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THE FREQUENCY OF THE ZEROS OF SOME DIFFERENTIAL POLYNOMIALS

Abstract. Let $\rho_p(f)$ and $\sigma_p(f)$ denote respectively the iterated p-order and the iterated p-type of an entire function f. In this paper, we study the iterated order and the fixed points of some differential polynomials generated by solutions of the differential equation

$$f'' + A_1(z)f' + A_0(z)f = 0,$$

where $A_1(z)$, $A_0(z)$ are entire functions of finite iterated p-order such that $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$.

1. Introduction and statement of results

In this paper, it is assumed that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [9, 15]). For the definition of the iterated order of an entire function, we use the same definition as in [10], [4, p. 317], [11, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$(1.1) \qquad \rho_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \geq 1 \text{ is an integer}),$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire

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function, then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r} \ (p \geq 1 \text{ is an integer}),$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order (see [9, 15, 18]).

Definition 1.2. (See [4, 11].) The finiteness degree of the order of an entire function f is defined by

(1.2)

$$i(f) = \begin{cases} 0, & \text{for } f \text{ polynomial,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{for } f \text{ transcendental for} \\ & \text{which some } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.3. [7] Let f be a meromorphic function. Then the iterated p-type of f, with iterated p-order $0 < \rho_p(f) < \infty$ is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ is an integer}). \tag{1.3}$$

If f is an entire function, then the iterated p-type of f, with iterated p-order $0 < \rho_p(f) < \infty$ is defined by [3]

$$\sigma_{M,p}(f) = \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ is an integer}).$$

For $p = 1$, this notation is called the type of f (see [13]).

Remark 1.1. For entire function, we can have $\sigma_{M,1}(f) \neq \sigma_1(f)$. For example, if $f(z) = e^z$, then we have $\sigma_{M,1}(f) = 1$ and $\sigma_1(f) = \frac{1}{\pi}$. However, it follows by Proposition 2.2.2 in [11] that $\sigma_{M,p}(f) = \sigma_p(f)$ for $p \geq 2$.

Definition 1.4. (See [10, 12].) Let f be a meromorphic function. Then the iterated convergence exponent of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r, 1/f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.4}$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$, and the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r, 1/f)}{\log r} \quad (p \geq 1 \text{ is an integer}),$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

Definition 1.5. (See [12].) Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of fixed points of $f(z)$ is defined by

$$(1.5)\quad \tau_p(f)=\lambda_p(f-z)=\limsup_{r\to+\infty}\frac{\log_p N\left(r,\frac{1}{f-z}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}),$$

and the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\limsup_{r\to+\infty}\frac{\log_p \overline{N}\left(r,\frac{1}{f-z}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}).$$

Thus $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions (see [17]). However, there are few studies on the fixed points of solutions of differential equations. It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [8]). In [14], Liu and Zhang investigated fixed points and hyper order of some higher order linear differential equations with meromorphic coefficients. In [16], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients.

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G}\subset\mathbb{C}$. If $\mathbf{G}=\mathbb{C}$, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{C})$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho}=\{g \text{ meromorphic: } \rho_{p+1}(g)<\rho\},$$

where ρ is a positive constant. In [12], Laine and Rieppo gave an improvement of the results of [16] by considering fixed points and iterated order and obtained the following result.

Theorem A. [12] *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A)=\rho>0$ such that $\delta(\infty,A)=\lim\limits_{r\to+\infty}\frac{m(r,A)}{T(r,A)}=\delta$ >0, and let f be a transcendental meromorphic solution of the equation*

$$(1.6)\qquad f''+A(z)f=0.$$

Suppose, moreover, that either:

(i) *all poles of* f *are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_{p+1}(f) = \rho_p(A) = \rho$. *Moreover, let*

$$P[f] = P(f, f', \dots, f^{(m)}) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.7}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, *assuming that at least one of the coefficients* p_j *does vanish identically. Then for the fixed points of* $P[f]$, *we have* $\overline{\tau}_{p+1}(P[f]) = \rho$, *provided that neither* $P[f]$ *nor* $P[f] - z$ *vanishes identically.*

Remark 1.2. (See [12, p. 904].) In Theorem A, in order to study $P[f]$, the authors consider $m \leq 1$. Indeed, if $m \geq 2$, we obtain, by repeated differentiation of (1.6), that $f^{(k)} = q_{k,0} f + q_{k,1} f'$, $q_{k,0}$, $q_{k,1} \in \mathcal{L}_{p+1,\rho}$ for $k = 2, \dots, m$. Substitution into (1.7) yields the required reduction.

Recently, the author has studied the relationship between solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \quad (k \geq 2), \tag{1.8}$$

and entire functions with finite iterated p-order and have obtained the following result.

Theorem B. [3] *Let* $A_0(z), \dots, A_{k-1}(z)$ *be entire functions, and let* $i(A_0) = p$ $(1 \leq p < \infty)$. *Assume that* $\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} \leq \rho_p(A_0) = \rho (0 < \rho < +\infty)$ *and* $\max\{\sigma_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. *If* $\varphi(z) \not\equiv 0$ *is an entire function with finite iterated* p*-order* $\rho_p(\varphi) < +\infty$, *then every solution* $f \not\equiv 0$ *of equation* (1.8) *satisfies*

$$\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f) = +\infty \tag{1.9}$$

and

$$\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho. \tag{1.10}$$

Consider the linear differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0, \tag{1.11}$$

where $A_1(z)$, $A_0(z)$ are entire functions of finite iterated p-order.

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations. The main purpose of this paper is to investigate the growth and the fixed points of the linear differential polynomial $g_f = d_1 f' + d_0 f$, where $d_0(z), d_1(z)$ are entire functions, generated by solutions of equation (1.11). Instead of looking at the zeros of $g_f - z$, we proceed to a slight generalization by considering zeros of $g_f - \varphi$, where φ is an entire function of finite iterated p-order, while the solution of respective differential equation is of infinite iterated p-order. We obtain some estimates of their iterated order and fixed points.

Theorem 1.1. *Let $A_1(z)$, $A_0(z)$ be entire functions, and let $i(A_j) = p$ $(j = 0, 1)$, $(1 \le p < \infty)$ such that $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Let $d_0(z), d_1(z)$ be entire functions such that at least one of $d_0(z), d_1(z)$ does not vanish identically with $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$, and let $\varphi(z) \not\equiv 0$ be an entire function with $\rho_p(\varphi) < \infty$. If $f \not\equiv 0$ is a solution of equation* (1.11)*, then the differential polynomial $g_f = d_1 f' + d_0 f$ satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty, \tag{1.12}$$

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho. \tag{1.13}$$

Applying Theorem 1.1 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.1. *Under the hypotheses of Theorem* 1.1*. If $f \not\equiv 0$ is a solution of equation* (1.11)*, then the differential polynomial $g_f = d_1 f' + d_0 f$ satisfies $\overline{\tau}_p(g_f) = \tau_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\tau}_{p+1}(g_f) = \tau_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$.*

In the following, we obtain a result which is an application of Theorem 1.1.

Theorem 1.2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $|b_n| > |a_n|$. Let $h_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\max\{\rho_p(h_j) : j = 0, 1\} < n$ $(1 \le p < \infty)$. Let $d_0(z), d_1(z)$ be entire functions such that at least one of $d_0(z), d_1(z)$ does not vanish identically with $\max\{\rho_p(d_j) : j = 0, 1\} < n$, and let $\varphi(z) \not\equiv 0$ be an entire function with $\rho_p(\varphi) < +\infty$. If $f \not\equiv 0$ is a solution of the equation*

$$f'' + h_1(z) \exp_p\{P(z)\} f' + h_0(z) \exp_p\{Q(z)\} f = 0, \tag{1.14}$$

then the differential polynomial $g_f = d_1 f' + d_0 f$ satisfies

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty, \tag{1.15}$$

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = n. \tag{1.16}$$

Applying Theorem 1.2 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.2. *Under the assumptions of Theorem* 1.2, *if* $f \not\equiv 0$ *is a solution of equation* (1.14), *then the differential polynomial* $g_f = d_1 f' + d_0 f$ *satisfies* $\overline{\tau}_p(g_f) = \tau_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$ *and* $\overline{\tau}_{p+1}(g_f) = \tau_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = n$.

2. Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 2.1. (See Remark 1.3 of [10].) *If* f *is a meromorphic function with* $i(f) = p \geq 1$, *then* $\rho_p(f) = \rho_p(f')$.

Lemma 2.2. [12] *If* f *is a meromorphic function with* $0 < \rho_p(f) < \rho$ $(p \geq 1)$, *then* $\rho_{p+1}(f) = 0$.

Lemma 2.3. [2, 5] *Let* A_0, $A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ *be finite iterated* p-*order meromorphic functions. If* f *is a meromorphic solution with* $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho < +\infty$ *of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{2.1}$$

then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Lemma 2.4. *Let* f, g *be meromorphic functions with iterated* p-*orders* $0 < \rho_p(f), \rho_p(g) < \infty$ *and iterated* p-*types* $0 < \sigma_p(f), \sigma_p(g) < \infty$ $(1 \leq p < \infty)$. *Then the following statements hold:*

(i) *If* $\rho_p(g) < \rho_p(f)$, *then*

$$\sigma_p(f+g) = \sigma_p(fg) = \sigma_p(f). \tag{2.2}$$

(ii) *If* $\rho_p(f) = \rho_p(g)$ *and* $\sigma_p(g) \neq \sigma_p(f)$, *then*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f). \tag{2.3}$$

Proof. (i) By the definition of the iterated p-type, we have

$$\begin{aligned} \sigma_p(f+g) &= \limsup_{r\to+\infty} \frac{\log_{p-1} T(r, f+g)}{r^{\rho_p(f+g)}} \\ &\leq \limsup_{r\to+\infty} \frac{\log_{p-1}(T(r,f) + T(r,g) + O(1))}{r^{\rho_p(f+g)}}. \end{aligned} \tag{2.4}$$

Since $\rho_p(g) < \rho_p(f)$, then $\rho_p(f+g) = \rho_p(f)$. Thus, from (2.4), we obtain

$$\begin{aligned} \sigma_p(f+g) &\leq \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} + \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,g) + O(1)}{r^{\rho_p(f)}} \\ &= \sigma_p(f). \end{aligned} \tag{2.5}$$

On the other hand since

$$\rho_p(f+g) = \rho_p(f) > \rho_p(g), \tag{2.6}$$

then by (2.5), we get

$$\sigma_p(f) = \sigma_p(f+g-g) \leq \sigma_p(f+g). \tag{2.7}$$

Hence by (2.5) and (2.7), we obtain $\sigma_p(f+g) = \sigma_p(f)$. Now we prove $\sigma_p(fg) = \sigma_p(f)$. Since $\rho_p(g) < \rho_p(f)$, then $\rho_p(fg) = \rho_p(f)$. By the definition of the iterated p-type, we have

$$\begin{aligned} \sigma_p(fg) &= \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,fg)}{r^{\rho_p(fg)}} \\ &\leq \limsup_{r\to+\infty} \frac{\log_{p-1}(T(r,f)+T(r,g))}{r^{\rho_p(f)}} \\ &\leq \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} + \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,g)+O(1)}{r^{\rho_p(f)}} \\ &= \sigma_p(f). \end{aligned} \tag{2.8}$$

Since

$$\rho_p(fg) = \rho_p(f) > \rho_p(g) = \rho_p\left(\frac{1}{g}\right), \tag{2.9}$$

then by (2.8), we obtain

$$\sigma_p(f) = \sigma_p\left(fg\frac{1}{g}\right) \leq \sigma_p(fg). \tag{2.10}$$

Thus, by (2.8) and (2.10), we obtain $\sigma_p(fg) = \sigma_p(f)$.

(ii) Without loss of generality, we suppose that $\rho_p(f) = \rho_p(g)$ and $\sigma_p(g) < \sigma_p(f)$. Then, we have

$$\rho_p(f+g) \leq \max\{\rho_p(f), \rho_p(g)\} = \rho_p(f) = \rho_p(g). \tag{2.11}$$

If we suppose that $\rho_p(f+g) < \rho_p(f) = \rho_p(g)$, then by (2.2), we get

$$\sigma_p(g) = \sigma_p(f+g-f) = \sigma_p(f)$$

and this is a contradiction. Hence $\rho_p(f+g) = \rho_p(f) = \rho_p(g)$. Now, we prove that $\rho_p(fg) = \rho_p(f) = \rho_p(g)$. Also we have

$$\rho_p(fg) \leq \max\{\rho_p(f), \rho_p(g)\} = \rho_p(f) = \rho_p(g). \tag{2.12}$$

If we suppose that $\rho_p(fg) < \rho_p(f) = \rho_p(g) = \rho_p\left(\frac{1}{f}\right)$, then by (2.2), we can write

$$\sigma_p(g) = \sigma_p\left(fg\frac{1}{f}\right) = \sigma_p\left(\frac{1}{f}\right) = \sigma_p(f)$$

and this is a contradiction. Hence $\rho_p(fg) = \rho_p(f) = \rho_p(g)$. ■

Lemma 2.5. *Let $A_1(z)$, $A_0(z)$ be entire functions, and let $i(A_j) = p$ $(j = 0, 1)$, $(1 \leq p < \infty)$ such that $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Let B_j $(j = 0, 1, 2)$ be entire functions such that at least one of $B_0(z)$, $B_1(z)$, $B_2(z)$ does not vanish identically with $\rho_p(B_j) < \rho_p(A_0)$ $(j = 0, 1, 2)$. Then*

$$h = B_0 A_0 + B_1 A_1 + B_2 \not\equiv 0. \tag{2.13}$$

Proof. First, we suppose that $B_0 \not\equiv 0$. Then by Lemma 2.4, we have $\rho_p(h) = \rho_p(A_0) = \rho > 0$. Thus $h \not\equiv 0$.

If $B_0 \equiv 0$, $B_1 \not\equiv 0$, then $h = B_1 A_1 + B_2$ and $\rho_p(h) = \rho_p(A_1) > 0$. Hence $h \not\equiv 0$.

Finally, if $B_0 \equiv 0$, $B_1 \equiv 0$, $B_2 \not\equiv 0$, then we have $h = B_2 \not\equiv 0$. ∎

Lemma 2.6. [3, Corollary 1.8] *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, and let $i(A_j) = p(j = 0, \ldots, k-1)$, $(1 \leq p < \infty)$. Assume that $\rho_p(A_j) = \rho$ $(j = 0, \ldots, k-1)$, $(0 < \rho < +\infty)$ and $\max\{\sigma_p(A_j) : j = 1, 2, \ldots, k-1\} < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Then every solution $f \not\equiv 0$ of (1.8) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho$.*

Lemma 2.7. *Let $A_1(z)$, $A_0(z)$ be entire functions, and let $i(A_j) = p$ $(j = 0, 1)$, $(1 \leq p < \infty)$ such that $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Let $d_0(z)$, $d_1(z)$ be entire functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$. If $f \not\equiv 0$ is a solution of (1.11), then the differential polynomial*

$$g_f = d_1 f' + d_0 f \tag{2.14}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty, \quad \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho. \tag{2.15}$$

Proof. Suppose that $f \not\equiv 0$ is a solution of equation (1.11). Then by Lemma 2.6, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Differentiating both sides of equation (2.14) and replacing f'' with $f'' = -A_1 f' - A_0 f$, we obtain

$$g_f' = (d_1' + d_0 - d_1 A_1) f' + (d_0' - d_1 A_0) f. \tag{2.16}$$

Set

$$\alpha_0 = d_0' - d_1 A_0, \qquad \alpha_1 = d_1' + d_0 - d_1 A_1. \tag{2.17}$$

Then by (2.14), (2.16) and (2.17), we have

$$d_1 f' + d_0 f = g_f, \tag{2.18}$$

$$\alpha_1 f' + \alpha_0 f = g_f'. \tag{2.19}$$

Set

$$h = d_1\alpha_0 - d_0\alpha_1 = d_1(d_0' - d_1A_0) - d_0(d_1' + d_0 - d_1A_1). \tag{2.20}$$

Then from (2.20), we can write

$$h = B_0A_0 + B_1A_1 + B_2,$$

where B_j $(j = 0, 1, 2)$ are entire functions such that at least one of $B_0(z)$, $B_1(z)$, $B_2(z)$ does not vanish identically with $\rho_p(B_j) < \rho_p(A_0)$ $(j = 0, 1, 2)$. Thus, by Lemma 2.5, we have $h \not\equiv 0$. By $h \not\equiv 0$ and (2.18)–(2.20), we obtain

$$f = \frac{d_1 g_f' - \alpha_1 g_f}{h}. \tag{2.21}$$

If $\rho_p(g_f) < \infty$, then by (2.21) and Lemma 2.1, we get $\rho_p(f) < \infty$ and this is a contradiction. Hence $\rho_p(g_f) = \infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (2.14), Lemma 2.1 and Lemma 2.2, we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (2.21) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. ∎

Lemma 2.8. [3] *Let $h_j(z)$ $(j = 0, 1, \dots, k-1)$ $(k \geq 2)$ be entire functions with $h_0 \not\equiv 0$, $\rho_p(h_j) < n$ $(1 \leq p < \infty)$, and let $A_j(z) = h_j(z)\exp_p(P_j(z))$, where $P_j(z) = \sum_{i=0}^{n} a_{ji}z^i$ $(j = 0, \dots, k-1)$ are polynomials with degree $n \geq 1$, a_{jn} $(j = 0, 1, \dots, k-1)$ are complex numbers. If $|a_{0n}| > \max\{|a_{jn}| : j = 1, \dots, k-1\}$, then every solution $f \not\equiv 0$ of equation (1.8) satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = n$.*

Lemma 2.9. [10] *Let f be a meromorphic function for which $i(f) = p \geq 1$ and $\rho_p(f) = \rho$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}), \tag{2.22}$$

outside of a possible exceptional set E_1 of finite linear measure.

To avoid some problems caused by the exceptional set we recall the following lemma.

Lemma 2.10. [1, p. 68] *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

3. Proof of Theorem 1.1

Suppose that $f \not\equiv 0$ is a solution of equation (1.11). Then by Lemma 2.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Set $w(z) = d_1 f' + d_0 f - \varphi$. Since $\rho_p(\varphi) < \infty$, then by Lemma 2.7 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho$, we need to prove only $\overline{\lambda}_p(w) = \lambda_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho$. By $g_f = w + \varphi$, we get from (2.21)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where

$$\psi = \frac{d_1 \varphi' - \alpha_1 \varphi}{h}. \tag{3.2}$$

Substituting (3.1) into equation (1.11), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -(\psi'' + A_1(z)\psi' + A_0(z)\psi) = A, \tag{3.3}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho_p(\phi_j) < \infty$ $(j = 0, 1, 2)$.

Now, we prove that $\psi(z) \not\equiv 0$. Assume that $\psi(z) \equiv 0$. Then from (3.2) and (2.17), we obtain that

$$d_1' + d_0 - d_1 A_1 = d_1 \frac{\varphi'}{\varphi}. \tag{3.4}$$

First, if $d_1 \equiv 0$, then by (3.4), we get $d_0 \equiv 0$ and this is a contradiction. Suppose that $d_1 \not\equiv 0$. Since $d_1 \not\equiv 0$ and $\rho_p(\varphi) = \alpha < \infty$, we obtain by using Lemma 2.9 and the equation (3.4) that

$$\begin{aligned} T(r, d_1' + d_0 - d_1 A_1) &= m(r, d_1' + d_0 - d_1 A_1) \\ &\leq m(r, d_1) + O(\exp_{p-2}\{r^{\alpha+\varepsilon}\}) \\ &= T(r, d_1) + O(\exp_{p-2}\{r^{\alpha+\varepsilon}\}) \end{aligned} \tag{3.5}$$

holds for all r outside a set $E \subset (0, +\infty)$ with a finite linear measure $m(E) < +\infty$. Then by (3.5) and Lemma 2.10, we obtain

$$\rho_p(A_1) = \rho_p(d_1' + d_0 - d_1 A_1) \leq \rho_p(d_1) < \rho_p(A_0)$$

and this is a contradiction. Hence $\psi(z) \not\equiv 0$. By $\psi(z) \not\equiv 0$ and $\rho_p(\psi) < \infty$, it follows by Lemma 2.6 that $A \not\equiv 0$. Then by $h \not\equiv 0$ and Lemma 2.3, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w) = \rho_p(A_0) = \rho$, that is, $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. ∎

4. Proof of Theorem 1.2

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1.14). Then by Lemma 2.8, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = n$. By using Theorem 1.1, we obtain Theorem 1.2. ∎

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