



On the Growth of Solutions of Some Second Order Linear Differential Equations With Entire Coefficients

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Abstract

In this paper, we investigate the order and the hyper-order of growth of solutions of the linear differential equation

$$f'' + Q(e^{-z})f' + (A_1e^{a_1z} + A_2e^{a_2z})^n f = 0,$$

where $n \geq 2$ is an integer, $A_j(z) (\neq 0)$ ($j = 1, 2$) are entire functions with $\max\{\sigma(A_j) : j = 1, 2\} < 1$, $Q(z) = q_m z^m + \dots + q_1 z + q_0$ is a nonconstant polynomial and a_1, a_2 are complex numbers. Under some conditions, we prove that every solution $f(z) \neq 0$ of the above equation is of infinite order and hyper-order 1.

1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [8], [13]). Let $\sigma(f)$ denote the order of growth of an entire function f and the hyper-order $\sigma_2(f)$ of f is defined by (see [9], [13])

$$\sigma_2(f) = \limsup_{r \rightarrow +\infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log \log \log M(r, f)}{\log r},$$

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where $T(r, f)$ is the Nevanlinna characteristic function of f and $M(r, f) = \max_{|z|=r} |f(z)|$.

In order to give some estimates of fixed points, we recall the following definition.

Definition 1.1 ([3], [10]) Let f be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\bar{\tau}(f) = \bar{\lambda}(f - z) = \limsup_{r \rightarrow +\infty} \frac{\log \bar{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$. We also define

$$\bar{\lambda}(f - \varphi) = \limsup_{r \rightarrow +\infty} \frac{\log \bar{N}\left(r, \frac{1}{f-\varphi}\right)}{\log r}$$

for any meromorphic function $\varphi(z)$.

In [11], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem A ([11]) Let $A_j(z) (\neq 0) (j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f \neq 0$ of the equation

$$f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0$$

has infinite order and $\sigma_2(f) = 1$.

The main purpose of this paper is to extend and improve the results of Theorem A to some second order linear differential equations. In fact we will prove the following results.

Theorem 1.1 Let $n \geq 2$ be an integer, $A_j(z) (\neq 0) (j = 1, 2)$ be entire functions with $\max\{\sigma(A_j) : j = 1, 2\} < 1$, $Q(z) = q_m z^m + \dots + q_1 z + q_0$ be nonconstant polynomial and a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$. If (1) $\arg a_1 \neq \pi$ and $\arg a_1 \neq \arg a_2$ or (2) $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ and

$|a_2| > n|a_1|$ or (3) $a_1 < 0$ and $\arg a_1 \neq \arg a_2$ or (4) $-\frac{1}{n}(|a_2| - m) < a_1 < 0$,
 $|a_2| > m$ and $\arg a_1 = \arg a_2$, then every solution $f \neq 0$ of the equation

$$f'' + Q(e^{-z})f' + (A_1e^{a_1z} + A_2e^{a_2z})^n f = 0 \quad (1.1)$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$.

Theorem 1.2 Let $A_j(z)$ ($j = 1, 2$), $Q(z)$, a_1 , a_2 , n satisfy the additional hypotheses of Theorem 1.1. If $\varphi \neq 0$ is an entire function of order $\sigma(\varphi) < +\infty$, then every solution $f \neq 0$ of equation (1.1) satisfies

$$\bar{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty,$$

$$\bar{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = 1.$$

Theorem 1.3 Let $A_j(z)$ ($j = 1, 2$), $Q(z)$, a_1 , a_2 , n satisfy the additional hypotheses of Theorem 1.1. If $\varphi \neq 0$ is an entire function of order $\sigma(\varphi) < 1$, then every solution $f \neq 0$ of equation (1.1) satisfies

$$\bar{\lambda}(f - \varphi) = \bar{\lambda}(f' - \varphi) = +\infty.$$

Furthermore, if (i) $(2n + 2)a_1 \neq (2 - p)a_1 + pa_2 - k$ ($p = 0, 1, 2$; $k = 0, 1, \dots, 2m$), $(n + 2 - p)a_1 + pa_2 - k$ ($p = 0, 1, \dots, n + 2$; $k = 0, 1, \dots, m$) or (ii) $(2n + 2)a_2 \neq (2 - p)a_1 + pa_2 - k$ ($p = 0, 1, 2$; $k = 0, 1, \dots, 2m$), $(n + 2 - p)a_1 + pa_2 - k$ ($p = 0, 1, \dots, n + 2$; $k = 0, 1, \dots, m$), then

$$\bar{\lambda}(f'' - \varphi) = +\infty.$$

Corollary 1.1 Let $A_j(z)$ ($j = 1, 2$), $Q(z)$, a_1 , a_2 , n satisfy the additional hypotheses of Theorem 1.1. If $f \neq 0$ is any solution of equation (1.1), then f , f' all have infinitely many fixed points and satisfy

$$\bar{\tau}(f) = \bar{\tau}(f') = \infty.$$

Furthermore, if (i) $(2n + 2)a_1 \neq (2 - p)a_1 + pa_2 - k$ ($p = 0, 1, 2$; $k = 0, 1, \dots, 2m$), $(n + 2 - p)a_1 + pa_2 - k$ ($p = 0, 1, \dots, n + 2$; $k = 0, 1, \dots, m$) or (ii) $(2n + 2)a_2 \neq (2 - p)a_1 + pa_2 - k$ ($p = 0, 1, 2$; $k = 0, 1, \dots, 2m$), $(n + 2 - p)a_1 + pa_2 - k$ ($p = 0, 1, \dots, n + 2$; $k = 0, 1, \dots, m$), then f'' has infinitely many fixed points and satisfies

$$\bar{\tau}(f'') = \infty.$$

2 Preliminary lemmas

To prove our theorems, we need the following lemmas.

Lemma 2.1 ([7]) *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$, $H = \{(k_1, j_1), (k_2, j_2), \dots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ ($i = 1, \dots, q$) and let $\varepsilon > 0$ be a given constant. Then,*

(i) *there exists a set $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2}]$ with linear measure zero, such that, if $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}] \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$, such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \quad (2.1)$$

(ii) *there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \quad (2.2)$$

(iii) *there exists a set $E_3 \subset (0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma+\varepsilon)}. \quad (2.3)$$

Lemma 2.2 ([4]) *Suppose that $P(z) = (\alpha + i\beta)z^n + \dots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)$ ($\neq 0$) is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_4 \cup E_5)$, there is $R > 0$, such that for $|z| = r > R$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\}; \quad (2.4)$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\}, \quad (2.5)$$

where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3 ([11]) *Suppose that $n \geq 1$ is a positive entire number. Let $P_j(z) = a_{jn}z^n + \dots$ ($j = 1, 2$) be nonconstant polynomials, where a_{jq} ($q = 1, \dots, n$) are complex numbers and $a_{1n}a_{2n} \neq 0$. Set $z = re^{i\theta}$, $a_{jn} = |a_{jn}|e^{i\theta_j}$, $\theta_j \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\delta(P_j, \theta) = |a_{jn}| \cos(\theta_j + n\theta)$, then there is a set $E_6 \subset [-\frac{\pi}{2n}, \frac{3\pi}{2n})$ that has linear measure zero. If $\theta_1 \neq \theta_2$, then there exists a ray $\arg z = \theta$, $\theta \in (-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_6 \cup E_7)$, such that*

$$\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0 \tag{2.6}$$

or

$$\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0, \tag{2.7}$$

where $E_7 = \{\theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}) : \delta(P_j, \theta) = 0\}$ is a finite set, which has linear measure zero.

Remark 2.1 ([11]) In Lemma 2.3, if $\theta \in (-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_6 \cup E_7)$ is replaced by $\theta \in (\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_6 \cup E_7)$, then we obtain the same result.

Lemma 2.4 ([5]) *Suppose that $k \geq 2$ and B_0, B_1, \dots, B_{k-1} are entire functions of finite order and let $\sigma = \max\{\sigma(B_j) : j = 0, \dots, k-1\}$. Then every solution f of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \dots + B_1f' + B_0f = 0 \tag{2.8}$$

satisfies $\sigma_2(f) \leq \sigma$.

Lemma 2.5 ([7]) *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_8 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j ($0 \leq i < j \leq k$), such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we have*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B \left\{ \frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right\}^{j-i}. \tag{2.9}$$

Lemma 2.6 ([2]) *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\sigma(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = F, \tag{2.10}$$

then f satisfies

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = +\infty.$$

Lemma 2.7 ([1]) *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation (2.10) with $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma$, then f satisfies*

$$\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma. \tag{2.11}$$

Lemma 2.8 ([6], [13]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;
 - (ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
 - (iii) For $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o\{T(r, e^{g_h(z)-g_k(z)})\}$ ($r \rightarrow \infty, r \notin E_g$), where E_g is a set with finite linear measure.
- Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.9 ([12]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv f_{n+1}$;
- (ii) If $1 \leq j \leq n+1, 1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2, 1 \leq j \leq n+1, 1 \leq h < k \leq n$, and the order of f_j is less than the order of $e^{g_h - g_k}$. Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n+1$).

3 Proof of Theorem 1.1

Assume that $f (\neq 0)$ is a solution of equation (1.1).

First step: We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. We rewrite (1.1) as

$$\frac{f''}{f} + Q(e^{-z}) \frac{f'}{f} + A_1^n e^{na_1 z} + A_2^n e^{na_2 z} + \sum_{p=1}^{n-1} C_n^p A_1^{n-p} e^{(n-p)a_1 z} A_2^p e^{pa_2 z} = 0. \quad (3.1)$$

By Lemma 2.1, for any given ε ,

$$0 < \varepsilon < \min \left\{ \frac{|a_2| - n|a_1|}{2[(2n-1)|a_2| + n|a_1|]}, \frac{1}{2(2n-1)} \right\},$$

there exists a set $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2}]$ of linear measure zero, such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}] \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_0$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq r^{j(\sigma-1+\varepsilon)} \quad (j = 1, 2). \quad (3.2)$$

Let $z = re^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$. We know that $\delta(pa_1 z, \theta) = p\delta(a_1 z, \theta)$ and $\delta(pa_2 z, \theta) = p\delta(a_2 z, \theta)$, where $p > 0$.

Case 1: Assume that $\arg a_1 \neq \pi$ and $\arg a_1 \neq \arg a_2$, which is $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$.

By Lemma 2.2 and Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$ (where E_6 and E_7 are defined as in Lemma 2.3, $E_1 \cup E_6 \cup E_7$ is of the linear measure zero), and satisfying

$$\delta(a_1z, \theta) > 0, \delta(a_2z, \theta) < 0$$

or

$$\delta(a_1z, \theta) < 0, \delta(a_2z, \theta) > 0.$$

a) When $\delta(a_1z, \theta) > 0, \delta(a_2z, \theta) < 0$, for sufficiently large r , we get by Lemma 2.2

$$|A_1^n e^{na_1z}| \geq \exp\{(1 - \varepsilon)n\delta(a_1z, \theta)r\}, \quad (3.3)$$

$$|A_2^n e^{na_2z}| \leq \exp\{(1 - \varepsilon)n\delta(a_2z, \theta)r\} < 1, \quad (3.4)$$

$$\begin{aligned} |A_1^{n-p} e^{(n-p)a_1z}| &\leq \exp\{(1 + \varepsilon)(n - p)\delta(a_1z, \theta)r\} \\ &\leq \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\}, \quad p = 1, \dots, n - 1, \end{aligned} \quad (3.5)$$

$$|A_2^p e^{pa_2z}| \leq \exp\{(1 - \varepsilon)p\delta(a_2z, \theta)r\} < 1, \quad p = 1, \dots, n - 1. \quad (3.6)$$

For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$\begin{aligned} |Q(e^{-z})| &= |q_m e^{-mz} + \dots + q_1 e^{-z} + q_0| \\ &\leq |q_m| |e^{-mz}| + \dots + |q_1| |e^{-z}| + |q_0| \\ &\leq |q_m| e^{-mr \cos \theta} + \dots + |q_1| e^{-r \cos \theta} + |q_0| \leq M, \end{aligned} \quad (3.7)$$

where $M > 0$ is a some constant. By (3.1) – (3.7), we get

$$\begin{aligned} &\exp\{(1 - \varepsilon)n\delta(a_1z, \theta)r\} \leq |A_1^n e^{na_1z}| \\ &\leq \left| \frac{f''}{f} \right| + |Q(e^{-z})| \left| \frac{f'}{f} \right| + |A_2^n e^{na_2z}| + \sum_{p=1}^{n-1} C_n^p |A_1^{n-p} e^{(n-p)a_1z}| |A_2^p e^{pa_2z}| \\ &\leq r^{2(\sigma-1+\varepsilon)} + Mr^{\sigma-1+\varepsilon} + 2^n \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\} \\ &\leq M_1 r^{M_2} \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\}, \end{aligned} \quad (3.8)$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. By $0 < \varepsilon < \frac{1}{2(2n-1)}$ and (3.8), we have

$$\exp\left\{\frac{1}{2}\delta(a_1z, \theta)r\right\} \leq M_1 r^{M_2}. \quad (3.9)$$

By $\delta(a_1z, \theta) > 0$ we know that (3.9) is a contradiction.

b) When $\delta(a_1z, \theta) < 0$, $\delta(a_2z, \theta) > 0$, using a proof similar to the above, we can also get a contradiction.

Case 2: Assume that $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ and $|a_2| > n|a_1|$, which is $\theta_1 \neq \pi$ and $\theta_1 = \theta_2$ and $|a_2| > n|a_1|$.

By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_1z, \theta) > 0$. Since $|a_2| > n|a_1|$ and $n \geq 2$, then $|a_2| > |a_1|$, thus $\delta(a_2z, \theta) > \delta(a_1z, \theta) > 0$. For sufficiently large r , we have by using Lemma 2.2

$$|A_2^n e^{na_2z}| \geq \exp\{(1 - \varepsilon)n\delta(a_2z, \theta)r\}, \quad (3.10)$$

$$|A_1^n e^{na_1z}| \leq \exp\{(1 + \varepsilon)n\delta(a_1z, \theta)r\}, \quad (3.11)$$

$$\left| A_1^{n-p} e^{(n-p)a_1z} \right| \leq \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\}, \quad p = 1, \dots, n - 1, \quad (3.12)$$

$$|A_2^p e^{pa_2z}| \leq \exp\{(1 + \varepsilon)(n - 1)\delta(a_2z, \theta)r\}, \quad p = 1, \dots, n - 1. \quad (3.13)$$

By (3.1), (3.2), (3.7) and (3.10) – (3.13) we get

$$\begin{aligned} & \exp\{(1 - \varepsilon)n\delta(a_2z, \theta)r\} \leq |A_2^n e^{na_2z}| \\ & \leq \left| \frac{f''}{f} \right| + |Q(e^{-z})| \left| \frac{f'}{f} \right| + |A_1^n e^{na_1z}| + \sum_{p=1}^{n-1} C_n^p \left| A_1^{n-p} e^{(n-p)a_1z} \right| |A_2^p e^{pa_2z}| \\ & \leq r^{2(\sigma-1+\varepsilon)} + Mr^{\sigma-1+\varepsilon} + \exp\{(1 + \varepsilon)n\delta(a_1z, \theta)r\} \\ & \quad + 2^n \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\} \exp\{(1 + \varepsilon)(n - 1)\delta(a_2z, \theta)r\} \\ & \leq M_1 r^{M_2} \exp\{(1 + \varepsilon)n\delta(a_1z, \theta)r\} \exp\{(1 + \varepsilon)(n - 1)\delta(a_2z, \theta)r\}. \quad (3.14) \end{aligned}$$

Therefore, by (3.14), we obtain

$$\exp\{\alpha r\} \leq M_1 r^{M_2}, \quad (3.15)$$

where

$$\alpha = [1 - \varepsilon(2n - 1)]\delta(a_2z, \theta) - (1 + \varepsilon)n\delta(a_1z, \theta).$$

Since $0 < \varepsilon < \frac{|a_2| - n|a_1|}{2[(2n-1)|a_2| + n|a_1|]}$, $\theta_1 = \theta_2$ and $\cos(\theta_1 + \theta) > 0$, then

$$\begin{aligned} \alpha &= [1 - \varepsilon(2n - 1)]|a_2| \cos(\theta_2 + \theta) - (1 + \varepsilon)n|a_1| \cos(\theta_1 + \theta) \\ &= \{|a_2| - n|a_1| - \varepsilon[(2n - 1)|a_2| + n|a_1|]\} \cos(\theta_1 + \theta) \end{aligned}$$

$$> \frac{|a_2| - n|a_1|}{2} \cos(\theta_1 + \theta) > 0.$$

Hence (3.15) is a contradiction.

Case 3: Assume that $a_1 < 0$ and $\arg a_1 \neq \arg a_2$, which is $\theta_1 = \pi$ and $\theta_2 \neq \pi$.

By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_2z, \theta) > 0$. Because $\cos \theta > 0$, we have $\delta(a_1z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos \theta < 0$. For sufficiently large r , we obtain by Lemma 2.2

$$|A_2^n e^{na_2z}| \geq \exp\{(1 - \varepsilon)n\delta(a_2z, \theta)r\}, \quad (3.16)$$

$$|A_1^n e^{na_1z}| \leq \exp\{(1 - \varepsilon)n\delta(a_1z, \theta)r\} < 1, \quad (3.17)$$

$$\left| A_1^{n-p} e^{(n-p)a_1z} \right| \leq \exp\{(1 - \varepsilon)(n - p)\delta(a_1z, \theta)r\} < 1, \quad p = 1, \dots, n - 1, \quad (3.18)$$

$$|A_2^p e^{pa_2z}| \leq \exp\{(1 + \varepsilon)(n - 1)\delta(a_2z, \theta)r\}, \quad p = 1, \dots, n - 1. \quad (3.19)$$

Using the same reasoning as in Case 1(a), we can get a contradiction.

Case 4. Assume that $-\frac{1}{n}(|a_2| - m) < a_1 < 0$, $|a_2| > m$ and $\arg a_1 = \arg a_2$, which is $\theta_1 = \theta_2 = \pi$ and $|a_1| < \frac{1}{n}(|a_2| - m)$, then $|a_2| > n|a_1| + m$, hence $|a_2| > n|a_1|$.

By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, then $\cos \theta < 0$, $\delta(a_1z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos \theta > 0$, $\delta(a_2z, \theta) = |a_2| \cos(\theta_2 + \theta) = -|a_2| \cos \theta > 0$. Since $|a_2| > n|a_1|$ and $n \geq 2$, then $|a_2| > |a_1|$, thus $\delta(a_2z, \theta) > \delta(a_1z, \theta) > 0$, for sufficiently large r , we get (3.10) – (3.13) hold. For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ we have

$$|Q(e^{-z})| \leq M e^{-mr \cos \theta}. \quad (3.20)$$

By (3.1), (3.2), (3.10) – (3.13) and (3.20), we get

$$\begin{aligned} & \exp\{(1 - \varepsilon)n\delta(a_2z, \theta)r\} \leq |A_2^n e^{na_2z}| \\ & \leq \left| \frac{f''}{f} \right| + |Q(e^{-z})| \left| \frac{f'}{f} \right| + |A_1^n e^{na_1z}| + \sum_{p=1}^{n-1} C_n^p \left| A_1^{n-p} e^{(n-p)a_1z} \right| |A_2^p e^{pa_2z}| \\ & \leq r^{2(\sigma-1+\varepsilon)} + M r^{\sigma-1+\varepsilon} e^{-mr \cos \theta} + \exp\{(1 + \varepsilon)n\delta(a_1z, \theta)r\} \\ & \quad + 2^n \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\} \exp\{(1 + \varepsilon)(n - 1)\delta(a_2z, \theta)r\} \end{aligned}$$

$$\leq M_1 r^{M_2} e^{-mr \cos \theta} \exp \{ (1 + \varepsilon) n \delta(a_1 z, \theta) r \} \exp \{ (1 + \varepsilon) (n - 1) \delta(a_2 z, \theta) r \}. \quad (3.21)$$

Therefore, by (3.21), we obtain

$$\exp \{ \beta r \} \leq M_1 r^{M_2}, \quad (3.22)$$

where

$$\beta = [1 - \varepsilon (2n - 1)] \delta(a_2 z, \theta) - (1 + \varepsilon) n \delta(a_1 z, \theta) + m \cos \theta.$$

Since $|a_2| - n|a_1| - m > 0$, then

$$2[(2n - 1)|a_2| + n|a_1|] > |a_2| - n|a_1| - m > 0.$$

Therefore,

$$\frac{|a_2| - n|a_1| - m}{2[(2n - 1)|a_2| + n|a_1|]} < 1.$$

Then, we can take $0 < \varepsilon < \frac{|a_2| - n|a_1| - m}{2[(2n - 1)|a_2| + n|a_1|]}$. Since $0 < \varepsilon < \frac{|a_2| - n|a_1| - m}{2[(2n - 1)|a_2| + n|a_1|]}$, $\theta_1 = \theta_2 = \pi$ and $\cos \theta < 0$, then

$$\begin{aligned} \beta &= -\cos \theta \{ |a_2| - n|a_1| - m - \varepsilon [(2n - 1)|a_2| + n|a_1|] \} \\ &> -\frac{1}{2} (|a_2| - n|a_1| - m) \cos \theta > 0. \end{aligned}$$

Hence, (3.22) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = +\infty$.

Second step: We prove that $\sigma_2(f) = 1$. By

$$\max \{ \sigma(Q(e^{-z})), \sigma((A_1 e^{a_1 z} + A_2 e^{a_2 z})^n) \} = 1$$

and the Lemma 2.4, we get $\sigma_2(f) \leq 1$. By Lemma 2.5, we know that there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we get

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B [T(2r, f)]^{j+1} \quad (j = 1, 2). \quad (3.23)$$

Case 1: $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_1 z, \theta) > 0, \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0, \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1z, \theta) > 0$, $\delta(a_2z, \theta) < 0$, for sufficiently large r , we get (3.3) – (3.7) holds. By (3.1), (3.3) – (3.7) and (3.23), we obtain

$$\begin{aligned} & \exp\{(1 - \varepsilon)n\delta(a_1z, \theta)r\} \leq |A_1^n e^{na_1z}| \\ & \leq \left| \frac{f''}{f} \right| + |Q(e^{-z})| \left| \frac{f'}{f} \right| + |A_2^n e^{na_2z}| + \sum_{p=1}^{n-1} C_n^p |A_1^{n-p} e^{(n-p)a_1z}| |A_2^p e^{pa_2z}| \\ & \leq B [T(2r, f)]^3 + MB [T(2r, f)]^2 + 2^n \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\} \\ & \leq M_1 \exp\{(1 + \varepsilon)(n - 1)\delta(a_1z, \theta)r\} [T(2r, f)]^3. \end{aligned} \quad (3.24)$$

By $0 < \varepsilon < \frac{1}{2(2n-1)}$ and (3.24), we have

$$\exp\left\{\frac{1}{2}\delta(a_1z, \theta)r\right\} \leq M_1 [T(2r, f)]^3. \quad (3.25)$$

By $\delta(a_1z, \theta) > 0$ and (3.25), we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

b) When $\delta(a_1z, \theta) < 0$, $\delta(a_2z, \theta) > 0$, using a proof similar to the above, we can also get $\sigma_2(f) = 1$.

Case 2: $\theta_1 \neq \pi$, $\theta_1 = \theta_2$ and $|a_2| > n|a_1|$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2z, \theta) > \delta(a_1z, \theta) > 0$$

and for sufficiently large r , we get (3.7) and (3.10) – (3.13) hold. By (3.1), (3.7), (3.10) – (3.13) and (3.23), we get

$$\exp\{\alpha r\} \leq M_1 [T(2r, f)]^3, \quad (3.26)$$

where

$$\alpha = [1 - \varepsilon(2n - 1)]\delta(a_2z, \theta) - (1 + \varepsilon)n\delta(a_1z, \theta) > 0.$$

By $\alpha > 0$ and (3.26), we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

Case 3: $a_1 < 0$ and $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2z, \theta) > 0 \text{ and } \delta(a_1z, \theta) < 0$$

and for sufficiently large r , we get (3.16) – (3.19) hold. Using the same reasoning as in second step (**Case 1** (a)), we can get $\sigma_2(f) = 1$.

Case 4: $-\frac{1}{n}(|a_2| - m) < a_1 < 0$, $|a_2| > m$ and $\theta_1 = \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$$

and for sufficiently large r , we get (3.10)–(3.13) hold. By (3.1), (3.10)–(3.13), (3.20) and (3.23) we obtain

$$\exp\{\beta r\} \leq M_1 [T(2r, f)]^3, \quad (3.27)$$

where

$$\beta = [1 - \varepsilon(2n - 1)]\delta(a_2 z, \theta) - (1 + \varepsilon)n\delta(a_1 z, \theta) + m \cos \theta > 0.$$

By $\beta > 0$ and (3.27), we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$. Concluding the above proof, we obtain $\sigma_2(f) = 1$. The proof of Theorem 1.1 is complete.

Example 1.1 Consider the differential equation

$$f'' + (-4e^{-3z} - 4ie^{-z} - 1)f' + (ie^z + 2e^{-z})^2 f = 0, \quad (3.28)$$

where $Q(z) = -4z^3 - 4iz - 1$, $a_1 = 1$, $a_2 = -1$, $A_1(z) = i$ and $A_2(z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.28).

Example 1.2 Consider the differential equation

$$f'' + (-8e^{-2z} - 12e^{i\frac{\pi}{3}}e^{-z} - 1 - 6e^{i\frac{2\pi}{3}})f' + (e^{i\frac{\pi}{3}}e^{\frac{2}{3}z} + 2e^{-\frac{1}{3}z})^3 f = 0, \quad (3.29)$$

where $Q(z) = -8z^2 - 12e^{i\frac{\pi}{3}}z - 1 - 6e^{i\frac{2\pi}{3}}$, $a_1 = \frac{2}{3}$, $a_2 = -\frac{1}{3}$, $A_1(z) = e^{i\frac{\pi}{3}}$ and $A_2(z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.29).

Example 1.3 Consider the differential equation

$$f'' + (-e^{-3z} - 4e^{i\frac{\pi}{4}}e^{-2z} - 6ie^{-z} - 1 - 4e^{i\frac{3\pi}{4}})f' + (e^{-\frac{1}{2}z} + e^{i\frac{\pi}{4}}e^{\frac{1}{2}z})^4 f = 0, \quad (3.30)$$

where $Q(z) = -z^3 - 4e^{i\frac{\pi}{4}}z^2 - 6iz - 1 - 4e^{i\frac{3\pi}{4}}$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$, $A_1(z) = 1$ and $A_2(z) = e^{i\frac{\pi}{4}}$. Obviously, the conditions of Theorem 1.1 (3) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.30).

4 Proof of Theorem 1.2

We prove that $\bar{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty$ and $\bar{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = 1$. First, setting $\omega = f - \varphi$. Since $\sigma(\varphi) < \infty$, then we have $\sigma(\omega) = \sigma(f) = +\infty$. From (1.1), we have

$$\omega'' + Q(e^{-z})\omega' + (A_1e^{a_1z} + A_2e^{a_2z})^n\omega = H, \quad (4.1)$$

where $H = -[\varphi'' + Q(e^{-z})\varphi' + (A_1e^{a_1z} + A_2e^{a_2z})^n\varphi]$. Now we prove that $H \not\equiv 0$. In fact if $H \equiv 0$, then

$$\varphi'' + Q(e^{-z})\varphi' + (A_1e^{a_1z} + A_2e^{a_2z})^n\varphi = 0. \quad (4.2)$$

Hence φ is a solution of equation (1.1) with $\sigma(\varphi) = \infty$ and by Theorem 1.1, it is a contradiction. Since $\sigma(f) = \infty$, $\sigma(\varphi) < \infty$ and $\sigma_2(f) = 1$, we get $\sigma_2(\omega) = \sigma_2(f - \varphi) = \sigma_2(f) = 1$. By the Lemma 2.6 and Lemma 2.7, we have $\bar{\lambda}(\omega) = \lambda(\omega) = \sigma(\omega) = \sigma(f) = +\infty$ and $\bar{\lambda}_2(\omega) = \lambda_2(\omega) = \sigma_2(\omega) = \sigma_2(f) = 1$, i.e., $\bar{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty$ and $\bar{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = 1$.

5 Proof of Theorem 1.3

Suppose that $f \not\equiv 0$ is a solution of equation (1.1), then $\sigma(f) = +\infty$ by Theorem 1.1. Since $\sigma(\varphi) < 1$, then by Theorem 1.2, we have $\bar{\lambda}(f - \varphi) = +\infty$. Now we prove that $\bar{\lambda}(f' - \varphi) = \infty$. Set $g_1(z) = f'(z) - \varphi(z)$, then $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$. Set $B(z) = Q(e^{-z})$ and $R(z) = A_1e^{a_1z} + A_2e^{a_2z}$, then $B'(z) = -e^{-z}Q'(e^{-z})$ and $R' = (A'_1 + a_1A_1)e^{a_1z} + (A'_2 + a_2A_2)e^{a_2z}$. Differentiating both sides of equation (1.1), we have

$$f''' + Bf'' + (B' + R^n)f' + nR'R^{n-1}f = 0. \quad (5.1)$$

By (1.1), we have

$$f = -\frac{1}{R^n}(f'' + Bf'). \quad (5.2)$$

Substituting (5.2) into (5.1), we have

$$f''' + \left(B - n\frac{R'}{R}\right)f'' + \left(B' + R^n - nB\frac{R'}{R}\right)f' = 0. \quad (5.3)$$

Substituting $f' = g_1 + \varphi$, $f'' = g'_1 + \varphi'$, $f''' = g''_1 + \varphi''$ into (5.3), we get

$$g''_1 + E_1g'_1 + E_0g_1 = E, \quad (5.4)$$

where

$$E_1 = B - n \frac{R'}{R}, \quad E_0 = B' + R^n - nB \frac{R'}{R},$$

$$E = - \left\{ \varphi'' + \left(B - n \frac{R'}{R} \right) \varphi' + \left(B' + R^n - nB \frac{R'}{R} \right) \varphi \right\}.$$

Now we prove that $E \neq 0$. In fact, if $E \equiv 0$, then we get

$$\frac{\varphi''}{\varphi} R + \frac{\varphi'}{\varphi} (BR - nR') + B'R - nBR' + R^{n+1} = 0. \quad (5.5)$$

Obviously $\frac{\varphi''}{\varphi}$, $\frac{\varphi'}{\varphi}$ are meromorphic functions with $\sigma\left(\frac{\varphi''}{\varphi}\right) < 1$, $\sigma\left(\frac{\varphi'}{\varphi}\right) < 1$.

We can rewrite (5.5) in the form

$$\sum_{k=0}^m f_k e^{(a_1-k)z} + \sum_{l=0}^m h_l e^{(a_2-l)z} + \sum_{p=1}^n C_{n+1}^p A_1^{n+1-p} A_2^p e^{[(n+1-p)a_1+pa_2]z}$$

$$+ A_1^{n+1} e^{(n+1)a_1 z} + A_2^{n+1} e^{(n+1)a_2 z} = 0, \quad (5.6)$$

where f_k ($k = 0, 1, \dots, m$) and h_l ($l = 0, 1, \dots, m$) are meromorphic functions with $\sigma(f_k) < 1$ and $\sigma(h_l) < 1$. Set $I = \{a_1 - k$ ($k = 0, 1, \dots, m$), $a_2 - l$ ($l = 0, 1, \dots, m$), $(n+1-p)a_1 + pa_2$ ($p = 1, 2, \dots, n$), $(n+1)a_1$, $(n+1)a_2\}$. By the conditions of the Theorem 1.1, it is clear that $(n+1)a_1 \neq a_1$, $(n+1)a_2$, $(n+1-p)a_1 + pa_2$ ($p = 1, 2, \dots, n$).

(i) If $(n+1)a_1 \neq a_1 - k$ ($k = 1, \dots, m$), $a_2 - l$ ($l = 0, 1, \dots, m$), then we write (5.6) in the form

$$A_1^{n+1} e^{(n+1)a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq I \setminus \{(n+1)a_1\}$. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, it is a contradiction.

(ii) If $(n+1)a_1 = \gamma$ such that $\gamma \in \{a_1 - k$ ($k = 1, \dots, m$), $a_2 - l$ ($l = 0, 1, \dots, m$)}, then $(n+1)a_2 \neq \beta$ for all $\beta \in I \setminus \{(n+1)a_2\}$. Hence, we write (5.6) in the form

$$A_2^{n+1} e^{(n+1)a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq I \setminus \{(n+1)a_2\}$. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $E \neq 0$ is proved. We know that the functions E_1 , E_0 and E are of finite order. By Lemma 2.6 and (5.4), we have $\bar{\lambda}(g_1) = \bar{\lambda}(f' - \varphi) = \infty$.

Now we prove that $\bar{\lambda}(f'' - \varphi) = \infty$. Set $g_2(z) = f''(z) - \varphi(z)$, then $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$. Differentiating both sides of equation (1.1), we have

$$\begin{aligned} f^{(4)} + Bf''' + (2B' + R^n)f'' + (B'' + 2nR'R^{n-1})f' \\ + n[R''R^{n-1} + (n-1)R'^2R^{n-2}]f = 0. \end{aligned} \quad (5.7)$$

Combining (5.2) with (5.7), we get

$$\begin{aligned} f^{(4)} + Bf''' + \left(2B' + R^n - n\frac{R''}{R} - n(n-1)\frac{R'^2}{R^2}\right)f'' \\ + \left(B'' + 2nR'R^{n-1} - nB\frac{R''}{R} - n(n-1)B\frac{R'^2}{R^2}\right)f' = 0. \end{aligned} \quad (5.8)$$

Now we prove that $B' + R^n - nB\frac{R'}{R} \not\equiv 0$. Suppose that $B' + R^n - nB\frac{R'}{R} \equiv 0$, then we have

$$B'R + R^{n+1} - nBR' = 0. \quad (5.9)$$

We can write (5.9) in the form (5.6), then by the same reasoning as in the proof of $\bar{\lambda}(f' - \varphi) = \infty$ we get a contradiction. Hence $B' + R^n - nB\frac{R'}{R} \not\equiv 0$ is proved. Set

$$\psi(z) = B'R + R^{n+1} - nBR', \quad (5.10)$$

$$S_1 = 2B'R^2 + R^{n+2} - nR''R - n(n-1)R'^2, \quad (5.11)$$

$$S_2 = B''R^2 + 2nR'R^{n+1} - nBR''R - n(n-1)BR'^2, \quad (5.12)$$

$$S_3 = BR - nR'. \quad (5.13)$$

By (5.3), (5.10) and (5.13), we get

$$f' = -\frac{R}{\psi(z)} \left(f''' + \frac{S_3}{R} f'' \right). \quad (5.14)$$

By (5.14), (5.11), (5.12) and (5.8), we obtain

$$f^{(4)} + \left(B - \frac{S_2}{R\psi(z)} \right) f''' + \left(\frac{S_1}{R^2} - \frac{S_2S_3}{R^2\psi(z)} \right) f'' = 0. \quad (5.15)$$

Substituting $f'' = g_2 + \varphi$, $f''' = g_2' + \varphi'$, $f^{(4)} = g_2'' + \varphi''$ into (5.15) we get

$$g_2'' + H_1g_2' + H_0g_2 = H, \quad (5.16)$$

where

$$H_1 = B - \frac{S_2}{R\psi(z)}, \quad H_0 = \frac{S_1}{R^2} - \frac{S_2S_3}{R^2\psi(z)},$$

$$-H = \varphi'' + \varphi' H_1 + \varphi H_0.$$

We can get

$$H_1 = \frac{L_1(z)}{R\psi(z)}, H_0 = \frac{L_0(z)}{R\psi(z)}, \quad (5.17)$$

where

$$L_1(z) = B'BR^2 + BR^{n+2} - nB^2R'R - B''R^2 - 2nR'R^{n+1} \\ + nBR'R + n(n-1)BR'^2, \quad (5.18)$$

$$L_0(z) = 2B'^2R^2 + 3B'R^{n+2} - 2nB'BR'R + R^{2n+2} - 3nBR'R^{n+1} \\ - nB'R''R - nR''R^{n+1} - n(n-1)B'R'^2 + (n^2+n)R'^2R^n - B''BR^2 \\ + nB^2R''R + n(n-1)B^2R'^2 + nB''R'R. \quad (5.19)$$

Therefore

$$\frac{-H}{\varphi} = \frac{1}{R\psi(z)} \left(\frac{\varphi''}{\varphi} R\psi(z) + \frac{\varphi'}{\varphi} L_1(z) + L_0(z) \right), \quad (5.20)$$

$$R\psi(z) = B'R^2 + R^{n+2} - nBR'R. \quad (5.21)$$

Now we prove that $-H \neq 0$. In fact, if $-H \equiv 0$, then by (5.20) we have

$$\frac{\varphi''}{\varphi} R\psi(z) + \frac{\varphi'}{\varphi} L_1(z) + L_0(z) = 0. \quad (5.22)$$

Obviously, $\frac{\varphi''}{\varphi}$ and $\frac{\varphi'}{\varphi}$ are meromorphic functions with $\sigma\left(\frac{\varphi''}{\varphi}\right) < 1$, $\sigma\left(\frac{\varphi'}{\varphi}\right) < 1$. By (5.18), (5.19) and (5.21), we can rewrite (5.22) in the form

$$A_1^{2n+2} e^{(2n+2)a_1 z} + A_2^{2n+2} e^{(2n+2)a_2 z} + \sum_{p=1}^{2n+1} C_{2n+2}^p A_1^{2n+2-p} A_2^p e^{[(2n+2-p)a_1 + pa_2]z} \\ + \sum_{\substack{0 \leq p \leq 2 \\ 0 \leq k \leq 2m}} f_{p,k} e^{[(2-p)a_1 + pa_2 - k]z} + \sum_{\substack{0 \leq p \leq n+2 \\ 0 \leq k \leq m}} h_{p,k} e^{[(n+2-p)a_1 + pa_2 - k]z} = 0, \quad (5.23)$$

where $f_{p,k}$ ($0 \leq p \leq 2, 0 \leq k \leq 2m$) and $h_{p,k}$ ($0 \leq p \leq n+2, 0 \leq k \leq m$) are meromorphic functions with $\sigma(f_{p,k}) < 1$ and $\sigma(h_{p,k}) < 1$. Set $J = \{(2n+2)a_1, (2n+2)a_2, (2n+2-p)a_1 + pa_2$ ($p = 1, 2, \dots, 2n+1$), $(2-p)a_1 + pa_2 - k$ ($p = 0, 1, 2; k = 0, \dots, 2m$), $(n+2-p)a_1 + pa_2 - k$ ($p = 0, 1, \dots, n+2; k = 0, 1, \dots, m$)}. By the conditions of Theorem 1.3, it is clear that $(2n+2)a_1 \neq (2n+2)a_2, (2n+2-p)a_1 + pa_2$ ($p = 1, 2, \dots, 2n+1$),

$2a_1, (n+2)a_1$ and $(2n+2)a_2 \neq (2n+2)a_1, (2n+2-p)a_1 + pa_2$ ($p = 1, 2, \dots, 2n+1$), $2a_2, (n+2)a_2$.

(1) By the conditions of Theorem 1.3 (i), we have $(2n+2)a_1 \neq \beta$ for all $\beta \in J \setminus \{(2n+2)a_1\}$, hence we write (5.23) in the form

$$A_1^{2n+2} e^{(2n+2)a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq J \setminus \{(2n+2)a_1\}$. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, it is a contradiction.

(2) By the conditions of Theorem 1.3 (ii), we have $(2n+2)a_2 \neq \beta$ for all $\beta \in J \setminus \{(2n+2)a_2\}$, hence we write (5.23) in the form

$$A_2^{2n+2} e^{(2n+2)a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq J \setminus \{(2n+2)a_2\}$. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $H \not\equiv 0$ is proved. We know that the functions H_1, H_0 and H are of finite order. By Lemma 2.6 and (5.16), we have $\bar{\lambda}(g_2) = \bar{\lambda}(f'' - \varphi) = \infty$. The proof of Theorem 1.3 is complete.

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