SEMI-FREDHOLM OPERATORS AND PURE CONTRACTIONS IN HILBERT SPACE

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ABSTRACT. In [9] W. E. Kaufman proved that the function Γ defined by $\Gamma(A) = A(I - A^*A)^{-1/2}$ maps the set $\mathcal{C}_0(H)$ of all pure contractions one-to-one onto the set $\mathcal{C}(H)$ of all closed and densely defined linear operators on Hilbert space H. In this paper, we gives some further properties of Γ, we establish the semi-Fredholmness and Fredholmness of unbounded operators in terms of bounded pure contractions, and we apply this results to an 2×2 upper triangular operator matrices. An application to linear delay differential equation is given.

AMS Subject Classification: 47A10, 47A53.
Keywords: Pure contractions; Semi-Fredholm operators; Upper triangular operator matrices; Linear delay differential equation.

1. Introduction

Let H be a complex Hilbert space endowed with the scalar product $< .; . >$ and the associated norm $\|.\|$. Denote by $\mathcal{L}(H)$ the Banach space of all bounded linear operators on H. For T densely, defined closed linear operator on H, the symbols $\mathcal{D}(T) \subset H$, $N(T)$ and $R(T)$ will denote the domain, null space and the range space of T, respectively, and T^* is the adjoint of T. Let I denote the identity operator on H. In [9] W. E. Kaufman showed that if T is a densely defined closed operator, then T is represented as $T = A(I - A^*A)^{-1/2}$ using a unique pure contraction A, i.e., an operator such that $\|Ax\| < \|x\|$ for all nonzero x in H. The function Γ defined by $\Gamma(A) = A(I - A^*A)^{-1/2}$ maps the set $\mathcal{C}_0(H)$ of all pure contractions one-to-one onto the set $\mathcal{C}(H)$ of all closed and densely defined linear operators on H, is used to reformulate questions about unbounded operators in terms of bounded ones:

- In [9, 11], Kaufman proved that the map Γ preserves many properties of operators: self-adjontness, nonnegative conditions, normality and quasinormality.
- In [5] Hirasawa showed that a pure contraction A is hyponormal if and only if $T = A(I - A^*A)^{-1/2}$ is formally hyponormal , and if A is quasinormal then $T^n = A^n(I - A^*A)^{-n/2}$ is quasinormal for all integers $n \geq 2$.
- In [2], Cordes and Labrousse prove that if a closed and densely defined operator T is semi Fredholm then so is the bounded operator $\Gamma^{-1}(T) = T(I + T^*T)^{-1/2}$.

In view of the works of Cordes and Labrousse and Kaufman's representation, the following natural question arises:
If a pure contraction A is a semi-Fredholm operator, does the densely defined operator $\Gamma(A)$ is a semi-Fredholm operator?

In this paper, we give an affirmative answer of this question. Furthermore, we establish some characterizations of Fredholm theory of unbounded operators in terms of bounded pure contractions.

Our paper is organized as follows.
In section 2, we express the Fredholm character for unbounded operators by means of that of pure contractions.
In section 3, we apply the previously obtained results to study the semi-Fredholmness and Fredholmness of class of upper triangular operator matrices. Finally, a special case of section 3 for a linear delay differential equation is discussed in section 4.

2. Main results

In this section we present some results concerning the essential spectrum of a unbounded operators in terms of bounded ones. We begin by introduce now some important classes of operators in Fredholm theory. In the sequel, for every closed and densely-defined operator T, let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined as $\alpha(T) := \dim N(T)$, and $\beta(T) := \text{codim} R(T)$. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. If T is either upper or lower semi-Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by $\text{ind}(T) := \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a called a Fredholm operator. In the following, A denotes a pure contraction, B and B_* the associated defect operators $(I - A^*A)^{1/2}$ and $(I - AA^*)^{1/2}$, respectively, and T the closed and densely-defined operator $\Gamma(A) = AB^{-1}$. Note that since A is a pure contraction, B and B^* are one-to-one and $\Gamma(A^*) = A^*B_*^{-1}$. Recall the following relations proved in [9]: $R(B) = D(T)$, $T^* = B^{-1}A^*$, $B = (I + T^*T)^{-1/2}$, and (thus) $T^*T = B^{-2} - I$.
The reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) = \inf_{x \in N(T)^\perp} \frac{\|Tx\|}{\|x\|}$$

If $T = 0$ then we take $\gamma(T) = \infty$. Note that (see [8]):

$$\gamma(T) > 0 \Leftrightarrow R(T) \text{ is closed}$$

Let M, N be two closed linear subspaces of the Hilbert space H. Denote by P_M and P_N the orthogonal projection onto M and N respectively. Set

$$\delta(M, N) = \|(I - P_N)P_M\|$$

Lemma 2.1. *([8])*

1. *If* $\delta(M, N) < 1$ *then* $\dim M \leq \dim N$.
2. $\delta(M, N) = \delta(N^\perp, M^\perp)$.

The main results of this paper read as follows:

Theorem 2.2. *Let $A \in \mathcal{C}_0(H)$. If A is upper semi-Fredholm operator then $\lambda I - \Gamma(A)$ is upper semi-Fredholm operator for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$.*

Proof. Let $A \in \mathcal{C}_0(H)$ and B denote the positive member $(I - A^*A)^{1/2}$ of $\mathcal{C}_0(H)$. For each nonzero x in H, $\|x\|^2 - \|Ax\|^2 = \|Bx\|^2$; thus

$$\|Bx\| \leq \|x\| + \|Ax\| \tag{2.1}$$

Let λ in $\mathbb{C}$ with $|\lambda| < 1$. We prove that if $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$ then $0 < \gamma(\lambda B - A) < \infty$ and hence $R(\lambda B - A)$ is closed. First if we use (2.1) with λx instead of x and the theorem 1a of [7], we obtain that $\gamma(\lambda B - A) > 0$ for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$. Now to prove that $\gamma(\lambda B - A) < \infty$, we proceed by contraposition. In fact $\gamma(\lambda B - A) = \infty$ implies that $(\lambda B - A)x = 0$ for all $x \in H$. Hence

$$\|Ax\| = |\lambda| \, \|Bx\| \leq |\lambda| \, (\|x\| + \|Ax\|)$$

and so

$$\gamma(A) \, \|x\| \leq \|Ax\| \leq \frac{|\lambda|}{1 - |\lambda|} \, \|x\| \tag{2.2}$$

for $x \in N(A)^{\perp}$ with $x \neq 0$. It follows that $|\lambda| \geq \frac{\gamma(A)}{1+\gamma(A)}$. We next prove that

$$\delta(N(\lambda B - A), N(A)) \leq \frac{|\lambda|}{(1 - |\lambda|)\gamma(A)}. \tag{2.3}$$

Let $x \in H$,

$$\gamma(A) \left\| (I - P_{N(A)}) P_{N(\lambda B - A)} x \right\| \leq \left\| A P_{N(\lambda B - A)} x \right\|.$$

Since $P_{N(\lambda B - A)} x \in N(\lambda B - A)$ by the same calculation given before we have

$$\gamma(A) \left\| (I - P_{N(A)}) P_{N(\lambda B - A)} x \right\| \leq \frac{|\lambda|}{1 - |\lambda|} \, \|x\|.$$

Recalling the definition of $\delta(N, M)$, this proves (2.3). The right side of (2.3) is smaller than one if $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$, thus Lemma 2.1 shows that

$$\alpha(\lambda B - A) \leq \alpha(A) \qquad \text{for } |\lambda| < \frac{\gamma(A)}{1 + \gamma(A)}. \tag{2.4}$$

We then conclude that $\lambda B - A$ is upper semi-Fredholm operator for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$

Since A is a pure contraction, B is one-to-one with dense range, and the fact that $\lambda I - \Gamma(A) = (\lambda B - A)B^{-1}$, it follows $\lambda B - A$ is upper semi-Fredholm operator then $\lambda I - \Gamma(A)$ is upper semi-Fredholm operator for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$. This is the statement of the theorem. □

Theorem 2.3. *Let $A \in \mathcal{C}_0(H)$ is a lower semi-Fredholm operator. Then $\lambda I - \Gamma(A)$ is a lower semi-Fredholm operator for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$.*

Proof. $R(A)$ is closed and by the first part of the proof of Theorem 2.2 $R(\lambda B - A)$ is closed and $R(\lambda B - A) = N(\lambda B^* - A^*)^{\perp}$ for all $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$. From (2.3) we deduce that

$$\delta(R(\lambda B - A)^{\perp}, R(A)^{\perp}) = \delta(N(\overline{\lambda} B^* - A^*), N(A^*)) \leq \frac{|\lambda|}{(1 - |\lambda|)\gamma(A)}$$

because $\gamma(A) = \gamma(A^*)$. Now by Lemma 2.1 we have

$$\beta(\lambda B - A) \leq \beta(A) \quad \text{ for } |\lambda| < \frac{\gamma(A)}{1 + \gamma(A)}.$$

Consequently $\lambda B - A$ is lower semi-Fredholm operator for all $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$ and hence $\lambda I - \Gamma(A)$ is lower semi-Fredholm operator for all $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$. □

Corollary 2.4. *If $A \in \mathcal{C}_0(H)$ is a semi-Fredholm operator (resp. Fredholm operator), then $\lambda I - \Gamma(A)$ is a semi-Fredholm operator (resp. Fredholm operator) for $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$.*

By [2, Lemma 5.2, p. 708] and Corollary 2.4 we obtain the following results

Theorem 2.5. *Let $A \in \mathcal{C}_0(H)$. Then A is a semi-Fredholm operator (resp. Fredholm operator) if and only if $\Gamma(A)$ is a semi-Fredholm operator (resp. Fredholm operator).*

Example. Let $H = l^2$ the space of complex square-summable sequences and the linear operator T defined by $[T(x)]_k = kx_k$, $k \in \mathbb{N}$. Then, $T = \Gamma(A)$ where $[A(x)]_k = \frac{k}{\sqrt{1+k^2}} x_k$, $A \in \mathcal{C}_0(l^2)$, (for more details see [3]).
From [4], for each scalar λ we have

$$\gamma(\lambda I - A) = \inf_{k \in E_\lambda} \left| \lambda - \frac{k}{\sqrt{1+k^2}} \right|^2$$

where E_λ is the set of integers k for which $\lambda \neq \frac{k}{\sqrt{1+k^2}}$. Again from [4], A is Fredholm operator with $\gamma(A) = 1/2$, $\alpha(A) = 1$ and $\beta(A) = 0$. Then by Corollary 2.4, $\lambda I - T$ is Fredholm operator for all $|\lambda| < 1/3$.

3. Application to Upper Triangular Operator Matrices

Throughout this section, let H and K be Hilbert spaces, ($\mathcal{L}(H;K)$ denote the set of bounded linear operators from H to K). When $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ are given we denote by M_C an operator acting on $H \oplus K$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where $C \in \mathcal{L}(K;H)$. Set

$$\mathcal{C}_0(H \oplus K) = \{M_C; \quad A \in \mathcal{C}_0(H), B \in \mathcal{C}_0(K)\},$$
$$\mathcal{C}(H \oplus K) = \{M_C; \quad A \in \mathcal{C}(H), B \in \mathcal{C}(K)\}$$

where $\mathcal{C}(H)$ (resp. $\mathcal{C}(K)$) is the space of closed densely defined linear operators on H (resp. on K).

Lemma 3.1. *For every $C \in \mathcal{L}(K;H)$, the formula*

$$\Gamma(M_C) = \begin{pmatrix} \Gamma(A) & C \\ 0 & \Gamma(B) \end{pmatrix} \qquad \text{for all } (A,B) \in \mathcal{C}_0(H) \times \mathcal{C}_0(K)$$

define a reversible function from $\mathcal{C}_0(H \oplus K)$ onto $\mathcal{C}(H \oplus K)$ with inverse function defined by

$$\Gamma^{-1}(M_C) = \begin{pmatrix} \Gamma^{-1}(A) & C \\ 0 & \Gamma^{-1}(B) \end{pmatrix} \qquad \text{for all } (A,B) \in \mathcal{C}(H) \times \mathcal{C}(K) \tag{3.1}$$

Proof. Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, with $(A,B) \in \mathcal{C}_0(H) \times \mathcal{C}_0(K)$ and $C \in \mathcal{L}(K;H)$. Observe that

$$\Gamma(M_C) = \begin{pmatrix} I & 0 \\ 0 & \Gamma(B) \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} \Gamma(A) & 0 \\ 0 & I \end{pmatrix}.$$

Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible for every $C \in \mathcal{L}(K;H)$, and since by Kaufman's theorem ([9, Theorem 2.]) $\begin{pmatrix} I & 0 \\ 0 & \Gamma(B) \end{pmatrix}$ and $\begin{pmatrix} \Gamma(A) & 0 \\ 0 & I \end{pmatrix}$ are tow invertible maps, it follows that $\Gamma(M_C)$ maps the set $\mathcal{C}_0(H \oplus K)$ one to one onto the set $\mathcal{C}(H \oplus K)$ with the inverse given by (3.1). □

Theorem 3.2. *Let $(A,B) \in \mathcal{C}_0(H) \times \mathcal{C}_0(K)$ are both semi-Fredholm operators (resp. Fredholm operators). Then for every $C \in \mathcal{L}(K;H)$, $\lambda I - \Gamma(M_C)$ is a semi-Fredholm operator (resp. Fredholm operator) for $|\lambda| < \min\{\frac{\gamma(A)}{1+\gamma(A)}, \frac{\gamma(B)}{1+\gamma(B)}\}$.*

Proof. For each $\lambda \in \mathbb{C}$ we have.

$$\begin{aligned}\lambda I - \Gamma(M_C) &= \begin{pmatrix} \lambda I - \Gamma(A) & C \\ 0 & \lambda I - \Gamma(B) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & \lambda I - \Gamma(B) \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda I - \Gamma(A) & 0 \\ 0 & I \end{pmatrix}.\end{aligned}$$

Since by Corollary 2.4 $\begin{pmatrix} I & 0 \\ 0 & \lambda I - \Gamma(B) \end{pmatrix}$ and $\begin{pmatrix} \lambda I - \Gamma(A) & 0 \\ 0 & I \end{pmatrix}$ are both semi-Fredholm operators (resp. Fredholm operators) for $|\lambda| < \frac{\gamma(B)}{1+\gamma(B)}$ and $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$ respectively, it follows that $\lambda I - \Gamma(M_C)$ is a semi-Fredholm operator (resp. Fredholm operator) for $|\lambda| < \min\{\frac{\gamma(A)}{1+\gamma(A)}, \frac{\gamma(B)}{1+\gamma(B)}\}$. □

4. Delay linear differential equations

In the following we consider the delay differential equations

$$\begin{cases} \dot{x_t} = Cx_t + Ax(t), \quad t \geq 0 \\ x_0 = \phi \\ x(0) = y \end{cases} \tag{4.1}$$

where

- $y \in H$, H is a Hilbert space,
- $A : D(A) \subseteq H \longrightarrow H$ is a linear, closed and densely defined operator which generates a strongly continuous semigroup,
- $\phi \in L^2([-1,0], H) = K$,
- $C : W^{1,2}([-1,0], H) \longrightarrow H$ is a linear, bounded operator, where $W^{1,2}([-1,0], H) = \{x_t \in K; \ \frac{d}{ds}x_t \in K\}$.
- $x : [-1,\infty) \longrightarrow H$ and $x_t : [-1,0] \longrightarrow H$ is defined by $x_t(s) = x(t+s)$.

The problem (4.1) is equivalent to the following abstract Cauchy problem in $H \times K$ with the vector function $z(t) = \binom{x(t)}{x_t}$:

$$\begin{cases} \dot{z}(t) = M_C z_t, \quad t \geq 0 \\ z(0) = \binom{y}{\phi} \end{cases} \tag{4.2}$$

where

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \ B = \frac{d}{ds}$$

with domain

$$D(M_C) = \{\binom{y}{\phi} \text{ such that } \phi \in W^{1,2}([-1,0], H), y \in D(A) \text{ and } \phi(0) = y\}$$

Under these assumptions, the operator M_C is already closed (see [1, Lemma 2.1.],). Following [4], we prove that the differential operator $\lambda I - B$ is Fredholm if and only if Re $\lambda \neq 0$. Then it follows from Theorem 3.2 the following result

Theorem 4.1. *If* $\Gamma^{-1}(A) = A(I + A^*A)^{-1/2} \in \mathcal{C}_0(H)$ *is a semi-Fredholm operator (resp. Fredholm operator), then* $\lambda I - \Gamma(M_C)$ *is a semi-Fredholm operator (resp. Fredholm operator) for* $|\lambda| < \frac{\gamma(\Gamma^{-1}(A))}{1+\gamma(\Gamma^{-1}(A))}$ *and* $Re\ \lambda \neq 0$.

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