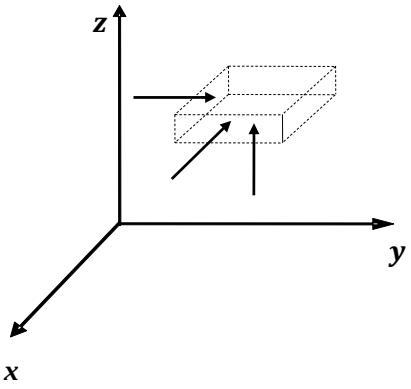

$$\frac{dQ}{dt} = -\lambda . S \frac{dT}{dx} \qquad \dots (II.7)$$

Unidirectional transfer

$$\varphi = -\lambda . S \frac{\partial T}{\partial x} \qquad \dots (II.8)$$

In a general case of a three-dimensional mechanism, we will express a heat flux along each of the principal directions of an orthonormal frame (O x,y,z).

If the medium is isotropic : $\boldsymbol{\lambda_x = \lambda_y = \lambda_z}$




Following **Ox** $\quad \boldsymbol{\varphi_x = -\lambda . S \frac{\partial T}{\partial x}}$

Following **Oy** $\quad \boldsymbol{\varphi_y = -\lambda . S \frac{\partial T}{\partial y}}$

Following **Oz** $\quad \boldsymbol{\varphi_z = -\lambda . S \frac{\partial T}{\partial z}}$

Or

$$\vec{\varphi} = -\lambda . S \overrightarrow{grad}(T) \qquad \text{... (II. 9)}$$

With

dQ : quantity of heat (joule).

φ: Heat flux transmitted by conduction (W)

dt : time (s)

dT : temperature (°C)

dx : thickness of the material through which the heat flow passes (m)

S : section traversed by the heat flow ($m^2$)

λ : thermal conductivity coefficient of the material (W $m^{-1}$ °$C^{-1}$).

We see that the temperature gradient (dT/dx) is negative in the equation above: this means that the heat flow goes from hot to cold, the sign of the gradient is opposite to that of the flow.

### II.3.1.1 Steady-state heat transfer by conduction

#### II.3.1.1.1 The heat equation

In its one-dimensional form, it describes unidirectional heat transfer through a flat wall:

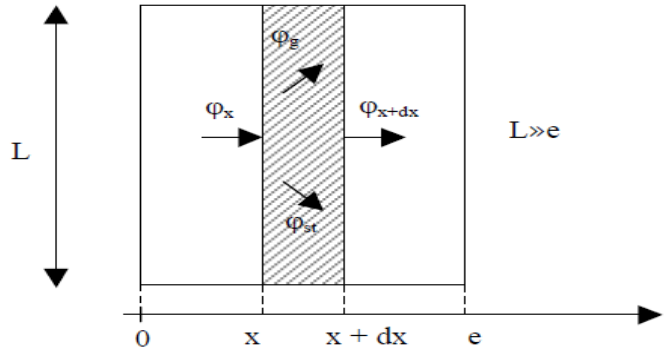



**Figure II. 2** Heat balance on an elementary system

Consider a system of thickness ***dx*** in the x direction and of cross-sectional area S perpendicular to the Ox direction. The energy balance on this system is written :

$$\boldsymbol{\varphi_x + \varphi_g = \varphi_{x+dx} + \varphi_{st}} \qquad \ldots (\mathbf{II.10})$$

$$\varphi_x = -\left(\lambda . S \frac{\partial T}{\partial x}\right)_x \qquad \varphi_{x+dx} = -\left(\lambda . S \frac{\partial T}{\partial x}\right)_{x+dx}$$

$$\varphi_g = \dot{q}\, S dx \qquad \varphi_{st} = \rho\, c\, S dx \frac{\partial T}{\partial t}$$

Substituting this into the energy balance and dividing by dx, we obtain:

$$\frac{\left(\lambda . S \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda . S \frac{\partial T}{\partial x}\right)_x}{dx} + \dot{q}\, S = \rho\, c\, S \frac{\partial T}{\partial t}$$

Either :

$$\frac{\partial}{\partial x}\left(\lambda . S \frac{\partial T}{\partial x}\right) + \dot{q}\, S = \rho\, c\, S \frac{\partial T}{\partial t} \qquad \ldots (\mathbf{II.11})$$

And in the three-dimensional case, we obtain **the heat equation** in the most general case:

$$\frac{\partial}{\partial x}\left(\lambda_x . \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y . \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z . \frac{\partial T}{\partial z}\right) + \dot{q} = \rho\, c \frac{\partial T}{\partial t} \qquad \ldots (\mathbf{II.12})$$

This equation can be simplified in several cases:

a) If the medium is isotropic: $\lambda x = \lambda y = \lambda z = \lambda$

b) If there is no energy generation within the system: $\dot{q} = 0$

c) If the medium is homogeneous, $\lambda$ is a function only of T.

Hypotheses a) + b) + c) allow us to write:

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{d\lambda}{\partial T}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2+\left(\frac{\partial T}{\partial z}\right)^2\right]=\rho\, c\ \frac{\partial T}{\partial t} \qquad \ldots(\mathbf{II.13})$$

d) If, in addition, λ is constant, we obtain **the Poisson equation**:

$$a\,\nabla^2 T=\frac{\partial T}{\partial t} \qquad \ldots(\mathbf{II.14})$$

The ratio $a=\frac{\lambda}{\rho\, c}$ is called **thermal diffusivity** ($m^2.s^{-1}$) which characterizes the speed at which heat flows through a material.

e) In steady state, we obtain **Laplace's equation**:

$$\nabla^2 T=0 \qquad \ldots(\mathbf{II.15})$$

### II.3.1.1.2 Example of applications of the Fourier equation (Simple wall)

We will consider the case where heat transfer is unidirectional and there is no energy generation or storage. We consider a wall of thickness e, thermal conductivity λ, and large transverse dimensions, whose extreme faces are at temperatures $T_1$ and $T_2$:

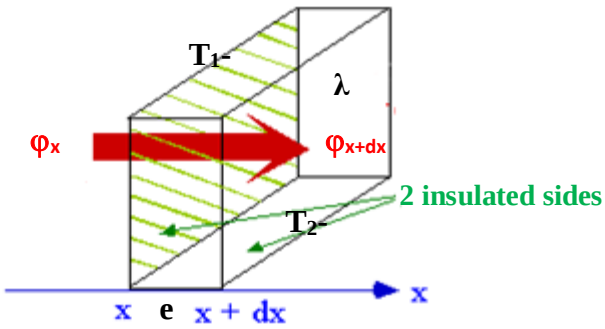



**Figure II. 3** Basic thermal balance on a simple wall

By performing a thermal balance on the elementary system consisting of the wall section between abscissas x and x + dx, we obtain:

$$\boldsymbol{\varphi_x=\varphi_{x+dx}} \Longrightarrow -\lambda\,.\,S\left(\frac{dT}{dx}\right)_x=-\lambda\,.\,S\left(\frac{dT}{dx}\right)_{x+dx}$$

Hence : $\frac{dT}{dx}=A\ \ and\ T(x)=Ax+B$

With the boundary conditions: $T(x=0)=T_1\ \ and\ \ T(x=e)=T_2$

$$\text{Hence}: T = T_1 - \frac{x}{e}(T_1 - T_2) \qquad \ldots (\mathbf{II.16})$$

The temperature profile is therefore linear. The heat flux density through the wall can be deduced from the following relationship:

$$\phi = -\lambda \frac{dT}{dx} \implies$$

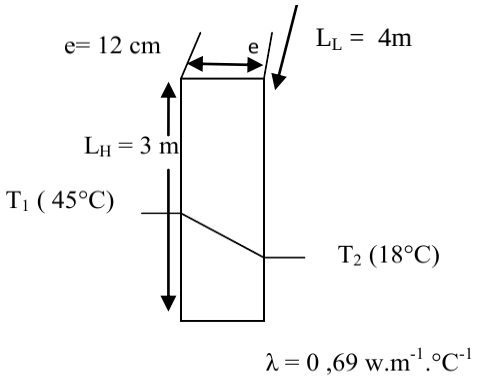
$$\phi = \lambda \frac{(T_1 - T_2)}{e} \qquad \ldots (\mathbf{II.17})$$

The heat fluid relationship

$$\varphi = \lambda . S \frac{(T_1 - T_2)}{e} \qquad \ldots (\mathbf{II.18})$$

***Example:*** Calculate the heat loss through a brick wall 12 cm thick, 3 m high, and 4 m wide. The temperatures of the two faces of the wall are 45°C and 18°C, respectively. ($\lambda$ = 0.69 $W.m^{-1}.°C^{-1}$).

**Correction :**




$\varphi$ is the steady-state conduction heat flow:

$$\varphi = -\lambda . S \frac{dT}{dx} \implies \varphi . dx = -\lambda . S . dT \implies \varphi \int_0^e dx = \int_{T1}^{T2} -\lambda . S . dT$$

$$\varphi . e = \lambda S (T_1 - T_2)$$

$$\varphi = \frac{\lambda S (T_1 - T_2)}{e}$$

$$\varphi = \frac{0.69(45 - 18) \times 3 \times 4}{0.12} = 1863W$$

### II.3.2 Convection

This is the heat transfer between a solid and a fluid, with energy being transmitted by the movement of the fluid. The underlying cause is still the agitation of fluid particles, but on a much less microscopic scale. The particles of matter in contact with the hot surface heat up by conduction; the movement of the fluid carries these particles back into the mass where they release some of the heat they received through mixing; other particles replace them at the surface, and so on.

Two types of convective heat transfer can be distinguished:

- **Forced convection**, in which the fluid flow is forced from the outside by a machine (pump, compressor, fan, etc.).
- **Natural convection**, in which the fluid movement is created by density differences, themselves caused by temperature differences due to the contact of the fluid with a warmer or colder surface.

#### II.3.2.1 Mechanism of convection

When a fluid comes into contact with a hot surface, its density changes, generating convection currents along the surface. This causes the hot area to move, thus transferring heat.

This mode of heat transfer can be described by a process involving the following three steps:

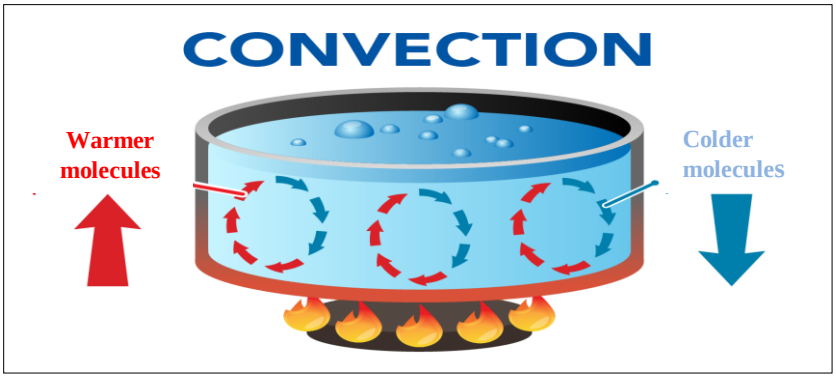



**Figure II. 4** Heat transfer process by convection

a) Contact of fluid particles with the wall to allow heat exchange

b) Transfer of heat from the wall to the particles with which it is in contact

c) Return of the particle to the fluid and exchange of heat with other particles.

### II.3.2.2 Flow regime

Two fluid flow regimes are distinguished: laminar and turbulent. In laminar flow, the fluid layers have regular trajectories and slide over one another without particles passing from one layer to another.

At high speeds, the movement of particles is chaotic, and the regime is called turbulent.

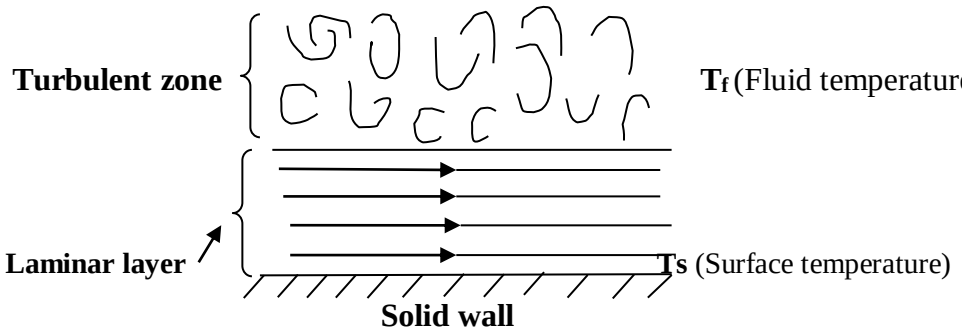



**Figure II. 5** Schematic representation of flow regimes

### II.3.2.3 Law of heat transfer by convection

#### II.3.2.3.1 Equation de Newton

Regardless of the type of convection (free or forced) and regardless of the fluid flow regime (laminar or turbulent), the heat flux by convection along a surface S, at surface temperature $T_s$, and immersed in an ambient medium at $T_f$, is expressed by the relation, known as **Newton's law**:

$$\boldsymbol{\varphi = h.S(T_S - T_f)} \qquad \textbf{... (II. 19)}$$

With

$\varphi$ : heat flux by convection (kcal/h, watt)

h : heat exchange coefficient (Kcal/h.m$^2$.°C, W.m$^{-2}$.K$^{-1}$)

S : exchange surface area (m$^2$) where the contact surface between the fluid and the solid

($T_s$-$T_f$) : temperature difference between the surface of the solid $T_s$ and that of the fluid $T_f$ far from the surface (°C).

h.S = $k_c$ thermal conductance by convection

1/h.S = $R_C$ thermal resistance by convection

### II.3.2.3.2 The overall heat transfer coefficient

Heat transfer coefficient ***(K)*** is a coefficient that quantifies the energy flow through a medium per unit area, volume, or length. It is the inverse of thermal resistance. It is an important term in the equation of heat transfer and indicates how easily thermal energy passes through an obstacle or medium. An overall heat transfer coefficient, **K** ($Wm^{-2}K^{-1}$), is calculated when heat flows through different layers, with both convective and conductive transfer occurring.

- **Example for the case of a flat wall composed**

This is the case for real walls made of several layers of different materials, where only the temperatures $T_{f1}$ and $T_{f2}$ of the fluids in contact with the two faces of the wall with lateral surface area S are known.

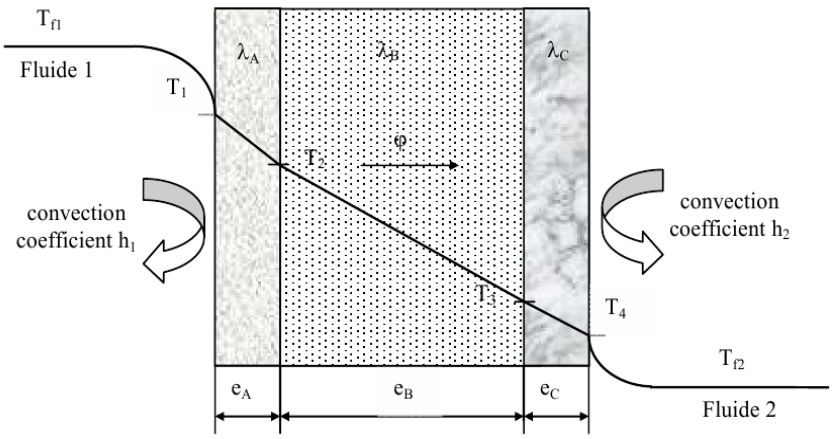



**Figure II. 6** Diagram of heat flow and temperature in a multilayer wall

$$\boldsymbol{\varphi} = h_1.S(T_{f1} - T_1) = \lambda_A.S\frac{(T_1 - T_2)}{e_A} = \lambda_B.S\frac{(T_2 - T_3)}{e_B} = \lambda_C.S\frac{(T_3 - T_4)}{e_C} = h_2.S(T_4 - T_{f2})$$

Hence :

$$\boldsymbol{\varphi} = \frac{(\mathbf{T_{f1} - T_{f2}})}{\frac{\mathbf{1}}{\mathbf{h_1.S}} + \frac{\mathbf{e_A}}{\boldsymbol{\lambda_A}.\mathbf{S}} + \frac{\mathbf{e_B}}{\boldsymbol{\lambda_B}.\mathbf{S}} + \frac{\mathbf{e_C}}{\boldsymbol{\lambda_C}.\mathbf{S}} + \frac{\mathbf{1}}{\mathbf{h_2.S}}} \qquad \ldots (\mathbf{II.20})$$

The overall transfer coefficient K is defined by the following relationship:

$$\frac{1}{K} = \frac{1}{h_1} + \frac{e_A}{\lambda_A} + \frac{e_B}{\lambda_B} + \frac{e_C}{\lambda_C} + \cdots + \frac{1}{h_2}$$

$e_A$, $e_B$, $e_C$…: thicknesses and $\lambda_A$, $\lambda_B$, $\lambda_C$…: conductivities.

$$K = \frac{1}{\frac{1}{h_1} + \sum_i \frac{e_i}{\lambda_i} + \frac{1}{h_2}} \qquad \text{... (II. 21)}$$

***Application example:***

The wall of a furnace consists of three layers of different materials placed side by side (in series):

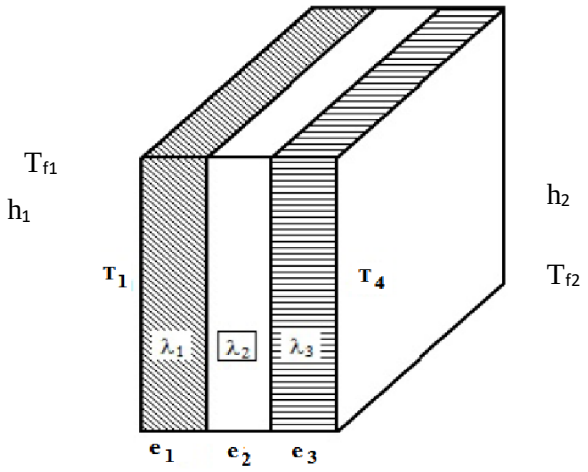
- A layer of refractory bricks ($\lambda$ = 1.21 W/m.°C)
- A layer of heat-insulating lining ($\lambda$ = 0.08 W/m.°C)
- A layer of bricks ($\lambda$ = 0.69 W/m.°C).

Each layer is 10 cm thick. The temperature is 872°C inside the kiln and 32°C outside. The corresponding surface heat transfer coefficients are $h_1$ = 100 W/m²°C and $h_2$ = 200 W/m²°C.

1. If the wall surface area is 42 m², calculate the heat loss by conduction over 24 hours in joules.
2. What is the temperature Tm at the center of the cladding?

**Answer:**

1) Calculation of heat loss



Heat transfer through fluid 1 occurs by convection.

$$\varphi = h_1 S(T_{f1} - T_1)$$

Is constant φ Conduction through a composite wall in steady state is given by Fourier's law:

$$\varphi = \phi . S = -\lambda . S \frac{dT}{dx}$$

For each thickness, we integrate the heat flux equation for each constant value of λ.

$$\Longrightarrow \varphi \int_0^e dx = \int_{T1}^{T2} -\lambda_1 S dT$$

The regime is steady, φ is constant, we can write :

$$\varphi . e_1 = \lambda_1 S(T_1 - T_2)$$

$$\varphi = \frac{(T_1 - T_2)}{\frac{e_1}{\lambda_1 S}}$$

The same applies to the other walls.

$$\varphi = \frac{(T_2 - T_3)}{\frac{e_2}{\lambda_2 S}} \quad and \quad \varphi = \frac{(T_3 - T_4)}{\frac{e_3}{\lambda_3 S}}$$

Heat transfer through fluid 2 occurs by convection.

$$\varphi = h_2 S(T_4 - T_{f2})$$

Eventually

$$\varphi = \frac{(T_{f1} - T_{f2})}{\frac{1}{h_1 S} + \frac{e_1}{\lambda_1 S} + \frac{e_2}{\lambda_2 S} + \frac{e_3}{\lambda_3 S} + \frac{1}{h_2 S}}$$

$$\varphi = \frac{(872 - 32)}{\frac{1}{100 \times 42} + \frac{0.1}{1.21 \times 42} + \frac{0.1}{0.08 \times 42} + \frac{0.1}{0.69 \times 42} + \frac{1}{200 \times 42}}, \varphi = 23.68 \text{ KW}$$

The heat loss by conduction over 24 hours

$$\varphi = 24 \times 23.68 = 568{,}32\ KW.h$$

$\varphi = 568{,}32 \times 3600 \times 1000 = 2.05 \times 10^9$ J

2) To calculate $T_m$ it is imperative to calculate $T_2$ and $T_3$ because $T_m$ is between the two temperatures.

$$T_2 = T_1 - \varphi\frac{e_1}{\lambda_1 S} \implies T_2 = 825°C$$

$$T_3 = T_4 - \varphi\frac{e_3}{\lambda_3 S} \Longrightarrow T_3 = 114.4°C$$

$$T_m = \frac{T_2 + T_3}{2} \Longrightarrow T_m = 469.7°C$$

### II.3.3 Radiation

This corresponds to the propagation of photons. The electromagnetic energy received by the system is absorbed and converted into thermal energy (heat). Thermal radiation can be considered a special case of electromagnetic radiation. The simplest example is solar radiation. Thermal radiation is the mode of transmission by which heat passes from a high-temperature body to a cooler one without the need for a material medium. It is therefore the only mode of heat transfer that can propagate through a vacuum. Thermal radiation is a surface phenomenon.

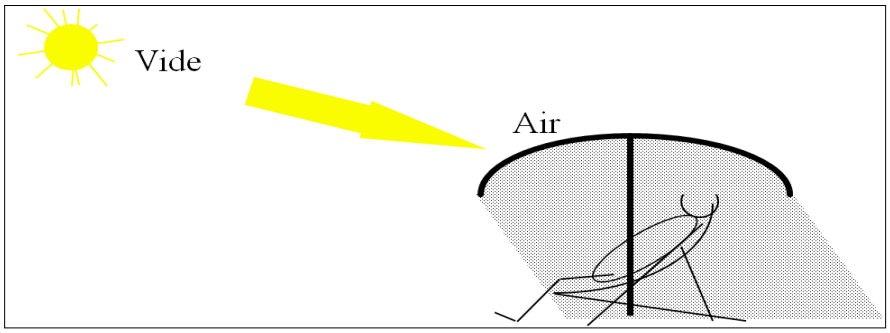



**Figure II. 7** Example of heat transfer by radiation

#### II.3.3.1 General information on heat transfer by radiation

- The mechanisms of heat transfer by conduction and convection require the presence of matter to transport thermal energy. This is not the case for heat transfer by radiation, where thermal energy is transmitted without a material medium by waves similar to those of light.

- This mode of transmission is significant when dealing with materials heated to high temperatures - above 700°C. It particularly concerns flames (furnaces, boilers, torches), and refractories lining combustion chambers.
- These bodies, due to the thermal agitation of their constituent particles (molecules and atoms), emit radiation that carries thermal energy. This radiation can be partially absorbed by the surface of a receiving body a furnace tube, for example resulting in heat absorption for the latter.
- Two main factors determine heat transmission by radiation:
  - Characteristics of the emitter : surface area, temperature, and material type
  - Characteristics of the receiver.

### II.3.3.2 Characteristics of radiant heat emission

Radiant heat emission is generally characterized by:

- The heat output rate φ (W or kcal/h)
- The external surface area of the emitter A ($m^2$).

We therefore define a heat flux density emitted by the relation:

**ϕ = Heat output rate / External emitting surface**

$$\boldsymbol{\phi = \frac{\varphi}{S}}$$

Apart from the surface area, the heat output also depends on:

- The temperature
- The nature of the emitter material.

We know that the heat transfer rate increases with temperature. At the same time, not all materials have the same heat emission capacity at the same temperature. To characterize this capacity, we define an ideal material that constitutes a **perfect emitter**, called a **black body**. The performance of other materials is then defined relative to that of the black body.

### II.3.3.3 Blackbody heat emission

A black body is a body that absorbs all the radiation it receives regardless of its thickness, temperature, angle of incidence and wavelength of the incident radiation; it is defined by the absorbing power: $\boldsymbol{\alpha_{\lambda T} = 1}$.

- ***Properties of the black body***

- All black bodies radiate in the same way.

- The black body radiates more than the non-black body at the same temperature.

- ***STEFAN-BOLTZMANN law***

The emission from a black body is governed by a physical law which shows that the emitted heat flux density is proportional to the fourth power of its absolute temperature: the **STEFAN-BOLTZMANN law**. This law is written as:

$$\boldsymbol{\phi = \sigma.\left(\frac{T}{100}\right)^4} \qquad \ldots (\mathbf{II.22})$$

$$\text{Or} \quad \boldsymbol{\varphi = \sigma.S.\left(\frac{T}{100}\right)^4} \qquad \ldots (\mathbf{II.23})$$

With

φ: the heat flux emitted by the black body (W)

ϕ: the flux density emitted by a black body (W/m²)

T: black body temperature (K)$^4$

σ: Stefan-Boltzmann constant, whose value depends on the system of units chosen to express the heat flux : $\sigma = 5.67 \times 10^{-8}$ W/m² K$^4$

$\sigma = 4.88$ Kcal/h.m$^2$ K$^4$

S : body surface (m²)

### II.3.3.4 Thermal Emission of Any Body

In practice, it is observed that the emission of materials is always less than that of a black body at the same temperature. To characterize their emission capacity, the **emission factor** or **emissivity (ε)** is defined as the ratio of the heat flux they emit to that of a black body at the same temperature.

**ε = Flux emitted by a material at T / Flux emitted by the black body at T**

High emissivity values characterize good emitters.

**Table II. 1** Emissivity for some materials

| **Materials** | **Emissivity** |
|---|---|
| Polished silver | 0,04 |
| Aluminum paints | 0,3 à 0,5 |
| New steels | 0,5 à 0,6 |
| Oxidized steels | 0,8 à 0,9 |
| Refractory insulators | 0,75 à 0,90 |
| Carbon black | 0,95 à 0,98 |
| Black body | 1,00 (by definition) |

Applying the definition of emissivity and Stefan-Boltzmann law, the heat flux emitted by any material can be written as:

$$\boldsymbol{\varphi = \sigma . \varepsilon . S \left(\frac{T}{100}\right)^4} \qquad \ldots (\mathbf{II.23})$$

ε : A factor that takes into account the emissivity factors of both bodies and their relative geometries. For a black body, $\boldsymbol{\varepsilon = 1}$

### II.3.3.5 Characteristics of heat absorption by radiation

Generally speaking, when an object is exposed to radiation, the energy it receives (incident energy) can be:

**Absorbed:** this results in the exposed object acquiring heat.

**Reflected:** that is, sent back by the object, as is the case with light striking a mirror.

**Transmitted:** through the object if it is transparent.

These different possibilities are represented in the following diagram:

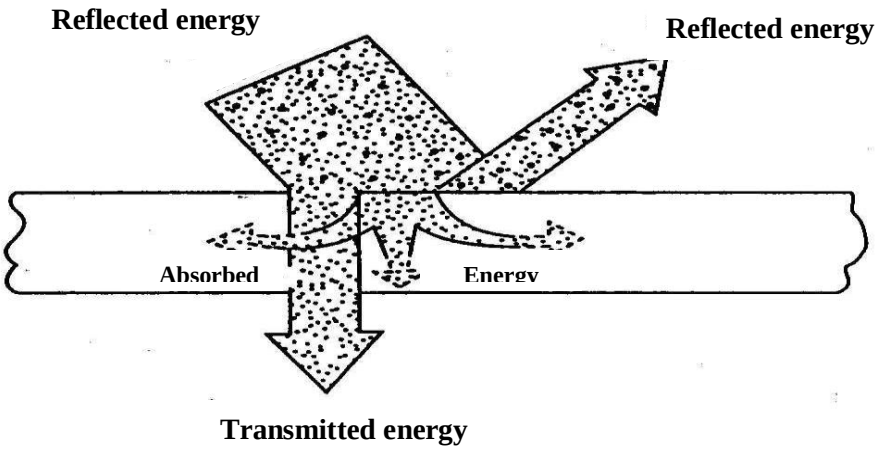



**Figure II. 8** Body exposed to radiation

To characterize a material's ability to absorb radiation, the **absorption coefficient** (**α**) is defined as the ratio of the absorbed heat rate to the incident heat rate:

**α = Absorbed heat rate / Incident heat rate**

In practice, we find that good emitters are good receivers of radiation, which means that materials with high emissivity have high absorption coefficient values.

Radiation incident on a surface is either absorbed, reflected, or transmitted.

Let $\varphi_i$ be the incident flux on a surface, $\varphi_a$ the absorbed flux, $\varphi_r$ the reflected flux, and $\varphi_t$ the transmitted flux.

Absorbance: $\alpha = \varphi_a / \varphi_i$

Reflectance: $\rho = \varphi_r / \varphi_i$

Transmittance: $\tau = \varphi_t / \varphi_i$

Conservation of energy principle : $\boldsymbol{\varphi_i = \varphi_a + \varphi_r + \varphi_t}$

and $\boldsymbol{\alpha + \rho + \tau = 1}$

# Chapter III :

# Mass Transfer

## III.1 Introduction

As soon as a chemical potential gradient of a species (a concentration gradient and/or pressure difference.) is present within a system, a movement of that species occurs in order to shift the chemical potential (concentration and/or pressure difference.) distribution towards the state in accordance with the principles of thermodynamics. This process is called **mass transfer**.

## III.2 Diffusive mass transfer (Matter transfer by diffusion)

If there is a difference in concentration in space, for example between two compartments separated by a permeable membrane (allowing the solvent and solute to pass through), there will be, over time, a movement of matter from the compartment with higher concentration to the one with lower concentration (see diagram). This transfer continues until concentration equilibrium is reached. This spontaneous phenomenon is called **diffusion**.

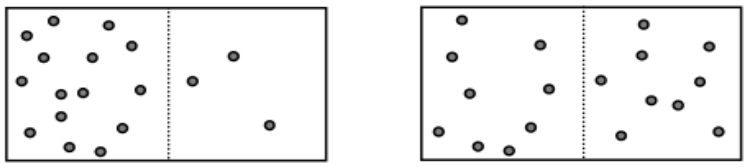


**Figure III. 1** Diffusion from the more concentrated medium to the less concentrated medium

### III.2.1 Definition of molecular diffusion

Diffusion is the microscopic transport (migration) of mass, such as atoms, molecules, and ions, within a medium under the influence of concentration differences, from areas of high concentration to areas of low concentration. It refers to the natural tendency of a system to homogenize the concentrations of chemical species within it. The phenomenon of mass transfer by diffusion occurs whenever there is a concentration difference. It is a spontaneous, irreversible, and unavoidable phenomenon (dispersive phenomenon) leading to equilibrium, occurring as a result of the thermal agitation of molecules. The kinetics of the phenomenon are quantified by diffusivity, also called the **diffusion coefficient D**.

### III.2.2 Fick's law

If a small amount of a volatile liquid is placed at the bottom of a test tube and left in contact with a stream of dry air, as illustrated in the diagram below, a linear concentration profile is established in the test tube at a steady state as time approaches infinity, and continuous evaporation of the liquid will occur. This phenomenon, in which mass transfer is caused by a non-uniform concentration distribution, is called diffusion.

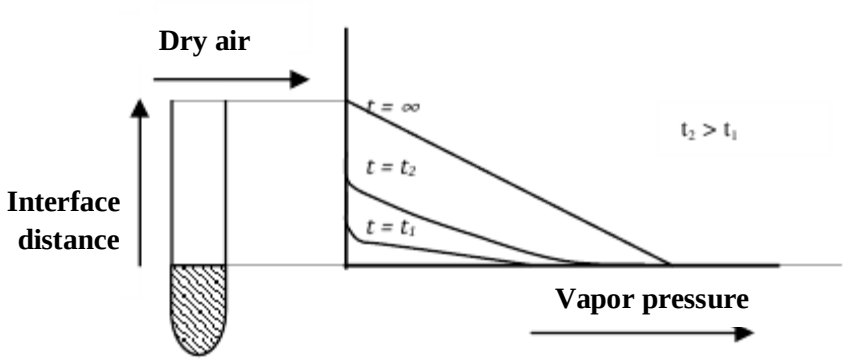



**Figure III. 2** Diffusion of a vapor in a gas

Indeed, mass transfer by diffusion is the transport mechanism of a constituent in a stagnant medium, due to a deviation in the spatial distribution of concentration from equilibrium. This diffusive molar flux of matter is proportional to the concentration gradient and to the surface area perpendicular to the direction of particle movement in the fluid.

In the general case and in a stationary medium, the transfer of matter by molecular diffusion of a species **A** in a fluid **B** is described by **Fick's first law**:

$$\boldsymbol{\phi_{Ay} = -\mathcal{D}_{AB}\ \frac{dC_A}{dy}} \qquad \text{... (III. 1)}$$

The subscript A designates the chemical species that diffuses, while y indicates the direction in which transport occurs.

In this case, $\boldsymbol{\phi_{Ay}}$ : the molar flux density transferred by diffusion,

$\mathbf{D_{AB}}$: the molecular diffusion coefficient (or molecular diffusivity) of species A within fluid B, ($m^2\ s^{-1}$).

$\mathbf{C_A}$: the concentration of species A, ($mol\ m^{-3}$).

$\frac{\mathbf{dC_A}}{\mathbf{dy}}$ : concentration gradient of A, along the diffusion axis y.

The minus sign (-) reminds us that matter diffuses from areas of high intensity (here of high concentration) to areas of low intensity, that is to say in the opposite direction to the gradient.

Writing Fick's law for three-way transport

$$\boldsymbol{\phi_{Ax}} = -\boldsymbol{\mathcal{D}_{AB}}\ \frac{\boldsymbol{\partial C_A}}{\boldsymbol{\partial x}}$$

$$\boldsymbol{\phi_{Ay}} = -\boldsymbol{\mathcal{D}_{AB}}\ \frac{\boldsymbol{\partial C_A}}{\boldsymbol{\partial y}}$$

$$\boldsymbol{\phi_{Az}} = -\boldsymbol{\mathcal{D}_{AB}}\ \frac{\boldsymbol{\partial C_A}}{\boldsymbol{\partial z}}$$

$$\overrightarrow{\boldsymbol{\phi_A}} = -\boldsymbol{\mathcal{D}}\ .\ \overrightarrow{\boldsymbol{grad}\ C_A} \qquad \text{... (III. 2)} \qquad \textbf{Three-way Transport}$$

Fick's law can be expressed using either mole fraction or mass fraction or partial pressure as follows:

As a function of mole fractions : $\boldsymbol{\phi_{Ax}} = -\boldsymbol{\mathcal{D}_{AB}}\ \mathbf{C}\ \frac{\mathbf{dX_A}}{\mathbf{dx}}$ ... (**III. 3**)

As a function of mass fractions : $\boldsymbol{\phi_{Ax}} = -\boldsymbol{\mathcal{D}_{AB}}\ \boldsymbol{\rho}\ \frac{\mathbf{d\omega_A}}{\mathbf{dx}}$ ... (**III. 4**)

As a function of partial pressures in the case of gases:

$$\boldsymbol{\phi_{Ax}} = -\boldsymbol{\mathcal{D}_{AB}}\ \frac{\mathbf{1}}{\mathbf{RT}}\ \frac{\mathbf{dP_A}}{\mathbf{dx}} \qquad \text{... (III. 5)}$$

With :

C: the total molar concentration

$X_A$: the mole fraction of species A

$\rho$: the total mass concentration

$\omega_A$: the mass fraction of species A

$\rho_A$: the mass concentration of species A

$P_A$: the partial pressure of species A

- ***Note***

When it comes to mass transport, three types of flow are defined:

- Volumetric flow rate is the volume per unit time; [L $s^{-1}$].
- Molar flow rate is the number of moles per unit time; [mol $s^{-1}$].
- Mass flow rate is the number of grams per unit time; [kg $s^{-1}$].

### III.2.3 Representation of concentrations in molar and mass terms

Concentration is expressed in terms of density, which measures the mass occupying a unit volume. For a mixture, we have:

- Partial density: the mass of compound (i) / unit volume

$$\boldsymbol{\rho_i = \frac{m_i}{V}} \quad \text{with} \quad \boldsymbol{\sum \rho_i = \rho}$$

$\rho$: the total mass concentration

Molar concentration $\mathbf{C_i}$ : This is the number of moles of compound (i), with molar mass $M_i$, per unit volume.

$$\boldsymbol{C_i = \frac{n_i}{V} = \frac{\rho_i}{M_i}}$$

$n_i$ : the number of moles of component i, defined by the following relation:

$$\boldsymbol{n_i = \frac{m_i}{M_i}}$$

With : $M_i$ the molar mass of species i.

For an ideal gas, and according to Dalton's law, the total pressure is equal to the sum of the partial pressures. That is:

$$\boldsymbol{P_{totl} = \sum P_i}$$

$$\boldsymbol{P_i = y_i P_{totl}}$$

### III.2.4 Average molar and mass velocities

In a diffusing mixture, the elements are moving at different speeds. Let $v_i$ be the speed of element i relative to a fixed frame of reference.

The average **mass velocity** $v$ for a mixture of **n** components is defined as follows:

$$v = \frac{\sum(\rho_i v_i)}{\sum \rho_i}$$

$$v = \frac{1}{\rho}\sum(\rho_i v_i) \qquad \text{... (III. 6)}$$

Similarly, the average **molar velocity** $v^*$ could be defined as follows:

$$v^* = \frac{\sum(C_i v_i)}{\sum C_i}$$

$$v^* = \frac{1}{C}\sum(\rho_i v_i) \qquad \text{... (III. 7)}$$

### III.2.5 Mass flux density

Matter transport is quantified by a mass flux (or mass flux density), $\phi$, which represents the mass (or number of moles) that passes through a unit area (perpendicular to the direction of transport) per unit time. Its unit is $\mathbf{kg \cdot m^{-2} \cdot s^{-1}}$ or $\mathbf{mol \cdot m^{-2} \cdot s^{-1}}$.

**$\phi$:** mass flux density (number of kg or mol transferred across the surface per unit time and area).

## III.3 Convective mass transfer (Matter transfer by convection)

Consider a phase containing a solute $A$. We speak of convective transport of $A$ within the phase if the phase is mobile. The solute $A$ is transported by the macroscopic movement of the fluid (flow).

### III.3.1 Constituent velocities and barycentric velocities

Consider a mobile phase, a mixture of two components A and B (a binary mixture). It is clear that each species is transported at its own velocity.
With respect to a fixed frame of reference, the molecules of A move (in the overall motion) at velocity $\boldsymbol{u_A}$ (the average velocity of a large number of molecules), and the molecules of B move at velocity $\boldsymbol{u_B}$ (the average velocity of a large number of molecules).
From these two velocities, we define two weighted averages called barycentric velocities.

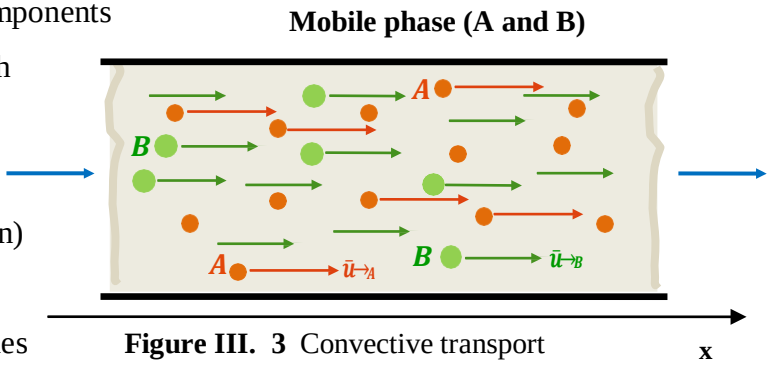



**Figure III. 3** Convective transport

The coefficients are the mass fractions for the first and the molar fractions for the second:

***Mass barycentric velocity:***

$$\boldsymbol{u = w_A u_A + w_B u_B} \qquad \text{... (III. 8)}$$

$\boldsymbol{w_A, w_B}$ : Mass fractions of constituents A and B

***Molar barycentric velocity:***

$$\boldsymbol{u^* = x_A u_A + x_B u_B} \qquad \text{... (III. 9)}$$

$\boldsymbol{x_A, x_B}$ : Molar fractions of constituents A and B

### III.3.2 Convective flux densities of the constituents

The expression for the volume $dV$ of phase that crosses the cross-section Ω during the time interval $dt$ is:

$$\mathbf{dV = \Omega . u . dt}$$

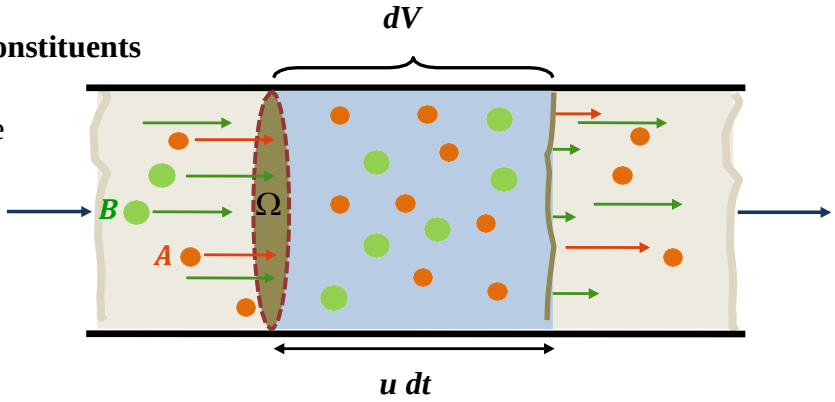



**Figure III. 4** Convective transport from A and B through the normal cross-section Ω

Let $\rho_A$ and $\rho_B$ denote the volumetric mass concentrations of $A$ and $B$ respectively in the mobile phase.

The mass of $A$ that passes through Ω during $dt$ is:

$$\mathbf{dV \times \rho_A = \Omega . u . dt \times \rho_A}$$

The mass of $A$ that crosses Ω per unit time, called the convective mass flux of $A$, is:

$$\frac{\mathbf{\Omega . u . dt \times \rho_A}}{\mathbf{dt}}$$

Either $\mathbf{\Omega . u \times \rho_A}$

The convective mass flux through a unit cross-section is called **the convective mass flux density** of A.

It is obtained by:

$$\frac{\mathbf{\Omega . u \times \rho_A}}{\mathbf{\Omega}}$$

Either

$$\mathbf{F_A = \rho_A \times u} \qquad \dots (\mathbf{III.10})$$

The same applies to the constituent B

$$\mathbf{F_B = \rho_B \times u} \qquad \dots (\mathbf{III.11})$$

The same approach is used to express **convective molar flux densities**:

The molar flux of A is the number of moles of A passing through Ω per unit time

$$\mathbf{C_A \times \Omega \ u^*}$$

$C_A$ : Molar concentration of A

Molar flux density of A = molar flux of A / Ω

$$\mathbf{F^*_A = C_A \ u^*} \qquad \dots (\mathbf{III.12})$$

The same applies to the constituent B

$$\mathbf{F^*_B = C_B \ u^*} \qquad \dots (\mathbf{III.13})$$

$C_A$ and $C_B$ being the molar volume concentrations.

### III.3.3 Relative velocities of constituents and diffusional fluxes

Molecules are in relative motion when observed in their movement relative to the mean flow (the overall motion). They therefore move with the following **relative velocities**:

For molecules of *A*: $\boldsymbol{u_A - u}$

For molecules of *B*: $\boldsymbol{u_B - u}$

The quantity of *A*, associated with this relative motion, that crosses a unit cross-section per unit time is called **the diffusional mass flux density**, denoted $\emptyset_A$. Its expression is formally identical to that of the convective flux (product of a velocity times a concentration):

$$\emptyset_A = \boldsymbol{\rho_A(u_A - u)} \qquad \text{... (III. 14)}$$

$$\emptyset_B = \boldsymbol{\rho_B(u_B - u)} \qquad \text{... (III. 15)}$$

Similarly for **the diffusional molar flux density**:

$$\emptyset^*_A = \boldsymbol{C_A(u_A - u^*)} \qquad \text{... (III. 16)}$$

$$\emptyset^*_B = \boldsymbol{C_B(u_B - u^*)} \qquad \text{... (III. 17)}$$

### III.3.4 Overall flux densities of constituents

The transport of a constituent by the average flow of the mixture is, in fact, the superposition of a convective flux and a diffusive flux; this is referred to as the overall flux of the constituent. This can be verified by rearranging equations (III.14) or (III.16):

$$\boldsymbol{\emptyset_A + \rho_A u = \rho_A u_A}$$

With :

$\emptyset_A$: Diffusive mass flux density

$F_A = \rho_A u$ : Convective mass flux density

$N_A = \rho_A u_A$ : Global mass flux density.

$$N_A = \emptyset_A + F_A = \emptyset_A + w_A N_T \qquad \text{... (III. 18)}$$

For molecules of *B*

$$N_B = \emptyset_B + F_B = \emptyset_B + w_B N_T \qquad \text{... (III. 19)}$$

Molar flux densities

$$N^*_A = \emptyset^*_A + F^*_A = \emptyset^*_A + x_A N_T \qquad \text{... (III. 20)}$$

For molecules of *B*

$$N^*_B = \emptyset^*_B + F^*_B = \emptyset^*_B + x_B N_T \qquad \text{... (III. 21)}$$

***Remarks***

- Adding (III.18) to (III.19) shows that the sum of the diffusional fluxes of the constituents is always zero $\emptyset_A + \emptyset_B = 0$ . Similarly $\emptyset^*_A + \emptyset^*_B = 0$.

- All the relationships established remain valid for mixtures of more than two constituents.

# Chapter IV :

# Momentum Transfer

## IV.1 Properties of fluids

### IV.1.1 Definition of a Fluid

A fluid can be considered a substance composed of a large number of very small, free-moving particles that can move relative to one another. It is therefore a continuous, deformable, and fluid medium that can flow. The cohesive forces between elementary particles are very weak, so the fluid is a body without a fixed shape that takes the form of its container. For example, molten metals are fluids that, through molding, allow for the creation of rough pieces of complex shapes.

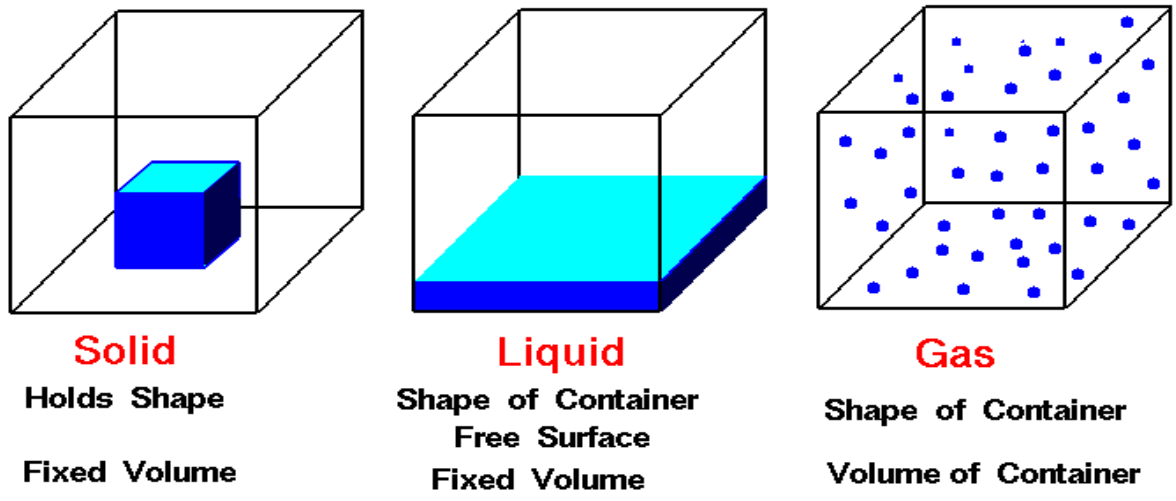



**Figure IV. 1** Idealized representation of the three states of matter

A **solid** has a defined shape and volume.

A **liquid** has a fixed volume but takes the shape of its container.

A **gas** takes on a defined volume and the shape of its container.

Thus, the term **fluid** applies to **liquids** and **gases**.

Fluids can be classified into two main families: ***Newtonian*** fluids (such as water, air, and most gases) and ***non-Newtonian*** fluids (virtually everything else... blood, gels, mud, pastes, suspensions, emulsions, etc.). Newtonian fluids have a constant viscosity, or one that varies only with temperature. The second family consists of non-Newtonian fluids, whose viscosity

varies according to the speed and stresses they experience as they flow. This chapter is limited to Newtonian fluids, which will be classified as follows.

### IV.1.2 Volume forces and surface forces

Like any problem in mechanics, the resolution of a fluid mechanics problem involves defining the material system S, fluid particles inside a closed surface limiting S

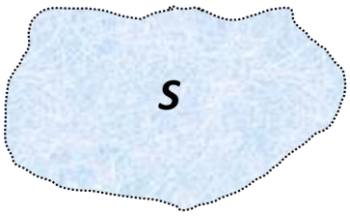



**Figure IV. 2** Sets of fluid particles inside a surface S

To this system, we apply the general principles and theorems of mechanics and thermodynamics:

Principle of **conservation of mass** (principle of continuity);

Principle of **conservation of momentum** (fundamental principle of dynamics);

Principle of **conservation of energy** (first law of thermodynamics).

Furthermore, the forces acting on the particles inside S can be classified into two categories: ***Surface forces:*** These are the forces that act only on the particles of the surface S. They are proportional to the surface elements, for example, pressure forces and friction forces.

***Volume forces:*** These are the forces that act on the molecules inside S; they are proportional to the volume elements. Examples include gravity, magnetic and electric forces. The only forces of this type that we will usually consider are gravitational forces.

### IV.1.3 Perfect Fluid

In fluid mechanics, a fluid is said to be perfect if it is possible to describe its motion without taking into account the effects of friction.

That is to say, when the tangential component is zero ($\overrightarrow{\boldsymbol{dF_T}} = \vec{\mathbf{0}}$). In other words, the interaction force is normal to the surface element dS ($\overrightarrow{\boldsymbol{dF}} = \overrightarrow{\boldsymbol{dF_N}}$).

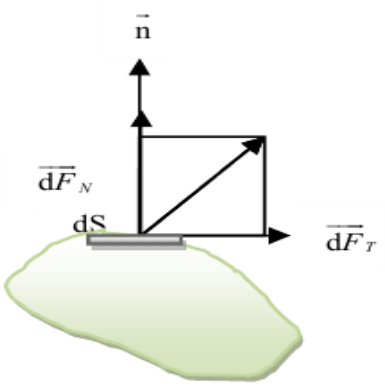



**Figure IV. 3** Explanatory diagram of a perfect or real fluid

### IV.1.4 Real Fluid

Unlike a perfect fluid, which is merely a model to simplify calculations and practically nonexistent in nature, a real fluid is a fluid whose movement is accompanied by a resistance called viscous force, opposing the sliding of fluid layers over one another. The fluid is then said to be viscous.

Only at rest will we assume that the real fluid behaves like a perfect fluid, assuming that the contact forces are perpendicular to the surface elements on which they act.

### IV.1.5 Constraint at a point

Consider a surface element *dS* drawn on *S* and surrounding a point *M* of *S*. The system of surface forces acting on *dS* can be reduced to a single force $\overrightarrow{\boldsymbol{dF}}$ applied at *M* and a couple $\overrightarrow{\boldsymbol{dC}}$. We assume that $\overrightarrow{\boldsymbol{dF}}$ is infinitely small.

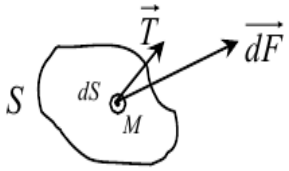



**Figure IV. 4** Constraint at a single point

As *dS* approaches zero, around a fixed point *M*, the vector $\frac{\overrightarrow{dF}}{dS}$ tends towards the vector $\vec{T}$, which is called the stress at *M* on the element *dS*.

The force $\overrightarrow{dF}$ is therefore given by: $\overrightarrow{dF} = \vec{T}\,dS$

### IV.1.6 Incompressible Fluid

A fluid is said to be incompressible when the volume occupied by a given mass does not change with respect to external pressure. Liquids can be considered incompressible fluids (water, oil, etc.).

### IV.1.7 Compressible Fluid

A fluid is said to be compressible when the volume occupied by a given mass changes with respect to external pressure. Gases are compressible fluids, for example, air, nitrogen, etc.

### IV.1.8 Physical Properties of the Fluid

#### IV.1.8.1 volumetric mass

The volumetric mass of a fluid is the mass of one unit volume of that fluid. It is calculated using the following relationship:

$$\rho = \frac{m}{V} \qquad \text{... (IV. 1)}$$

with

ρ: volumetric mass (Kg/m³),

*m*: Mass (Kg),

*V*: Volume (m³)

***Example: Calculating volumetric mass***

A bottle contains 200 mL of alcohol. The mass of this alcohol is 170 g. What is the volumetric mass of this alcohol ?

According to the definition:

$$\rho = \frac{m}{V} = \frac{170}{0.200} = 875 g/L$$

### IV.1.8.2 Density

Density is a dimensionless quantity defined by:

$$d = \frac{\text{volumetric mass of the fluid}}{\text{volumetric mass of a reference fluid}} = \frac{\boldsymbol{\rho}}{\boldsymbol{\rho_{ref}}} \qquad \ldots (\mathbf{IV.2})$$

For liquids, water will be used as the reference fluid.

For gases, air will be used as the reference fluid.

A densimeter is an instrument for measuring the density of liquids.

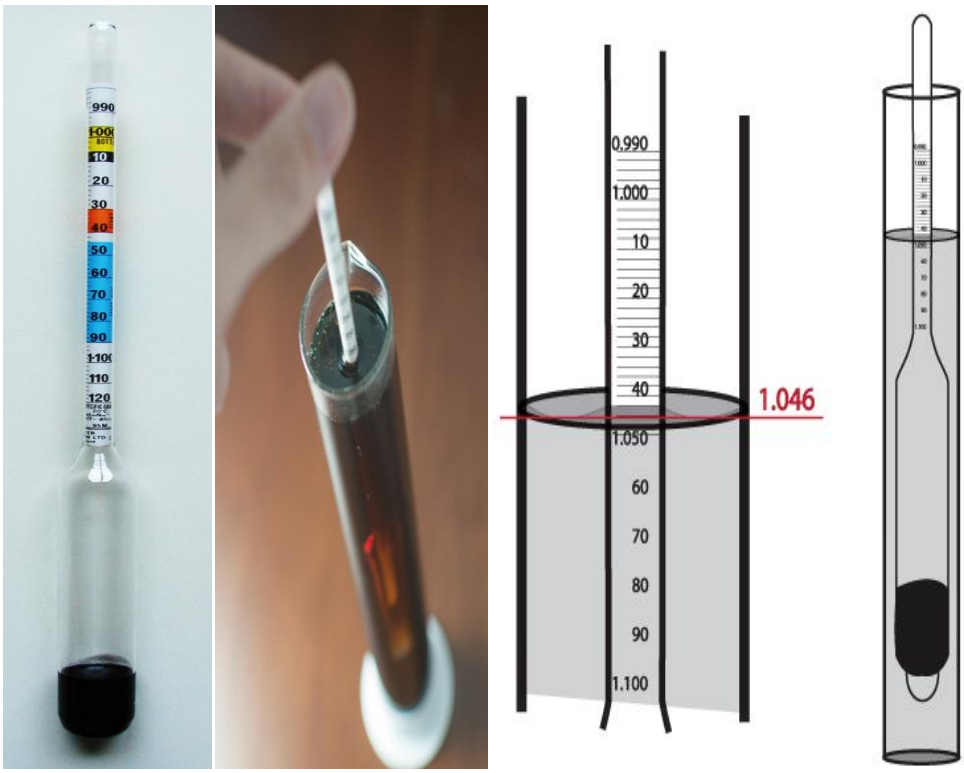



**Figure IV. 5** The Densimeter

***Example: Calculating Density***

Weighing 100 mL of oil gives a mass of 72 g. What is the density of the oil?

The density is:

$$\rho = \frac{m}{V} = \frac{72}{0.100} = 720 g/L$$

Since 1.0 L of water weighs 1000 g, we have:

$$d = \frac{\rho}{\rho_{ref}} = \frac{720}{1000} = 0.72$$

### IV.1.8.3 Volumetric Weight

$$\mathbf{w = \rho . g = \frac{m . g}{V}} \qquad \text{... (IV. 3)}$$

w: Volumetric Weight ($N/m^3$).

g: Acceleration due to gravity ($m/s^2$).

***Example: Calculating a Volumetric Weight***

Determine the volumetric Weight of gasoline given that its density d = 0.7. We are given:

- the acceleration due to gravity g = 9.81 $m/s^2$

- the volumetric mass of water ρ = 1000 $kg/m^3$

$$w = \rho . g = d \times \rho_{ref} \times g = 0.7 \times 1000 \times 9.81 = 6867 N/m^3$$

### IV.1.8.4 Viscosity

The viscosity of a fluid can be defined as the internal friction between its layers during flow. A significant consequence of this property is the fluid's adhesion to the surface of the solid in contact with it. Also, due to this viscous property of fluids, shear stresses (tangential sliding force expressed per unit area) exist.

***Dynamic viscosity***

Under the influence of the interaction forces between fluid molecules and the interaction forces between fluid molecules and those of the wall, each fluid molecule does not flow at the same speed. This is called a velocity profile.

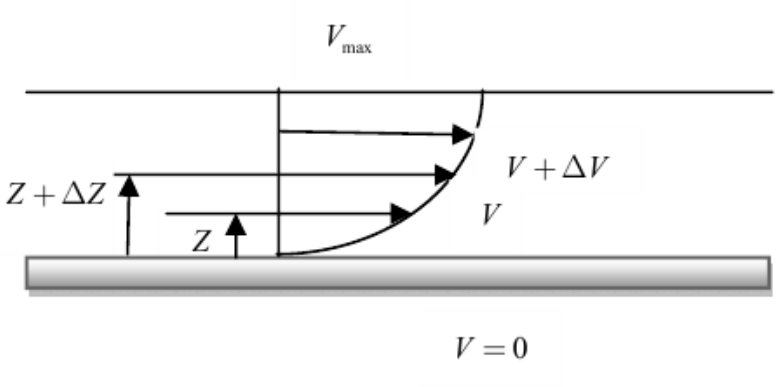



**Figure IV. 6** Particle velocity profile

If we represent the velocity of each particle located in a cross-section perpendicular to the overall flow by a vector, the curve formed by the endpoints of these vectors represents the velocity profile.

The fluid motion can be considered as resulting from the sliding of fluid layers over one another; the velocity of each layer is a function of the distance z.

The frictional force *F* acting at the interface between these two layers opposes the sliding of one layer over the other. It is proportional to the difference in velocity between the layers, $\Delta V$, to their surface area S, and inversely proportional to ΔZ.
The proportionality factor $\mu$ is the dynamic viscosity coefficient of the fluid.

$$\boldsymbol{F = -\mu . S . \frac{\Delta V}{\Delta Z}} \qquad \text{... (IV. 4)}$$

F: Sliding force between layers (N).

μ: Dynamic viscosity (kg/m·s).

S: Contact area between two layers (m²).

ΔV: Velocity difference between two layers (m/s).

ΔZ: Distance between two layers (m).

***Note:***

In the International System of Units (SI), the unit of dynamic viscosity is the Pascal second (**Pa.s**) or Poiseuille (**Pl**): **1 Pa.s = 1 Pl = 1 Kg/m.s.**

However, in the CGS system, the unit **g/cm·sec** is known as the poise (**P**).

***Kinematic viscosity***

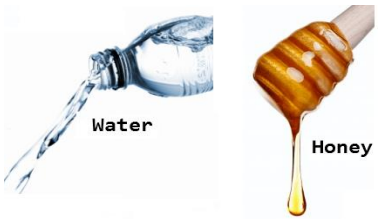
Kinematic viscosity is the ratio of dynamic viscosity $\mu$ to volumetric mass $\rho$:

$$\nu = \frac{\mu}{\rho} \qquad \dots (\mathbf{IV.5})$$

ν : the kinematic viscosity.

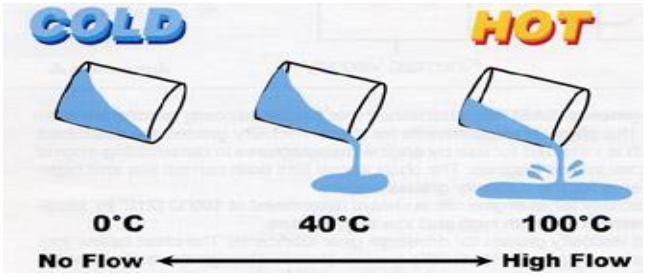
The popular unit used in the SI is Stokes: **1Stokes =$10^{-4}$ $m^2/s$**

In the CGS system, its unit is **cm/sec$^2$** known as Stokes, whose dimension corresponds to $\frac{L}{T^2}$




Note that the viscosity of gas **increases** with temperature, while that of liquids **decreases** according to a law of the form:

$$\log(\mu) = A + \frac{B}{T} \qquad \dots (\mathbf{IV.6})$$



***Example: Calculating Dynamic Viscosity***

Determine the dynamic viscosity of olive oil given that its density is 0.918 and its kinematic viscosity is 1.089 Stokes.

We have:

$\nu = 1.089st = 1.089 \times 10^{-4}\ m^2/s$

$d = \frac{\rho}{\rho_{ref}} \Rightarrow \rho = \rho_{ref} \times d = 918\ Kg/\ m^3$

So

$$\mu = \nu \times \rho = 1.089 \times 10^{-4} \times 918 = 0.1kg/m.s = 0.1Pa.s$$

***Example: Calculating Kinematic Viscosity***

Fuel heated to a temperature T = 20°C has a dynamic viscosity of $\mu = 95\times10^{-3}$ Pa·s.

Calculate its kinematic viscosity υ in stoichiometry, given that its density is d = 0.95. The volumetric mass of water is given as $\rho_{water}$ = 1000 Kg/m³.

We have:

$d = \frac{\rho}{\rho_{ref}} \Rightarrow \rho = \rho_{ref} \times d = 0.95 \times 1000 = 950\ Kg/\ m^3$

So

$$\nu = \frac{\mu}{\rho} = \frac{95 \times 10^{-3}}{950} = 10^{-4}\ m^2/s = 1st$$

## IV.2.1 Fluid Statics

### IIV.2.1 Definition

Fluid statics is the discipline that studies the equilibrium conditions of fluids at rest. That is to say, the fluid is subjected only to a gravitational field, which we will assume to be uniform. Since viscosity only manifests itself if there is relative movement of particles (deformation), the statics of viscous (or real) fluids is synonymous with the statics of perfect fluids. In this chapter, we will examine, in particular, the general equations of fluid statics, Pascal's theorem, Archimedes principle, and the quantity of pressure.

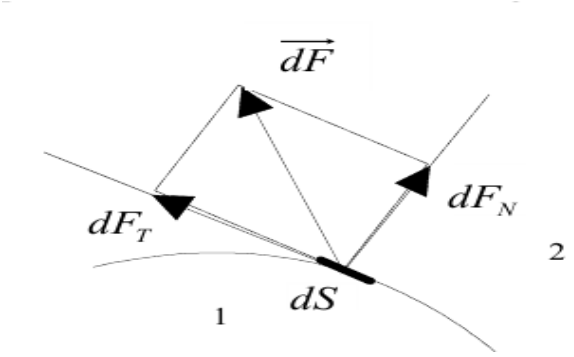



**Figure IV. 7** Diagram explaining the pressure force

In any medium, including a fluid, the force exerted by part (1) on part (2) through a real or fictitious surface element *dS* has an arbitrary direction.

However, this force $\overrightarrow{dF}$ can always be decomposed into:

- a **tangential component** $dF_T$

- a **normal component** $dF_N$

The quantity $dF_T/dS$ represents the tangential stress and $dF_N/dS$ the normal stress.

By definition, the normal stress is called **Pressure**:

$$\mathbf{P} = \frac{\boldsymbol{dF_N}}{\boldsymbol{dS}} \qquad \text{... (IV. 7)}$$

***Note:***

In fluid statics, only pressure forces $dF_N$, normal to the element $dS$, are involved. Tangential forces $dF_T$ appear only in fluid dynamics: they correspond to the viscous friction of the fluid layers moving relative to each other and to the pipe wall.

### IV.2.2 Pressure Units

Pressure, considered as force per unit area, is expressed in the SI system in

- Pascals (Pa), that is, in $N \cdot m^{-2}$.
- The bar (bar) and its submultiple, the millibar (mbar)
- The millimeter of mercury or Torr
- The millimeter of water column or meter of water column (mWC)
- The atmosphere (atm) The correspondence between these units is as follows:
  1 bar = $10^5$ Pa = 1000 mbar ≈ 760 mm of mercury ≈ 10.2 mWC ≈ 0.987 atm.

- ***Atmospheric pressure*** is the pressure exerted by the atmosphere at the Earth's surface. At sea level, this pressure is equivalent to that exerted by a column of approximately 760 mm of mercury. It varies slightly every day; however, it is always close to 1 bar.

- ***Absolute and relative pressure*** Absolute pressure is the pressure measured relative to a perfect vacuum (that is, the total absence of matter). It is always positive. Relative pressure is defined relative to the atmospheric pressure existing at the time of measurement: this pressure can therefore take a positive value if the pressure is greater than atmospheric pressure or a negative value if the pressure is less than atmospheric pressure.

### IV.2.3 Pressure at a point in a fluid

At every point in a fluid, there exists a certain pressure. Let *M* be a point in a fluid. If we consider an imaginary surface dS passing through *M*, the resultant of all the forces acting on dS from the fluid particles in disordered motion is perpendicular to this surface dS, and we can write: $\mathbf{P_M} = \frac{\|\overrightarrow{dF_N}\|}{dS}$

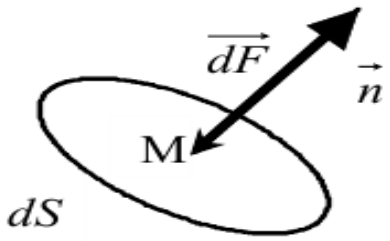



**Figure IV. 8** Diagram explaining the pressure force at a point

$\vec{n}$: being the unit vector normal to *dS*, oriented outwards.

$P_M$: Pressure at M (Pascals).

$dF_N$: Normal component of the elementary pressure force exerted on the surface (Newtons).

This force $\overrightarrow{\boldsymbol{dF}}$ obviously depends on the surface area *dS* considered, but the pressure $P_M$ at point M of the fluid does not depend on *dS*.

### IV.2.4 Fluid equilibrium in a gravitational field (Concept of driving pressure)

Consider the diagram opposite. At equilibrium, the resultant of the external forces acting on the fluid element of height is zero:

$$\overrightarrow{\boldsymbol{dF_1}} + \overrightarrow{\boldsymbol{dF_2}} + \overrightarrow{\boldsymbol{dF_3}} = \vec{\boldsymbol{0}}$$

The projection along the z-axis gives:

$$\boldsymbol{P(z).\Omega - P(z+dz).\Omega - \rho g\,\Omega\,dz = 0}$$

Or again:

$$\boldsymbol{P(z).\Omega - \left[P(z) + \frac{\partial P}{\partial z}dz\right].\Omega - \rho g\,\Omega\,dz = 0}$$

That is:

$$\boldsymbol{\rho g - \frac{\partial P}{\partial z} = 0} \qquad \text{... (IV. 8)}$$

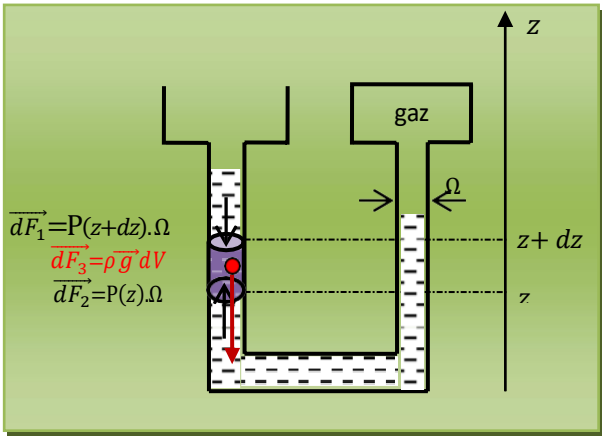



**Figure IV. 9** Diagram of the concept of driving pressure

More generally, this equation can be written in vector form :

$$\boldsymbol{\rho\vec{g} - \overrightarrow{grad}\,P = \vec{0}} \qquad \text{... (IV. 9)}$$

In the specific case of the **incompressible fluid** ( $\rho$=Cste ), relation (IV.8) provides:

$$\boldsymbol{P + \rho g z = Cste = \hat{P}} \qquad \ldots (\mathbf{IV.10}) \quad \textbf{Fundamental equation of hydrostatics}$$

$P$: is the “ordinary pressure” defined previously.

$\rho g z$ : is the "gravitational pressure," which represents the weight of the fluid column above the point considered.

$\hat{P}$ : is the “driving pressure,” which, as we will see, represents the potential energy of a unit volume of the fluid. Differences in this potential energy produce the movements.

Note that $\hat{P}$ is defined up to a constant. Its value depends on the origin of the vertical dimensions.

Isopressure (isobaric) surfaces within a continuous fluid are then composed of planes with constant elevation $z$.

### IV.2.5 Fundamental Equation of Fluid Statics

The objective of this section is to determine the pressure distribution within a fluid in equilibrium. To do this, we consider an arbitrary element *dv* of a fluid in a reservoir.

The objective of this section is to determine the pressure distribution within a fluid in equilibrium. To do this, we consider an arbitrary element (dv) of a fluid in a parallelepiped-reservoir (Figure IV.10). The tank and the fluid are in equilibrium, so we can write: Newton’s first law of motion:

$$\sum \overrightarrow{\boldsymbol{dF_{sys}}} = \vec{\mathbf{0}} \qquad \ldots (\mathbf{IV.11})$$

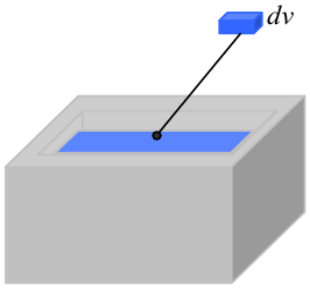


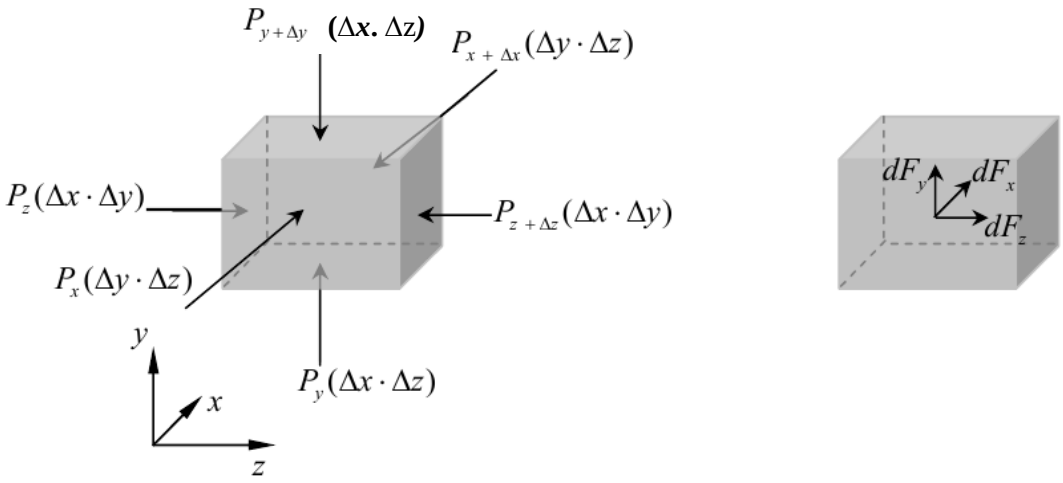

**Figure IV. 10** Tank containing a fluid at rest

The only forces acting on the *dv* system are:

-Pressure forces;

-Volume forces.

First, we begin by studying a fluid element with dimensions Δx, Δy, and Δz, then we deduce the desired differential equation. Figure IV.11 shows the forces acting on the six faces of the fluid element.




**(a) Pressure forces** **(b) Volume forces**

**Figure IV. 11** Forces acting on the six faces of the fluid element

Projecting equation (VI.11) onto the three axes gives:

$$\boldsymbol{P(x)dydz - P(x+dx)dydz + \rho F_x dxdydz = 0}$$

Which gives:

$$\boldsymbol{\rho F_x - \frac{\partial P}{\partial x} = 0} \qquad \boldsymbol{\ldots (IV.12)}$$

$$\boldsymbol{P(y)dxdz - P(y+dy)dxdz + \rho F_y dxdydz = 0}$$

$$\boldsymbol{\rho F_y - \frac{\partial P}{\partial y} = 0} \qquad \boldsymbol{\ldots (IV.13)}$$

$$\boldsymbol{P(z)dxdy - P(z+dz)dxdy + \rho F_z dxdydz = 0}$$

$$\boldsymbol{\rho F_z - \frac{\partial P}{\partial z} = 0} \qquad \text{... (IV. 14)}$$

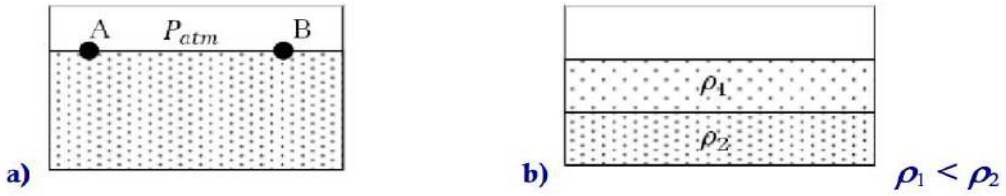
Equations (VI.12), (VI.13), (VI.14) are the fundamental equations of fluid statics.

In vector form:

$$\boldsymbol{\rho \vec{F} - \overrightarrow{grad}\, P = \vec{0}} \qquad \text{... (IV. 15)} \qquad \textbf{Fundamental equation of fluid statics}$$

### IV.2.6 Level Surface

A level surface is characterized by the same pressure at each of its points. It is the free surface of a liquid which, at each point, is perpendicular to gravity. The surface of the oceans is generally considered a level surface, that is, an equipotential surface of the gravitational field. Indeed, the surface of the oceans and seas is controlled primarily by the force of gravity, with some disturbing phenomena such as ocean currents, variations in salinity, tides, swell caused by winds, variations in atmospheric pressure, etc.




**(a) Free surface** **(b) Immiscible fluids.**

**Figure IV. 12** Level surface

### IV.2.7 Static pressure measurement

A **pressure gauge** typically measures the difference between the pressure inside a device and the external pressure; this difference is called the “effective pressure” or “gauge pressure”. The actual pressure inside the device is called the “absolute pressure” (psia in English).

### IV.2.7.1 Liquid column manometers

#### *IV.2.7.1.a Piezometric tube*

Consider a point *M* of a liquid with density $\rho$, in equilibrium in a gravitational field. A tube is opened at *M* in which the free surface of the liquid is fixed at a height *h* above *M*. The pressure is measured using the height h:

$\widehat{\boldsymbol{P}}_{\boldsymbol{M}} = \widehat{\boldsymbol{P}}_{\boldsymbol{N}}$ (continuous, incompressible fluid in equilibrium in a gravitational field).

$P_M + \rho g z_M = P_A + \rho g z_A$

With $P_A = P_{atm}$ ($A \in$ the free surface of the liquid)

$P_M = P_{atm} + \rho g(z_A - z_M)$

$P_M = P_{atm} + \rho g h$

(this is the effective pressure at point *M*)

$P_M - P_{atm} = \rho g h$

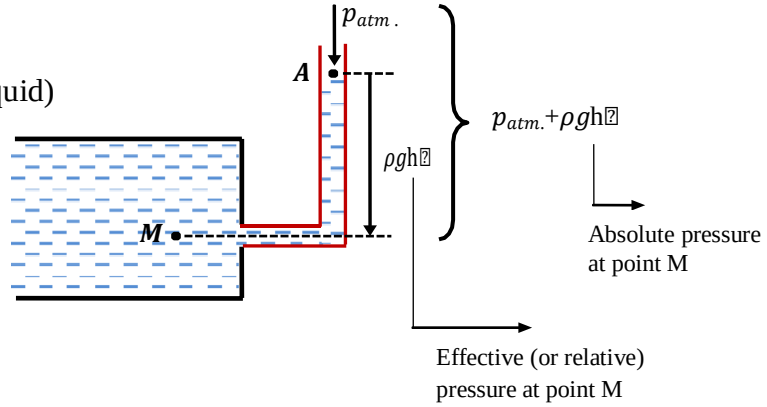



**Figure IV. 13** Piezometric tube

#### IV.2.7.1.b Two-liquid U-tube manometers

The principle is essentially the same as before,

but if we note that:

$$\frac{P_M - P_{atm}}{\rho g} = h_M$$

We can consider using a manometric fluid denser than the fluid whose pressure is being measured (mercury, for example).

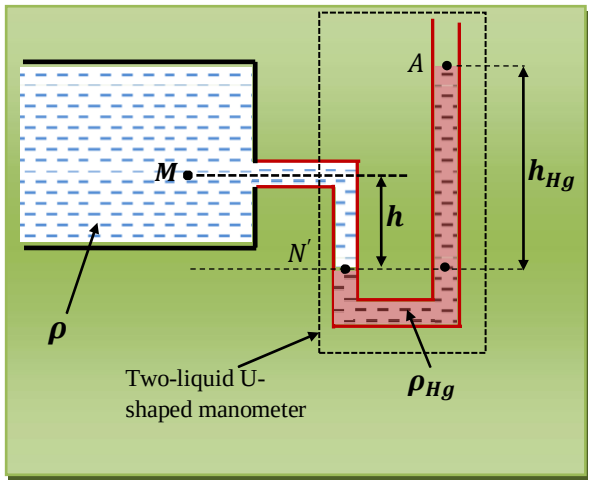



**Figure IV. 14** Two-liquid U-shaped manometer

We then obtain the device shown schematically opposite.

$\widehat{\boldsymbol{P}}_{\boldsymbol{M}} = \widehat{\boldsymbol{P}}_{\boldsymbol{N'}}$ (Fluid continuous, incompressible equilibrium in a gravitational field).

Hence : $P_M = P_{N'} + \rho g h$

$P_{N'} = P_N$ (horizontal plane of Hg)

$\widehat{\boldsymbol{P}}_{\boldsymbol{N}} = \widehat{\boldsymbol{P}}_{\boldsymbol{A}}$ (continuous, incompressible fluid in equilibrium in a gravitational field)

Hence : $P_N = P_{atm} + \rho_{Hg}\, g\, h_{Hg}$

Finally :

$P_M = P_{atm} + g(\rho_{Hg}\, h_{Hg} - \rho h)$

### IV.2.7.1.c Bourdon Tube Manometer

Manometers exist to measure a very wide pressure range, from vacuum to 4000 atm. Several types of manometers are available depending on the application. Among the most common is the Bourdon tube manometer. This type of manometer is used to measure the pressure of gases or inviscid liquids. Inside the manometer is a curved, flat tube: this is the Bourdon tube. The fluid whose pressure is being measured enters through the tube support and exerts pressure inside the flat tube (see figure below). Since the length of the outer arc is greater than that of the inner arc, the pressure exerts a greater outward force: the tube then tries to straighten. Under the effect of the internal pressure, the tube straightens and its free end moves. This movement is transmitted to a toothed sector via connecting rods. The rotation of the toothed sector drives a gear on which the needle is fixed.

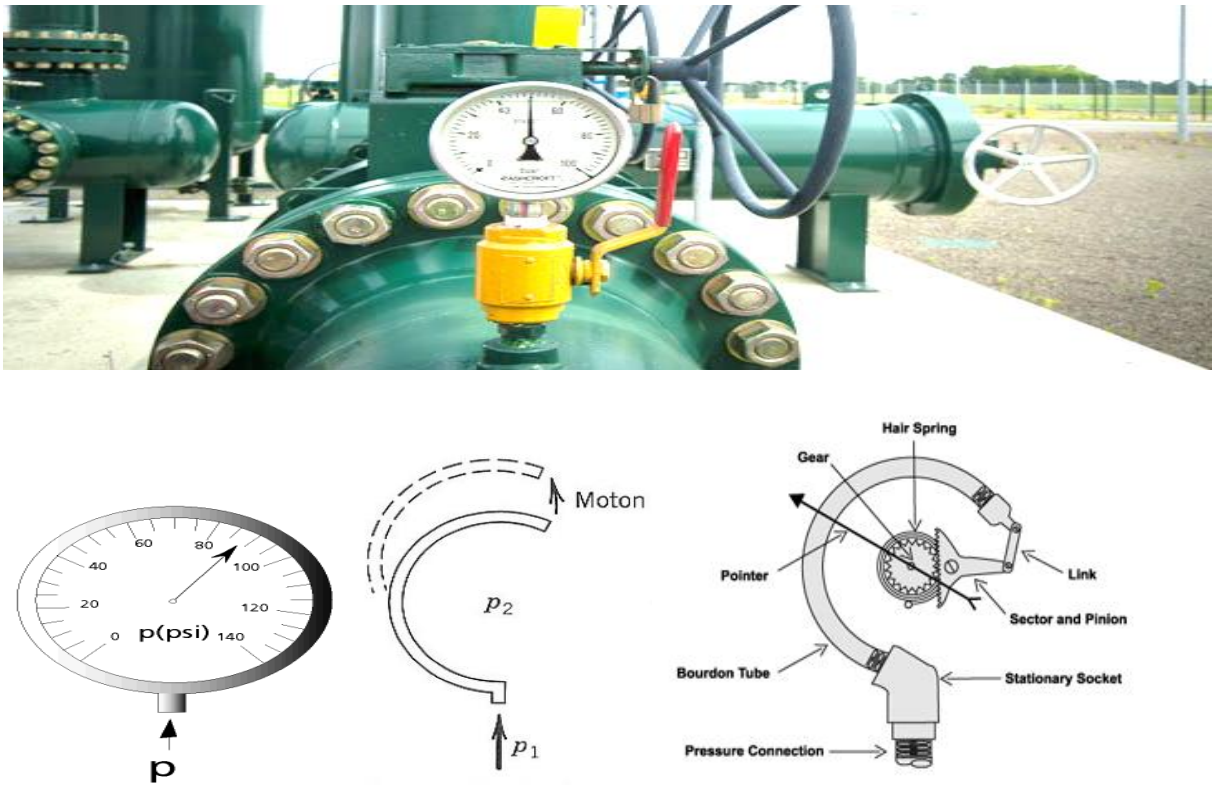



**Figure IV. 15** Bourdon-type pressure gauge

***Example:*** Consider a U-shaped tube closed at one end (Figure IV.16) containing two immiscible liquids.

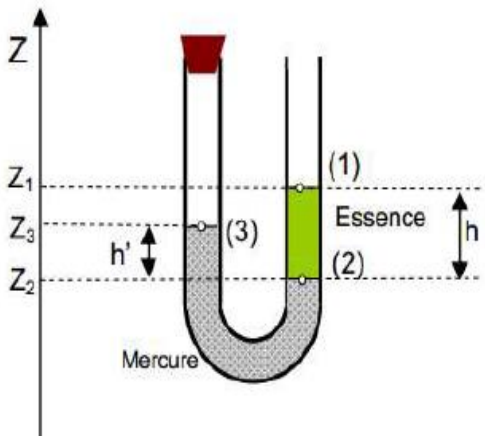



**Figure IV. 16** Application exercise: fundamental law of fluid statics

The surfaces:

- (1) and (2) represent gasoline with a volumetric mass $\rho_{essence}$ = 700 kg/m³.

- (2) and (3) represent mercury with a volumetric mass $\rho_{mercury}$ = 13600 kg/m³.

The pressure above the free surface (1) is $P_1 = P_{atm}$ = 1 bar. The acceleration due to gravity is g = 9.8 m/s².

The closed branch traps a gas at a pressure $P_3$, which we want to calculate.

1) By applying the fundamental law of fluid statics for essence, calculate the pressure $P_2$ (in mbar) at the separation surface (2), knowing that $h = (Z_1 - Z_2)$ = 728 mm.

2) Similarly, for mercury, calculate the pressure $P_3$ (in mbar) at the surface level (3) knowing that $h' = (Z_3 - Z_2)$ = 15 mm.

Answer:

1) For essence: $P_2 - P_1 = \rho_{essence} .g (Z_1 - Z_2)$

$P_2 = P_1 + \rho_{essence} .g.h$

$P_2 = 10^5 + 700 \times 9.8 \times 0.728 = 1.05 \times 10^5$ pascals = 1050 mbar

2) For mercury: $P_2 - P_3 = \rho_{mercury} \times (Z_3 - Z_2)$.

$P_3 = P_2 - \rho_{essence} .g.h'$

$P_3 = 1.05 \times 10^5 - 13600 \times 9.8 \times 0.15 = 1.03 \times 10^5$ pascals = 1030 mbar.

## IV.3 General conservation equations

### IV.3.1 Concept of Flow Rate

#### IV.3.1.1 Mass flow rate

The mass flow rate $\mathbf{Q_m}$ of a system is a measure of the mass of fluid passing through a point in the system per unit of time.

The mass flow rate of a fluid vein is the limit of the ratio dm/dt when dt tends towards 0.

$$\boldsymbol{Q_m = \frac{dm}{dt}} \qquad \text{... (IV. 16)}$$

$Q_m$: is the mass of fluid per unit time that passes through any cross-section of the pipe (Kg/s).

dm: elementary mass (Kg) that passes through the cross-section during a time interval dt

dt: time interval (s)

$$\boldsymbol{Q_m = \frac{dm}{dt} = \rho . S \frac{dx}{dt}} \quad avec \quad \boldsymbol{\frac{dx}{dt} = v}$$

Therefore:

$$\mathbf{Q_m = \rho . S . v} \qquad \text{... (IV. 17)}$$

ρ: Volumetric mass (Kg/m³)

S: Cross-sectional area of the fluid stream (m²)

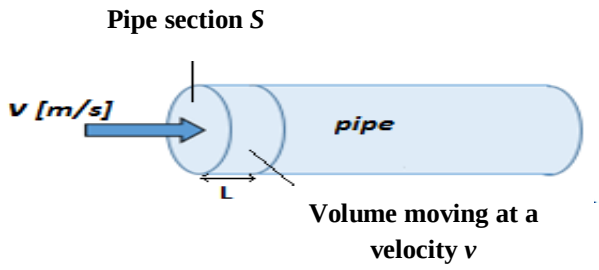
v: Average fluid velocity through (S) (m/s)

### IV.3.1.2 Volumetric flow rate

The volumetric flow rate $\mathbf{Q_v}$ of a system is a measure of the volume of fluid passing through a point in the system per unit of time.

The volumetric flow rate of a fluid vein is the limit of the ratio *dV/dt* when dt tends towards 0.

$$\boldsymbol{Q_v = \frac{dV}{dt}} \qquad \text{... (IV. 18)}$$




$Q_v$: Volume of fluid per unit time that passes through any cross-section of the pipe ($m^3/s$).

dV: Elementary volume, in m³, that has passed through a surface S during a time interval dt. Noting that

$$\boldsymbol{dV = \frac{dm}{\rho}}$$

Finally

$$\mathbf{Q_v = S.v} \qquad \text{... (IV. 19)}$$

***Example:***

An incompressible, perfect fluid (water) flows from a circular orifice located on the side of a tank with a volumetric flow rate $Q_v$ = 24 L/min. The diameter of the orifice is d = 15 mm.




1) Calculate the mass flow rate of water through the orifice in kg/s.
2) Determine the flow velocity at the orifice.

**Solution**

1) we know that

$Q_m = \rho . Q_v$ and $\rho_{eau}$ = 1000g/L

$$Q_m = 1000 \times \frac{24}{60} = 400 g/s = 0.4 kg/s$$

3) $Q_v = S.v \Rightarrow v = \frac{Q_v}{S} = \frac{Q_v}{\pi d^2} = 2.26\ m/s$

### IV.3.2 Dynamics of Incompressible Perfect Fluids

Fluid flow is a complex phenomenon. We are interested in the fundamental equations that govern the dynamics of incompressible, perfect fluids, in particular:

- The continuity equation (conservation of mass),

- Bernoulli's theorem (conservation of energy),

- Euler's theorem (conservation of momentum), from which we establish the equations giving the dynamic force exerted by fluids in motion (for example, water jets).

### IV.3.2.1 Continuity equation

Consider a vein of an incompressible fluid of density ρ undergoing steady flow.

A fluid flow is said to be steady if the velocity vector field of the fluid particles is constant over time.

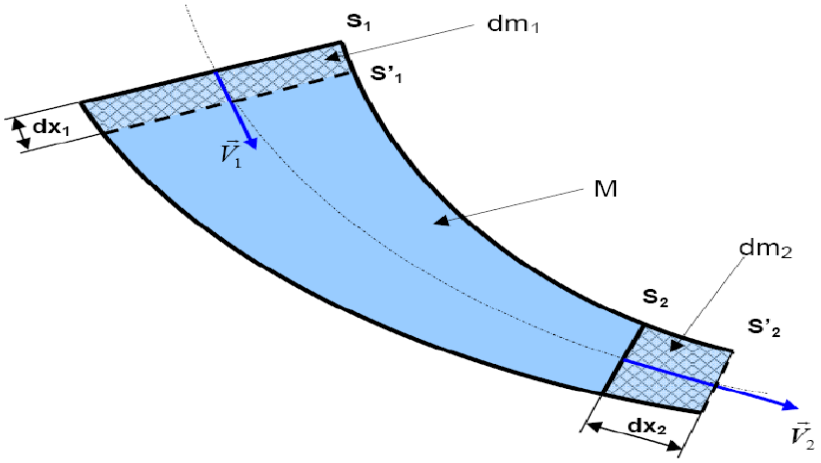



**Figure IV. 17** Explanatory diagram of a vein of an incompressible fluid

We denote by:

- $S_1$ and $S_2$ respectively the inlet and outlet sections of the fluid at time t,

- $S'_1$ and $S'_2$ respectively the inlet and outlet sections of the fluid at time t'= (t+dt),

- $V_1$ and $V_2$ the flow velocity vectors respectively through sections $S_1$ and $S_2$ of the vein.

- $dx_1$ and $dx_2$ are respectively the displacements of sections $S_1$ and $S_2$ during the time interval dt,

- $dm_1$: elementary mass entering between sections $S_1$ and $S'_1$,

- $dm_2$: elementary mass leaving between sections $S_2$ and $S'_2$,

- M: mass between $S_1$ and $S_2$,

- $dV_1$: elementary volume entering between sections $S_1$ and $S'_1$,

- $dV_2$: elementary volume leaving between sections $S_2$ and $S'_2$,

At time t: the fluid between $S_1$ and $S_2$ has a mass equal to ($dm_1$ + M)

At time t + dt: the fluid between $S'_1$ and $S'_2$ has a mass equal to $(M + dm_2)$.

By conservation of mass: $dm_1 + M = M + dm_2$, simplifying by m, we have: $dm_1 = dm_2$ Therefore, $\rho_1.dV_1 = \rho_2.dV_2$ or $\rho_1.S_1dx_1 = \rho_2.S_2dx_2$

Dividing by *dt*, we arrive at:

$\rho_1.S_1.dx_1/dt = \rho_2.S_2.dx_2/dt \Rightarrow \rho_1.S_1v_1 = \rho_2.S_2v_2$

Since the fluid is **incompressible**: $\boldsymbol{\rho_1 = \rho_2 = \rho}$ , we obtain:

$$\mathbf{S_1.v_1 = S_2.v_2} \qquad \text{... (IV. 20)}$$

### IV.3.2.2 Bernoulli's Theorem

Let's revisit the diagram Figure IV.17 with the same notation and introduce the following assumptions:

- The fluid is ideal and incompressible.
- The flow is steady.
- The flow is in a perfectly smooth pipe.

Consider a vertical axis Z directed upwards. Let $Z_1$, $Z_2$, and Z denote the altitudes of the centers of gravity of masses $dm_1$, $dm_2$, and M, respectively.

Let $F_1$ and $F_2$ denote the magnitudes of the fluid pressure forces acting at sections $S_1$ and $S_2$, respectively.

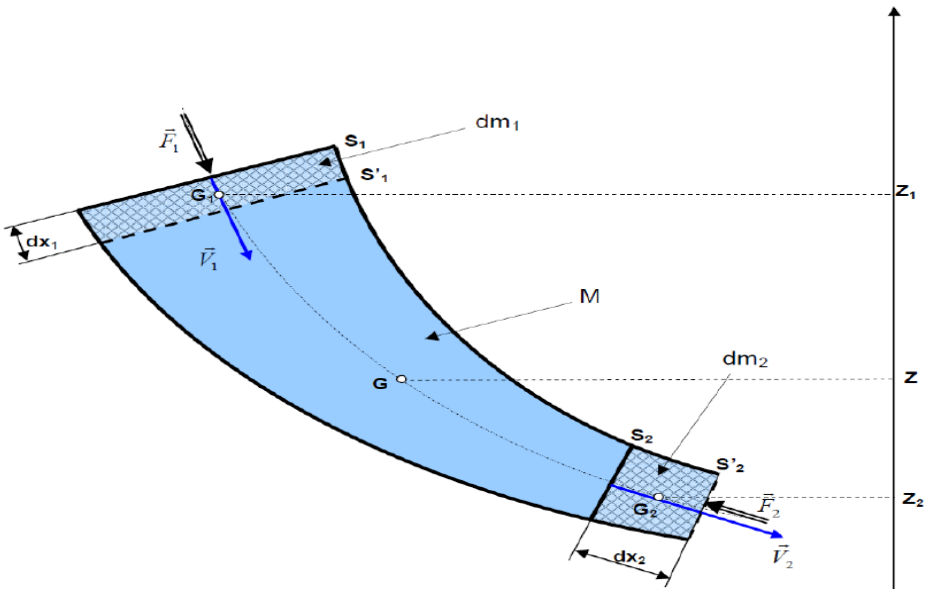



**Figure IV. 18** Explanatory diagram of Bernoulli's theorem

At time t, the fluid of mass ($dm_1$ + M) is located between $S_1$ and $S_2$. Its mechanical energy is:

$$\mathbf{E_{mec} = E_{pot} + E_{cin} = (dm_1\ g\ Z_1 + M\ g\ Z) + \frac{1}{2}(dm_1\ v_1^2 + \int_{S1}^{S2} \frac{V^2}{2})}$$

At time t, the fluid of mass ($dm_1$ + M) is located between $S_1$ and $S_2$. Its mechanical energy is:

$$\mathbf{E^1_{mec} = E^1_{pot} + E^1_{cin} = (M\ g\ Z + dm_2\ g\ Z_2) + \frac{1}{2}(dm_2\ v_2^2 + \int_{S1}^{S2} \frac{V^2}{2})}$$

Applying the work-energy theorem to this fluid between times t and t+Δt (the change in kinetic energy is equal to the sum of the work done by external forces: weight and pressure forces), we obtain:

$$\frac{v_1^2}{2} + \frac{P_1}{\rho} + gZ_1 = \frac{v_2^2}{2} + \frac{P_2}{\rho} + gZ_2 \qquad \dots (\mathbf{IV.21})$$

If we perform the same procedure on the different sections of the current tube, we obtain:

$$\frac{v^2}{2} + \frac{P}{\rho} + gZ = Cte \qquad \dots (\mathbf{IV.22})$$

Equation (IV.21) is ***Bernoulli's equation***. This equation is a special form of the theorem for the conservation of mechanical energy of a moving fluid.

By dividing all the terms of relation (IV.21) by g, we write all the terms in the dimension of a height (pressures expressed in meters of fluid column).

$$\frac{v^2}{2g} + \frac{P}{\rho g} + Z = H = Cte \qquad \dots (\mathbf{IV.23})$$

H: Total height

$\frac{v^2}{2g}$: Kinetic height

$\frac{P}{\rho g}$: Pressure height

z: the rating

$\frac{P}{\rho g} + z$ : Piezometric height.

Other formulations of Bernoulli's equation:

Per unit of pressure **(Pa)**:

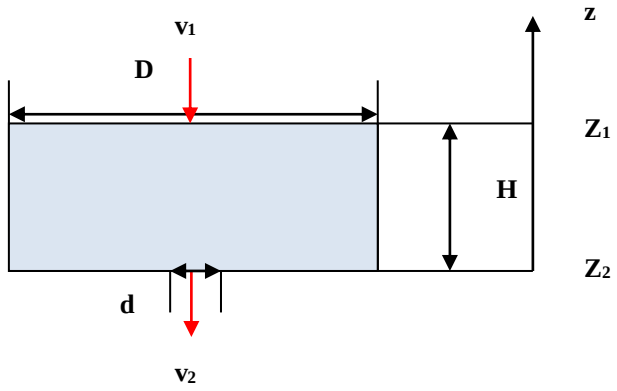
$$\frac{\rho\, v^2}{2} + P + \rho g Z = Cte \qquad \dots (\mathbf{IV.24})$$

Per unit of energy **(J/kg)**:

$$\frac{v^2}{2} + \frac{P}{\rho} + gZ = Cte \qquad \dots (\mathbf{IV.25})$$

***Example***

Consider a cylindrical tank with an internal diameter D = 4 m filled with water to a height H = 6 m. The bottom of the tank has an opening with a diameter d = 20 mm allowing the water to drain.




The bottom orifice is opened and a very small time dt is allowed to elapse. The water level H in the reservoir drops by a quantity dH. Let $v_1$ =dH/dt the rate of descent of the water level, and $v_2$ the flow velocity in the orifice.

The acceleration due to gravity is given as g = 9.81 m/s².

1) Deduce the expression for $v_1$ as a function of $v_2$, D, and d.

2) Write Bernoulli's equation. Assume that the fluid is ideal and incompressible.

3) From the answers to questions 1) and 2), establish the expression for the flow velocity $v_2$ as a function of g, H, D, and d.

4) Calculate the velocity $v_2$. Assume that the diameter d is negligible compared to D. That is, d << D

5) Deduce the volumetric flow rate $Q_V$.

**Solution**

1) By applying the continuity equation, we will obtain

$$Q_v = S_1.v_1 = S_2.v_2$$

$$\frac{\pi D^2}{4} \times v_1 = \frac{\pi d^2}{4} \times v_2 \Rightarrow v_1 = v_2 \frac{d^2}{D^2} \qquad \text{... (IV. 26)}$$

2)

$$\frac{v_1^2}{2} + \frac{P_1}{\rho} + gZ_1 = \frac{v_2^2}{2} + \frac{P_2}{\rho} + gZ_2$$

Now, $P_1 = P_2$ and $(Z_1\text{-}Z_2) = H \Rightarrow$ We obtain

$$\frac{v_2^2 - v_1^2}{2} = g \times H \qquad \text{... (IV. 27)}$$

3) By substituting equation (IV.26) into (IV.27), we obtain

$$\frac{v_2^2\left(1 - \frac{d^4}{D^4}\right)}{2} = g \times H$$

So, the speed:

$$v_2 = \sqrt{\frac{2gH}{1 - \frac{d^4}{D^4}}}$$

4) Considering d << D, then the expression for the flow velocity is written

$$v_2 = \sqrt{2gH}$$

$v_2 = \sqrt{2 \times 9.81 \times 6} \Rightarrow v_2 = 10.94$ m/s

5) $Q_v = S_2.v_2 = \frac{\pi d^2}{4} \times v_2 = \frac{\pi 0.02^2}{4} \times 10.94 = 3.29 \times 10^{-3} m^3/s$

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