DEMOCRATIC AND POPULAR ALGERIAN REPUBLIC

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FERHAT MADANI

LINEAR ALGEBRA
LESSONS AND CORRECTED EXERCISES

FIRST YEAR OF COMMON CORE IN SCIENCES AND TECHNOLOGY

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The teaching of mathematics has become an integral part of engineeiring studies in all across all disciplines. Therefore, special attention should be paid to certain areas of mathematics.

To introduce students to basic mathematical tools, especially linear algebra we have developed this manuscrit for first year students of the common core in sciences and technology. We begin with vector spaces and vector subspaces, followed by a section on linear maps and their properties. Subsequently, we explore matrices and determinants. Each lesson is accompanied by a series of detailed and carefully corrected exercises.

We welcome any feedback or suggestions you may have and thank you in advance for your remarks.

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Chapter 1

Vector space

1 Vector space

Definition1.1

We say that $(E,+,.)$ is a vector space on the commutative field K $(K - vector\ space)$ with $K = \mathbb{R}$ or $\mathbb{C}$ if $(E,+)$ is a commutative group

$$\forall x,y \in E{:}\, x+y \in E, \forall(\lambda,x) \in K \times E{:}\, \lambda x \in E$$
$$\forall x,y \in E:\ x+y = y+x$$

$$\forall x,y,z \in E:\ (x+y)+z = x+(y+z)$$

$$\forall x \in E:\ \ x+0_E = 0_E + x$$

$$\forall x \in E:\ \ x+(-x) = (-x)+x = 0_E$$

If there is an external composition law

$$K \times E \to E$$
$$(\lambda,x) \mapsto \lambda.x$$

verifying

$$(\lambda+\mu).x = \lambda.x + \mu.x$$

$$\lambda.(x+y) = \lambda.x + \lambda.y$$

$$(\lambda\mu).x = \lambda(\mu.x)$$

$$1.x = x$$

2 Product vector space

Let $E_1, E_2, E_3, \dots, E_n$ be n vector spaces on K. We call the product vector space of $E_1, E_2, E_3, \dots, E_n$, the vector space
$E = E_1 \times E_2 \times E_3 \times \dots \times E_n = \{X = (x_1, x_2, x_3, \dots, x_n){:}\, x_i \in E_i\}$

Such that

$$(x_1, x_2, x_3, \dots, x_n) + (y_1, y_2, y_3, \dots, y_n) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n)$$
$$\lambda(x_1, x_2, x_3, \dots, x_n) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n)$$

Special cases

K^n is a $K - vector\ space$.

If $E_1 = E_2 = E_3 = \ldots = E_n = \mathbb{R}$, so

$$\mathbb{R}^n = \{X = (x_1, x_2, x_3, \ldots, x_n): x_i \in \mathbb{R}\}$$

Is a $\mathbb{R} - vector\ space$.

Example 2.1

$E = \{x \in \mathbb{R}: x > 0\}$ provided with the following two operations

$$x + y = xy$$
$$\lambda . x = x^{\lambda}$$

Constitutes a vector space structure on $\mathbb{R}$, since

$$\forall x, y \in E: ((x > 0) \wedge (y > 0)) \Rightarrow x + y > 0 \Rightarrow x + y \in E$$

$$\forall x \in E, \forall \alpha \in \mathbb{R}\ : (\alpha x = x^{\alpha} > 0) \Rightarrow \alpha x \in E$$

On the other hand we have

$$\forall x, y \in E: x + y = xy = yx = y + x$$

$$\forall x, y, z \in E: (x + y) + z = xy + z = xyz = x + yz = x + (y + z)$$

$$\forall x \in E: x + 1 = x \times 1 = x$$

$$\forall x \in E: x + x' = 1 \Leftrightarrow xx' = 1 \Leftrightarrow x' = \frac{1}{x}$$

$$\forall x \in E, \forall \lambda, \mu \in R: (\lambda + \mu)x = x^{\lambda + \mu} = x^{\lambda} . x^{\mu} = x^{\lambda} + x^{\mu} = \lambda x + \mu x$$

$$\forall x, y \in E, \forall \lambda \in R: \lambda(x + y) = (x + y)^{\lambda} = (xy)^{\lambda} = x^{\lambda} . y^{\lambda} = x^{\lambda} + y^{\lambda} = \lambda x + \lambda y$$

$$\forall x \in E, \forall \lambda, \mu \in R: (\lambda\mu)x = x^{\lambda\mu} = (x^{\mu})^{\lambda} = \lambda x^{\mu} = \lambda(\mu x)$$

$$\forall x \in E: 1 \times x = x$$

So E provided with these two operations is a $\mathbb{R} - vector\ space$.

Example 2.2 (particular case of product vector space)

Let E a $K - vector\ space$. We define the vector space E^n on K by

$$E^n = \{X = (x_1, x_2, x_3, \ldots, x_n): x_i \in \mathrm{E}\}$$

Provided with the two operations

$$x + y = (x_1, x_2, x_3, \ldots, x_n) + (y_1, y_2, y_3, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n)$$
$$\lambda . x = \lambda . (x_1, x_2, x_3, \ldots, x_n) = (\lambda . x_1, \lambda . x_2, \lambda . x_3, \ldots, \lambda . x_n)$$

Theorem 2.1
Let E be a vector space on K, we have

$$\forall \alpha \in K, \alpha.0_E = 0_E$$

$$\forall x \in E, 0_K.x = 0_E$$

$$\forall x \in E, \forall \alpha \in K: (-\alpha)(-x) = \alpha x$$

$$\forall x \in E, \forall \alpha \in K: -(\alpha x) = (-\alpha)x = \alpha(-x)$$

3 Vector subspace

Let E be a space vector on K, any non-empty part F of E is called vector subspace of E, if it is also a vector space with respect addition and multiplication on E

Theorem 3.1
Let F a non-empty part of the vector space E, then we say that F is a vector subspace of E, if and only if

1- $\forall x, y \in F: x + y \epsilon F$

2- $\forall x \epsilon F, \forall \alpha \epsilon K: \alpha.x \epsilon F$

We can summarize the two conditions into one as follows
$$\forall x, y \in F \ \ , \forall \alpha, \beta \in K: \ \alpha.x + \beta.y \in F$$

Remark 3.1
In any vector space E, there exist at least two vector subspaces :$E \ and \ \{0_E\}$

Example 3.1
Let $F \subset \mathbb{R}^3$ such that
$$F = \{(x, y, x + y): x, y \in \mathbb{R}\}$$

Show that F is a vector subspace of $\mathbb{R}^3$

Solution
First $0_{\mathbb{R}^3} \in F$

We take any two vectors X and Y of F and any scalar α of $\mathbb{R}$

$$X = (x_1, y_1, x_1 + y_1) \text{ and } Y = (x_2, y_2, x_2 + y_2); x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Hence

$$X + Y = \left(x_1 + x_2, y_1 + y_2, (x_1 + x_2) + (y_1 + y_2)\right) \in F$$
and
$$\alpha X = \alpha(x_1, y_1, x_1 + y_1) = (\alpha.x_1, \alpha.y_1, \alpha.x_1 + \alpha.y_1) \in F$$

So F is a vector subspace of $\mathbb{R}^3$.

Example 3.2
Is $F = \{(x, y, z) \in: \mathbb{R}^3: x^2 + y^2 + z^2 \leq 1\}$ a vector subspace of $\mathbb{R}^3$?

Solution

$(1,0,0) \in F$ because $1^2 + 0^2 + 0^2 \leq 1$, but $2(1,0,0) = (2,0,0) \notin F$ because
$$2^2 + 0^2 + 0^2 = 4 \geq 1.$$
So F is not a vector subspace of $\mathbb{R}^3$.

Theorem 3.2
If F_1 et F_2 are two vector subspaces of E,
$F_1 \cap F_2$ is then a vector subspace of E .

Remark 3.2
If F_1 et F_2 are two vector subspaces E, then $F_1 \cup F_2$ in general is not a subspace of E .

Example 3.3
Let F_1 and F_2 be two vector subspaces of $\mathbb{R}^2$, such that

$$F_1 = \{(x, 0): x \in \mathbb{R}\} \; and \; F_2 = \{(0, y): y \in \mathbb{R}\}$$

We have
$$(1,0) \in F_1 \subset F_1 \cup F_2 \; and \; (0,1) \in F_2 \subset F_1 \cup F_2$$
But
$$(1,0) + (0,1) = (1,1) \notin F_1 \cup F_2$$

Thus $F_1 \cup F_2$ is not a vector subspace of $\mathbb{R}^2$.

4 Sum and direct sum

Definition 4.1
If E_1 and E_2 are two non-empty sets of the vector space E, we define the sum of $E_1$and E_2 as follows :

$$E_1 + E_2 = \{x + y: (x \in E_1) \wedge (y \in E_2)\}$$

Theorem 4.1
If E_1 and E_2 are two non-empty vector subspaces of the vector space E, then
$E_1 + E_2$ is also a vector subspace of E.

Définition 4.2

Let F_1 and F_2 be two vector subspaces of the vector space E, we call direct sum of F_1 and F_2, and we write $E = F_1 \oplus F_2$, if both conditions are verified

1- $E = F_1 + F_2$, .

2- $F_1 \cap F_2 = \{0_E\}$.

Example 4.1

Let F_1 and F_2 be two vector subspaces of $\mathbb{R}^3$, such that

$$F_1 = \{(0, y, z) : y, z \in \mathbb{R}\} \text{ and } F_2 = \{(x, y, z) : x = y = z\}$$

Show that

$$\mathbb{R}^3 = F_1 \oplus F_2$$

It is enough to show that

$$F_1 \cap F_2 = \{0_{\mathbb{R}^3}\} \quad \text{and} \quad \mathbb{R}^3 \subseteq F_1 + F_2$$

$$\begin{aligned} (x, y, z) \in F_1 \cap F_2 &\Rightarrow \big((x, y, z) \in F_1\big) \wedge \big((x, y, z) \in F_2\big) \\ &\Rightarrow \begin{cases} x = 0 \\ x = y = z = 0 \end{cases} \Rightarrow (x, y, z) = (0,0,0) \end{aligned}$$

So $F_1 \cap F_2 = \{0_{\mathbb{R}^3}\}$

Let $X = (x, y, z) \in \mathbb{R}^3$ then $X = (0, y - x, z - x) + (x, x, x) \in F_1 + F_2$

So $\mathbb{R}^3 \subseteq F_1 + F_2$ and consequently we have $\mathbb{R}^3 = F_1 \oplus F_2$

Theorem 4.2

Let F_1 and F_2 be two vector subspaces of E. If $E = F_1 \oplus F_2$, then any element $X \in E$ can be written in a unique way as follows

$$X = X_1 + X_2 \in F_1 + F_2 : \ X_1 \in F_1 \text{ et } X_2 \in F_2$$

5 Linear combination

Definition 5.1

Let E be a $K - vector\ space$, $B = \{x_1, x_2, \ldots, x_n\} \subseteq E$ and $X \in E$. It is said that X is a linear combination of the vectors of B, if it exist $\alpha_i \in K; i = 1,2, \ldots, n$ such that

$$X = \sum_{i=1}^{i=n} \alpha_i . x_i = \alpha_1 . x_1 + \alpha_2 . x_2 + \ldots + \alpha_n . x_n$$

Example 5.1

Show that the vector $X = (3,8,5)$ is a linear comnination of the two vectors :

$$x_1 = (1,2,-1) \text{ et } x_2 = (0,1,4)$$

We have

$$X = \alpha_1.x_1 + \alpha_2.x_2 \Leftrightarrow (\alpha_1, 2\alpha_1 + \alpha_2, -\alpha_1 + 4\alpha_2) = (3,8,5)$$
$$\Leftrightarrow \begin{cases} \alpha_1 = 3 \\ 2\alpha_1 + \alpha_2 = 8 \\ -\alpha_1 + 4\alpha_2 = 5 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 3 \\ \alpha_2 = 2 \end{cases}$$

Then

$$X = 3.x_1 + 2.x_2$$

6 Generating family

Definition 6.1

Let E a $K-vector\ space$ and the family $B = \{x_1, x_2, \ldots, x_n\} \subseteq E$. We say that B is a generating family of E, if every vector $X \in E$ is written in as a linear combination of the vectors of B and we say that B generates E.

Example 6.1

Show that the family $B = \{(1,2,1); (1,0,2); (1,1,0)\}$ generates $\mathbb{R}^3$.

Let the vector $X = (x, y, z) \in \mathbb{R}^3$. It is necessary to show that there exist $\alpha_1, \alpha_2, \alpha_3 \in K$ such that

$$\alpha_1.(1,2,1) + \alpha_2.(1,0,2) + \alpha_3.(1,1,0) = (x, y, z).$$

We obtain

$$\begin{cases} \alpha_1 = \frac{1}{3}(-2x + 2y + z) \\ \alpha_2 = \frac{1}{3}(x - y + z) \\ \alpha_3 = \frac{1}{3}(2x - y - 2z) \end{cases}$$

7 Linear dependence and Independence

Definition 7.1

Let E be a $K-vector\ space$ and the family $B = \{x_1, x_2, \ldots, x_n\} \subseteq E$. We say that B is a linearly independent, if they exist $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in K$ not all zero, such that

$$\sum_{i=1}^{i=n} \alpha_i.x_i = \alpha_1.x_1 + \alpha_2.x_2 + \ldots + \alpha_n.x_n = 0_E$$

Définition 7.2

Let E be a $K-vector\ space$. It is said that family $B = \{x_1, x_2, \ldots, x_n\} \subseteq E$ is linearly independent, if

$$(\alpha_1.x_1 + \alpha_2.x_2 + \ldots + \alpha_n.x_n = 0_E) \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

Example 7.1

Show that the following three vectors are linearly independent

$$x_1 = (1,1,0); x_2 = (1,0,1); x_3 =$$

We have

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0_{\mathbb{R}^3} \Rightarrow \alpha_1.(1,1,0) + \alpha_2.(1,0,1) + \alpha_3.(0,1,1) = (0,0,0)$$
$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -\alpha_2 \\ \alpha_1 = -\alpha_3 \\ \alpha_2 = -\alpha_3 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -\alpha_2 \\ \alpha_2 = \alpha_3 \\ \alpha_2 = -\alpha_3 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Theorem 7.3
Let $B = \{x_1, x_2, \dots, x_n\} \subseteq E$ be a family of no-zero vectors of a $K - vector\ space$. B is linearly dependent if and only if a vector x_j of B is a linear combination of others.

8 Bases and dimensions

Definition 8.1
Let E be a $K - vector\ space$ and $B = \{x_1, x_2, \dots, x_n\} \subseteq E$. We say that B constitutes a basis of E, if the following two conditions are verified :

1- B is linearly independent
2- B is a generating family.

Example 8.1
Show that family $B = \{x_1 = (1,-1,1); x_2 = (1,0,1); x_3 = (0,-1,1)\}$ is a basis of $\mathbb{R}^3$

We show that B is linearly independent.

$$\alpha_1.x_1 + \alpha_2.x_2 + \alpha_3.x_3 = 0_{\mathbb{R}^3} \Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ -\alpha_1 - \alpha_3 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_3 = 0 \\ \alpha_2 = 0 \end{cases}$$

We show that B is a generating family. Let us consider any vector $X = (a,b,c)$ and let us show that this vector can be written as a linear combination of vectors of B

$$(a,b,c) = \alpha_1.x_1 + \alpha_2.x_2 + \alpha_3.x_3 \Leftrightarrow \begin{cases} \alpha_1 + \alpha_2 = a \\ -\alpha_1 - \alpha_3 = b \\ \alpha_1 + \alpha_2 + \alpha_3 = c \end{cases} \Rightarrow \begin{cases} \alpha_1 = b + c \\ \alpha_2 = a - b - c \\ \alpha_3 = -2b - c \end{cases}$$

Then

$$X = (b+c)x_1 + (a-b-c)x_2 + (-2b-c)x_3$$

Therefore B constitute a basis of $\mathbb{R}^3$.

Theorem 8.1
Let E be a $K - vector\ space$ and $B = \{x_1, x_2, \dots, x_n\}$ be a basis of E and let $X \in E$. So X is written in unique way as a linear combination of the vectors of B, that is

$$\sum_{i=1}^{n} \alpha_i.x_i = \alpha_1.x_1 + \alpha_2.x_2 + \dots + \alpha_n.x_n \ , \alpha_i \in K, i \in \{1,2,\dots,n\}.$$

9 Coordinates of a vector with respect to a basis

Definition9.1
Let E be a $K - vector\ space$ and $B = \{x_1, x_2, \dots, x_n\}$ be a basis of E and let $X \in E$ such that

$$X = \alpha_1.x_1 + \alpha_2.x_2 + \ldots + \alpha_n.x_n \ , \alpha_i \in K, i \in \{1,2,\ldots,n\}$$

Then we write

$$X = (\alpha_1, \alpha_2, \ldots, \alpha_n)_{[B]}$$

$\alpha_i$are called the coordinates of the vector X with the respect to the basis B.

Example 9.1
Let $B = \{x_1, x_2, x_3\}$ such that

$$B = \{x_1 = (1,1,-1); x_2 = (2,1,0); x_3 = (0,1,2)\}.$$

Find the coordinates of the vector $X = (-3,0,5)$ with respect to the basis B.

We have $$X = (\alpha_1, \alpha_2, \alpha_3)_{[B]} \Leftrightarrow \alpha_1.x_1 + \alpha_2.x_2 + \alpha_3.x_3 = X$$

$$\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 = -3 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + 2\alpha_3 = 5 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -1 \\ \alpha_2 = -1 \\ \alpha_3 = 2 \end{cases}$$

So $X = (-1,-1,2)_{[B]}$.

Theorem 9.1
Let E be a $K - vector\ space$ and $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of E. Then any family that containts more than n vectors cannot be linearly independent and any linearly independent family containts at most n vectors.

10 Dimension of a vector space

Every $K - vector\ space$ is called a finite- dimensional space, if it containts a finite family of vectors $B = \{x_1, x_2, \ldots, x_n\}$ and which is a basis of E and its dimension is the numbers of vectors of this basis and we denote it $dim(E)$.

Example 10.1
$dim(\mathbb{R}^3) = 3$ and $dim(\mathbb{R}^4) = 4$

Example 10.2
Let $F = \{(x,y,z,t): y - 2z + t = 0\}$. Find a basis of F and $dim(F)$?

We have

$$y - 2z + t = 0 \Leftrightarrow y = 2z - t$$

$$(x,y,z,t) \in F \Leftrightarrow (x, 2z - t, z, t) = x.(1,0,0,0) + z.(0,2,1,0) + t.(0,-1,0,1)$$

$$= x.u_1 + z.u_2 + t.u_3$$

We must prove that $\{u_1, u_2, u_3\}$ is linearly independent

$$\alpha_1.u_1 + \alpha_2.u_2 + \alpha_3.u_3 = 0_{\mathbb{R}^4} \Rightarrow \begin{cases} \alpha_1 = 0 \\ 2\alpha_2 - \alpha_3 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

So $B = \{u_1, u_2, u_3\}$ is a basis and $dim(F) = 3$.

Theorem 10.1

Let E be a $K - vector\ space$ of finite dimension n. Then

1- If $B = \{x_1, x_2, \dots, x_n\}$ is linearly independent, then it is a basis of E.

2- If $B = \{x_1, x_2, \dots, x_n\}$ is a generating family of E, then it constitutes a basis of E.

11 Linear maps

Définition 11.1

Let $(E, +, .)$ and $(F, \oplus, \odot)$ be two $K - vector\ spaces$. We say that f is a linear map of E into F, if

1- $\forall (x, y) \in E^2 : f(x + y) = f(x) \oplus f(y)$

2- $\forall x \in E, \forall \lambda \in K : f(\lambda.x) = \lambda \odot f(x)$

We denote by $L(E, F)$ the set of all linear maps of E in F.

Theorem 11.1

If f is linear map from E in F and g is a linear map from F in G, then $g \circ f$ is a linear map from E in G.

Example 11.1

Show that the following map is linear.

$$\begin{gathered} f : \mathbb{R}^3 \to \mathbb{R}^2 \\ (x, y, z) \mapsto (x + y, z) \end{gathered}$$

1- $\forall (x, y, z); (x', y', z') \in \mathbb{R}^3$:

$$\begin{aligned} f\big((x, y, z) + (x', y', z')\big) = f(x + x', y + y', z + z') &= \big((x + x') + (y + y'), z + z'\big) \\ &= \big((x + y) + (x' + y'), z + z'\big) = (x + y, z) + (x' + y', z') \\ &= f(x, y, z) + f(x', y', z') \end{aligned}$$

2- $\forall (x, y, z); \forall \lambda \in K$

$$f\big(\lambda.(x, y, z)\big) = f(\lambda x, \lambda y, \lambda z) = (\lambda x + \lambda y, \lambda z) = \lambda(x, y, z) = \lambda f(x, y, z)$$

Theorem 11.2
Let $(E,+,.)$ and $(F,\oplus,\odot)$ be two $K-vector\ spaces$. We say that f is a linear map of E into F, if and only if

$$\forall(x,y)\in E^2;\forall\lambda,\mu\in K: f(\lambda x+\mu y)=\lambda\odot f(x)\oplus\mu\odot f(y)$$

Theorem 11.3
If E and F are two $K-vector\ spaces$ and f is a linear map from E into F, then

$$f(0_E)=0_F$$

12 Kernel and image of a linear map

Definition 12.1
Let E and F be two $K-vector\ spaces$ and f be a linear map of E into F. We call the kernel of f and we denote $Kerf$:

$$Kerf=\{x\in E: f(x)=0_F\}$$

Définition 12.2
Let E and F be two $K-vector\ spaces$ and f be a linear map of E into F. We call the image of f and we denote $Im\,f$

$$Im\,f=\{f(x): x\in E\}$$

Theorem 12.1
Let E and F be two $K-ev$ and f be a linear map of E into F. Then $Kerf$ is a vector subspace of E and $Im\,f$ is a vector subspace of F.

Example 12.1
Let the linear map

$$f:\mathbb{R}^3\to\mathbb{R}^3$$
$$(x,y,z)\mapsto(x,0,z)$$

Find $Kerf$ and $Im\,f$ and a basis for each as well as their dimensions.

$$Kerf=\{(x,y,z): f(x,y,z)=0_{R^3}\}$$

$$=\{(x,y,z):(x,0,z)=(0,0,0)\}$$

$$=\{(x,y,z): x=z=0\}=\{y(0,1,0): y\in\mathbb{R}\}$$

So $\{(0,1,0)\}$ is a basis for $Kerf$ and $\dim(kerf)=1$

$$Im\,f=\{f(x,y,z):(x,y,z)\in\mathbb{R}^3\}$$

$$=\{(x,0,z): x,z\in\mathbb{R}\}$$

$$=\{x(1,0,0)+z(0,0,1): x,z\in\mathbb{R}\}$$

So $\{(1,0,0);(0,0,1)\}$ is a basis for $Im\,f$ and $\dim(Imf)=2$.

13 Rank of a linear map

Définition 13.1

Let E and F be two $K - vector\ spaces$ of finite dimensions and f be a linear map of E into F. The rank of f is defined as follows :

$$rg(f) = dim(Im\, f)$$

Theorem 13.1

Let E and F be two $K - vector\ spaces$ of finite dimensions and f be a linear map of E into F. Then we have

$$dim(Ker f) + dim(Im\, f) = dim(E)$$

Property 13.1

Let E and F be two $K - vector\ spaces$ of finite dimensions and f be a linear map of E into F. Then we have

$$Ker f = \{0_E\} \Leftrightarrow f\ is\ i\text{njective.}$$

Example 13.1

$$f: \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \mapsto (x, y + z)$$

Find $Ker f$ and $Im\, f$and a basis for each as well as their dimensions.

Then check that

$$dim(Ker f) + dim(Im\, f) = dim(\mathbb{R}^3) = 3$$

We have

$$Ker f = \{(x, y, z): f(x, y, z) = 0_{R^2}\}$$

$$= \{(x, y, z): (x, y + z) = (0,0)\}$$

$$= \{(x, y, z): x = 0, y + z = 0\}$$

$$= \{(0, y, -y): y \in \mathbb{R}\} = \{y(0,1, -1): y \in \mathbb{R}\}$$

So $\{(0,1, -1)\}$ is a basis for $Ker f$ and $\dim(ker f) = 1$.

$$Im\, f = \{f(x, y, z): (x, y + z)\}$$

$$= \{(x, y + z): (x, y, z) \in \mathbb{R}^3\}$$

$$= \{(x, y + z): x, y, z \in \mathbb{R}\}$$

$$= \{x(1,0) + (y + z)(0,1): x, y, z \in \mathbb{R}\}$$

So $\{(1,0);(0,1)\}$ is a basis for $Im\,f$ and $dim(Im\,f)=2$.

And we have

$$dim(Kerf)+dim(Im\,f)=dim(\mathbb{R}^3)=3$$

EXERCISES

Exercise1
Which of the following subsets constitutes a vector subspace structure on ℝ ?

1- $F_1=\{(x,y,z)\in\mathbb{R}^3 : x+3y-4z=0\}$

2- $F_2=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=a\}$, discuss ?

3- $F_3=\{(x,y,z)\in\mathbb{R}^3 : x=3y\}$

4- $F_4=\{(x,y,z)\in\mathbb{R}^3 : x=y=0\}$

5- $F_5=\{(x,y,z)\in\mathbb{R}^3 : x>0\}$

Exercise2
Same questions as exercise2
1- $F_1=\{(x,0) : x\in\mathbb{R}\}$

2- $F_2=\{(x,y) : x^2+y^2=1\}$

3- $F_3=\{(x,2x) : x\in\mathbb{R}\}$

4- $F_4=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$

5- $F_5=\{(x,y,x+y) : x,y\in\mathbb{R}\}$

6- $F_6=\{(x,y,z)\in\mathbb{R}^3 : x=y\}$

Exercise3
Are the following families of vectors linearly independent ?
1- $(1,2,-1);(0,2,1);(1,1,1)$

2- $(1,2,-1,-2);(2,3,0,-1);(1,3,-1,0);(1,2,1,4)$

3- $(1,1,1);(1,1,0);(1,0,0)$

4- $(1,-1,1); (0,2,1); (2,-2,2)$

5- $(1,1,\alpha); (1,\alpha,1); (\alpha,1,1)$, Discuss according the to values of α.

6- $(1,-1,2); (0,5,3); (3,-4,7); (-1,2,3)$

Exercise4
Are the following families of vectors linearly independent ? are they generatring ? are they bases ? justify your answer ?

1/$(1,1); (2,1)$; 2/$(0,0); (2,3)$; 3/$(1,1); (3,3)$; 4/$(1,1); (0,1); (1,0)$
5/$(1,2); (3,1)$

Exercise5
Same questions as exercise 6

1/$(1,0,1); (0,1,0); (1,1,1)$; 2/$(1,0,0); (0,0,0); (0,0,1)$; 3/$(1,2,0); (0,1,0); (1,2,3)$
4/$(1,1,1); (2,1,0); (2,0,1); (0,0,1)$; 5/$(1,0,1); (2,1,0)$

Exercise6
Write the vector $(2,3,4)$ as a linear combination of the vectors :

$$(1,1,0); (0,1,1); (1,0,1)$$

Same questions for the vector $(4,3,2)$ as a linear combination of the the vectors :

$$(1,2,3); (1,1,2); (1,-1,1)$$

Exercise7
Let the family $\{X,Y,Z\}$ of vectors linearly independent in vector space E. Show that the family $\{X+Y,X+Z,Y+Z\}$ is linearly independent and the family $\{X-Y,X-Z,Y-Z\}$ is linearly dependent

Exercise8
Let E is a $K-vector\ space.$:
Show that $\{X,Y,Z\}$ is linearly independent if and only if $\{X+\alpha Y+\beta Z,Y,Z\}$ is a linearly independent ?$(\alpha,\beta \in K)$

Exercise9
Let the family $\{X,Y,Z\}$ be a linearly independent on a $K-vector\ space\ E.$.
Show the that family $\{X,X+Y,X+Y+Z\}$ is also linearly independent.

Exercise10
Détermine the real numbers a and b so that the vector $(-2,a,b,3)$ belongs to the vector subspace of $\mathbb{R}^4$ generated by $\{(1,-1,1,2); (-1,2,3,1)\}$.

Exercise11

Show that the following maps are linear maps :

1-

$$f:\mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (3x-y,x)$$

2- $f:\mathbb{R} \to \mathbb{R}^2$

3- $x \mapsto (2x,4x)$

Exercise12

Let the map

$$f:\mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto (y+z,x+y+z,x)$$

1- Show that f is a linear map .

2- Find $Kerf$ et $Im\,f$. Then give a basis and the dimension for each of these two subspaces

Exercise13

Same questions for the following maps :

$$f:\mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto (x,x+y,x+y+z)$$

Exercise 14

Are the following maps linear ?

1- $f(x,y) = |x-y|$

2- $f(x,y) = (x^2,y^2)$

3- $f(x,y,z) = (|xy|,|z|)$

SOLUTIONS

Exercise1

1- We show that $F_1 = \{(x, y, z): x + 3y - 4z = 0\}$ is a vector subspace.

$$F_1 \neq \Phi \text{ because } (0,0,0) \in F_1 \; (0 + 3(0) - 4(0) = 0)$$

Now we prove that

$$\forall (x, y, z); (x', y', z') \in F_1, \forall \alpha, \beta \in R: \alpha(x, y, z) + \beta(x', y', z') \in F_1$$

$$((x, y, z) \in F_1 \Rightarrow x + 3y - 4z = 0 \text{) and (} (x', y', z') \in F_1 \Rightarrow x' + 3y' - 4z' = 0)$$

Then we have

$$\alpha.(x, y, z) + \beta.(x', y', z') = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in F_1$$

Because

$$\alpha x + \beta x' + 3(\alpha y + \beta y') - 4(\alpha z + \beta z') = \alpha(x + 3y - 4z) + \beta(x' + 3y' - 4z') = 0$$

So $F_1 = \{(x, y, z): x + 3y - 4z = 0\}$ is a vector subspace.

2- Let $F_2 = \{(x, y, z): x + y + z = a\}$

If $a \neq 0$ the zero vector $(0,0,0) \notin F_2$ and therefore F_2 is not a vector subspace of $\mathbb{R}^3$

If $a = 0$ the zero vector $(0,0,0) \in F_2$ and therefore $F_2 \neq \Phi$.

Now we prove that

$$\forall (x, y, z); (x', y', z') \in F_2, \forall \alpha, \beta \in R: \alpha(x, y, z) + \beta(x', y', z') \in F_2$$

We have

$$(x, y, z) \in F_2 \Rightarrow x + y + z = 0 \text{ and } (x', y', z') \in F_2 \Rightarrow x' + y' + z' = 0$$

$$\alpha(x, y, z) + \beta(x', y', z') = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in F_2, \text{ because}$$

$$(\alpha x + \beta x') + (\alpha y + \beta y') + (\alpha z + \beta z') = \alpha(x + y + z) + \beta(x' + y' + z') = 0$$

Then we have $F_2 = \{(x, y, z): x + y + z = 0\}$ is a vector subspace of $\mathbb{R}^3$

3- Let $F_3 = \{(x, y, z): x = 3y\}$

$F_3 \neq \Phi$ because $(0,0,0) \in F_3$, $(0 = 3(0))$

Now we prove that

$$\forall (x, y, z); (x', y', z') \in F_3, \forall \alpha, \beta \in R: \alpha(x, y, z) + \beta(x', y', z') \in F_3$$

$$(x, y, z) \in F_3 \Rightarrow x = 3y \text{ et } (x', y', z') \in F_3 \Rightarrow x' = 3y'$$

$$\alpha(x, y, z) + \beta(x', y', z') = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in F_3$$

So
$$\alpha x + \beta x' = \alpha(3y) + \beta(3y') = 3(\alpha y + \beta y')$$

Then we have $F_3 = \{(x, y, z): x = 3y\}$ is a vector subspace of $\mathbb{R}^3$.

4- Let $F_4 = \{(x, y, z): x = y = 0\}$

$F_4 \neq \Phi$ because $(0,0,0) \in F_4$,$(0 = 0)$

Now we prove that

$$\forall (x, y, z); (x', y', z') \in F_4, \forall \alpha, \beta \in R; \alpha(x, y, z) + \beta(x', y', z') \in F_4$$

$$(x, y, z) \in F_4 \Rightarrow x = y = 0 \text{ et } (x', y', z') \in F_4 \Rightarrow x' = y' = 0$$

$$\alpha(x, y, z) + \beta(x', y', z') = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in F_4$$

So
$$\alpha x + \beta x' = \alpha y + \beta y' = 0$$

Then we have $F_4 = \{(x, y, z): x = y = 0\}$ is a vector subspace of $\mathbb{R}^3$.

Same reasoning for $F_5 = \{(x, y, z): x > 0\}$.

Exercise3

1- We show that $F_1 = \{(x, 0): x \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^2$

$F_1 \neq \Phi$, because $(0,0) \in F_2$

Now we prove that

$$\forall (x, y); (x', y') \in F_1, \forall \alpha, \beta \in \mathbb{R}: \alpha(x, y) + \beta(x', y') \in F_1$$

$$(x, 0) \in F_1 \text{ and } (x', 0) \in F_1$$

So

$$\alpha(x, 0) + \beta(x', 0) = (\alpha x + \beta x', 0) \in F_1,$$

Then we have $F_1 = \{(x, 0): x \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^2$.

2- Let $F_2 = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\}$

$F_2 \neq \Phi$, because $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in F_2$

We have

$$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) \in F_2 \text{ but } \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)=\left(\frac{1+\sqrt{3}}{2},\frac{\sqrt{3}+1}{2}\right) \notin F_2$$

Since

$$\left(\frac{1+\sqrt{3}}{2}\right)^2+\left(\frac{\sqrt{3}+1}{2}\right)^2 \neq 1$$

So $F_2 = \{(x,y): x^2+y^2=1\}$ is not a vector subspace of $\mathbb{R}^2$.

We can easly check that F_3, F_5 and F_6 are vector subspace, while F_4 is not

Exercise4

We say that the vectors $X_1, X_2, \ldots, X_n$ of a vector space E are linearly independent, if

$$\alpha_1 X_1+\alpha_2 X_2+\ldots+\alpha_n X_n = 0_E : \alpha_1=\alpha_2=\ldots=\alpha_n=0$$

1- Let's check if the following vectors $(1,2,-1); (0,2,1); (1,1,1)$ are linearly independent ?

$$\alpha_1.(1,2,-1)+\alpha_2.(0,2,1)+\alpha_3.(1,1,1)=(0,0,0) \Rightarrow \begin{cases} \alpha_1+\alpha_3=0 \\ 2\alpha_1+2\alpha_2+\alpha_3=0 \\ -\alpha_1+\alpha_2+\alpha_3=0 \end{cases}$$

$$\Longrightarrow \begin{cases} \alpha_1=-\alpha_2 \\ \alpha_3=0 \\ \alpha_1=\alpha_2 \end{cases} \Longrightarrow \alpha_1=\alpha_2=\alpha_3=0$$

2- Same thing for vectors $(1,2,-1,-2); (2,3,0,-1); (1,3,-1,0); (1,2,1,4)$

$$\alpha_1.\begin{pmatrix}1\\2\\-1\\-2\end{pmatrix}+\alpha_2.\begin{pmatrix}2\\3\\0\\-1\end{pmatrix}+\alpha_3.\begin{pmatrix}1\\3\\-1\\0\end{pmatrix}+\alpha_4.\begin{pmatrix}1\\2\\1\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1+2\alpha_2+\alpha_3+\alpha_4=0 \\ 2\alpha_1+3\alpha_2+3\alpha_3+2\alpha_4=0 \\ -\alpha_1-\alpha_2+\alpha_3=0 \\ -2\alpha_1-\alpha_2+4\alpha_4=0 \end{cases}$$

$$\Longrightarrow \begin{cases} \alpha_1=4\alpha_4 \\ \alpha_2=-9\alpha_4 \\ \alpha_3=-5\alpha_4 \\ 5\alpha_4=0 \end{cases} \Leftrightarrow \alpha_1=\alpha_2=\alpha_3=\alpha_4=0$$

Same thing for vectors $(1,1,1); (1,1,0); (1,0,0)$.

While the following vectors $(1,-1,1); (0,2,1); (2,-2,2)$ are linearly dependent, because

$$(2,-2,2)=2(1,-1,1)$$

Same reasoning for the vectors $(1,-1,2); (0,5,3); (3,-4,7); (-1,2,3)$ are linearly dependent

3- For the vectors $(1,1,\alpha);(1,\alpha,1);(\alpha,1,1)$, we will discuss their independence according to the values of the parameter α .

$$\lambda.\begin{pmatrix}1\\1\\\alpha\end{pmatrix}+\mu.\begin{pmatrix}1\\\alpha\\1\end{pmatrix}+\eta.\begin{pmatrix}\alpha\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}\lambda+\mu+\mu\alpha=0.....(1)\\\lambda+\mu\alpha+\eta=0.....(2)\\\lambda\alpha+\mu+\eta=0.....(3)\end{cases}$$

$$(1)+(2)+(3)\Rightarrow(\alpha+2)(\lambda+\mu+\eta)=0$$

If $\alpha=-2$, those vectors are linearly dependent

Because $(1,1,-2)=-(1,-2,1)-(-2,1,1)$

If $\alpha=1$,those vectors are linearly dependent

If $(\alpha\neq-2)\wedge(\alpha\neq1)$, the three vectors are linearly independent

Exercise5

1- $B_1=\{(1,1);(2,1)\}$

$$\alpha.\begin{pmatrix}1\\1\end{pmatrix}+\beta.\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}\alpha+2\beta=0\\\alpha+\beta=0\end{cases}\Longrightarrow\alpha=\beta=0$$

Since the family of vectors $\{(1,1);(2,1)\}$ is linearly independent and containts two vectors and $dim(\mathbb{R}^2)=2$, then it is a basis of $\mathbb{R}^2$

2- The following family $B_2=\{(0,0);(2,3)\}$, $B_3=\{(1,1);(3,3)\}$,

$B_4=\{(1,1);(0,1),(1,0)\}$ are not bases for $\mathbb{R}^2$

By identical reasoning, we will show that $B_5=\{(1,2);(3,1)\}$ is basis for $\mathbb{R}^2$

Exercise6

Write the vector $(2,3,4)$ as a linear combination of vectors

$$(1,1,0);(0,1,1);(1,0,1)$$

Same question for the vector $(4,3,2)$ as a linear combination of vectors

$$(1,2,3);(1,1,2);(1,-1,1)$$

$$\begin{pmatrix}2\\3\\4\end{pmatrix}=\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\0\\1\end{pmatrix}\Leftrightarrow\begin{cases}\alpha+\gamma=2\\\alpha+\beta=3\\\beta+\gamma=4\end{cases}\Leftrightarrow\begin{cases}\beta=\frac{5}{2}\\\alpha=\frac{1}{2}\\\gamma=\frac{3}{2}\end{cases}$$

So $$\begin{pmatrix}2\\3\\4\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}+\frac{5}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}+\frac{3}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

For the vector $(4,3,2)$ and the base $B=\{(1,2,3);(1,1,2);(1,-1,1)\}$, we have

$$\begin{pmatrix}4\\3\\2\end{pmatrix}=\alpha\begin{pmatrix}1\\2\\3\end{pmatrix}+\beta\begin{pmatrix}1\\1\\2\end{pmatrix}+\gamma\begin{pmatrix}1\\-1\\1\end{pmatrix}\Leftrightarrow\begin{cases}\alpha+\beta+\gamma=4\\2\alpha+\beta-\gamma=3\\3\alpha+2\beta+\gamma=2\end{cases}\Leftrightarrow\begin{cases}\beta=20\\\alpha=-11\\\gamma=-5\end{cases}$$

Then we have $$\begin{pmatrix}4\\3\\2\end{pmatrix}=-11\begin{pmatrix}1\\2\\3\end{pmatrix}+20\begin{pmatrix}1\\1\\2\end{pmatrix}-5\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Exercise7

Given the family $\{X,Y,Z\}$ of linearly independent vectors in a vector space E, show that the family $\{X+Y,X+Z,Y+Z\}$ is a linearly independent and the family $\{X-Y,X-Z,Y-Z\}$ is a linearly dependent

$$\{X,Y,Z\}\text{ linearly independent}\Longrightarrow\{X+Y,X+Z,Y+Z\}\text{ linearly independent}$$

$$\alpha(X+Y)+\beta(X+Z)+\gamma(Y+Z)=0\Rightarrow(\alpha+\beta)X+(\alpha+\gamma)Y+(\beta+\gamma)Z=0$$

Since $\{X,Y,Z\}$ is linearly independent, we have

$$\begin{cases}\alpha+\beta=0\\\alpha+\gamma=0\\\beta+\gamma=0\end{cases}\Leftrightarrow\begin{cases}\beta=\gamma\\\beta=-\gamma\\\alpha=-\beta\end{cases}\Leftrightarrow\alpha=\beta=\gamma=0$$

Given $\{X,Y,Z\}$ is linearly independent, we want to show that $\{X-Y,X-Z,Y-Z\}$ is not ?

$$\alpha(X-Y)+\beta(X-Z)+\gamma(Y-Z)=0\Rightarrow(\alpha+\beta)X+(-\alpha+\gamma)Y-(\beta+\gamma)Z=0$$

Since $\{X,Y,Z\}$ is linearly independent, we have

$$\begin{cases}\alpha+\beta=0\\-\alpha+\gamma=0\\-\beta-\gamma=0\end{cases}\Leftrightarrow\begin{cases}\beta=-\gamma\\\gamma=\gamma\\\alpha=\gamma\end{cases}$$

So the family $\{X-Y,X-Z,Y-Z\}$ is not linearly independent

Exercise8

$$\lambda(X+\alpha Y+\beta Z)+\mu Y+\eta Z=0\Rightarrow\lambda X+(\lambda\alpha+\mu)Y+(\lambda\beta+\eta)Z=0$$

Since $\{X,Y,Z\}$ is a linearly independent, we have

$$\begin{cases}\lambda=0\\\lambda\alpha+\mu=0\\\lambda\beta+\eta=0\end{cases}\Leftrightarrow\begin{cases}\lambda=0\\\mu=0\\\eta=0\end{cases}$$

We have

$$\{X,Y,Z\} \text{ is linearly independent} \Longrightarrow \{X+\alpha Y+\beta Z, Y, Z\} \text{ is linearly independent}$$

Exercise9

Same reasoning as in exercise8.

Exercise10

Détermine the reals a et b so that the vector $(-2,a,b,3)$ belongs to the vector subspace of $\mathbb{R}^4$ generated by $\{(1,-1,1,2);(-1,2,3,1)\}$.

We have

$$(-2,a,b,3)=\alpha(1,-1,1,2)+\beta(-1,2,3,1)$$

$$\Leftrightarrow \begin{cases} \alpha-\beta=-2 \\ -\alpha+2\beta=a \\ \alpha+3\beta=b \\ 2\alpha+\beta=3 \end{cases} \Leftrightarrow \begin{cases} \alpha=\dfrac{1}{3} \\ \beta=\dfrac{7}{3} \\ a=-\dfrac{1}{3}+\dfrac{14}{3}=\dfrac{13}{3} \\ b=\dfrac{1}{3}+\dfrac{21}{3}=\dfrac{22}{3} \end{cases}$$

Exercise11

Let us show that the following maps are linear.

1-
$$\begin{aligned} f:\mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto (3x-y,x) \end{aligned}$$

$Let\ (x,y),(x',y') \in \mathbb{R}^2\ and\ \alpha,\beta \in \mathbb{R}$.

$$\begin{aligned} f(\alpha.(x,y)+\beta.(x',y')) &= f(\alpha x+\beta x', \alpha y+\beta y') \\ &= (3(\alpha x+\beta x')-(\alpha y+\beta y'), \alpha x+\beta x') \\ &= (\alpha(3x-y)+\beta(3x'-y'), \alpha x+\beta x') \\ &= (\alpha(3x-y), \alpha x)+(\beta(3x'-y'), \beta x') \\ &= \alpha(3x-y,x)+\beta(3x'-y',x') \\ &= \alpha f(x,y)+\beta f(x',y') \end{aligned}$$

Thus f is a linear map.

2-
$$\begin{aligned} f:\mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto (2x,4x) \end{aligned}$$

$Let\ (x,y),(x',y') \in \mathbb{R}^2\ and\ \alpha,\beta \in \mathbb{R}$.

$$f(\alpha(x,y)+\beta(x',y')) = f(\alpha x+\beta x', \alpha y+\beta y')$$

$$= (2(\alpha x + \beta x'), 4(\alpha x + \beta x'))$$

$$= (\alpha(2x) + \beta(2x'), \alpha(4x) + \beta(4x'))$$

$$= (\alpha(2x), \alpha(4x)) + (\beta(2x'), \beta(4x'))$$

$$= \alpha(2x, 4x) + \beta(2x', 4x')$$

$$= \alpha f(x, y) + \beta f(x', y')$$

Thus f is a linear map

Exercise12

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \mapsto (y + z, x + y + z, x)$$

1- Let us show that f is a linear map.

$$\forall (x, y, z), (x', y', z') \in \mathbb{R}^3, \forall \alpha, \beta \in \mathbb{R}:$$

$$f\big(\alpha(x, y, z) + \beta(x', y', z')\big) = f(\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z')$$

$$= \big((\alpha y + \beta y') + (\alpha z + \beta z'), (\alpha x + \beta x') + (\alpha y + \beta y') + (\alpha z + \beta z'), \alpha x + \beta x'\big)$$

$$= \alpha(y + z, x + y + z, x) + \beta(y' + z', x' + y' + z', x')$$

$$= \alpha f(x, y, z) + \beta f(x', y', z')$$

2- Let's find $Kerf$ and $Im\, f$.

$$Kerf = \{(x, y, z): f(x, y, z) = (0,0,0)\}$$

$$f(x, y, z) = (0,0,0) \Leftrightarrow \begin{cases} y + z = 0 \\ x + y + z = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ z = -y \end{cases}$$

So

$$Kerf = \{(0, y, -y): y \in \mathbb{R}\} = \{y(0,1,-1): y \in \mathbb{R}\}$$

Then $Kerf$ admits $B_1 = \{(0,1,-1)\}$ as a basis and consequently $dim(Kerf) = 1$

So

$$Im\, f = \{(X, X + Z, Z): X, Z \in \mathbb{R}\}$$

$$= \{(X, X, 0) + (0, Z, Z): X, Z \in \mathbb{R}\}$$

$$= \{X(1,1,0) + Z(0,1,1): X, Z \in \mathbb{R}\}$$

Then $Im\, f$ admits $B_2 = \{(1,1,0), (0,1,1)\}$ as a basis and consequently $dim(Im\, f) = 2$

Exercise13

Same questions for the following map :

$$\begin{aligned} f:\mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (x,x+y,x+y+z) \end{aligned}$$

Preuve
We can easily show that the map

$f:(x,y,z)\mapsto(x,x+y,x+y+z)$ is linear and $Kerf=\{(0,0,0)\}$ and therefore

$$dim(Kerf)=0$$

$$Im\,f=\{(x,x+y,x+y+z): x,y,z\in R\}=\{x(1,1,1)+y(0,1,1)+z(0,0,1): x,y,z\in R\}$$

The basis of $Im\,f$ is $B=\{(1,1,1),(0,1,1),(0,0,1)\}$ and $dim(Im\,f)=3$.

Exercise14
We will check if the following maps are linear.

1- The map :$f(x,y)=|x-y|$ is not linear, because

$$\begin{aligned} f\big(\alpha(x,y)+\beta(x',y')\big) &= f(\alpha x+\beta x',\alpha y+\beta y') \\ &= |(\alpha x+\beta x')-(\alpha y+\beta y')| \\ &= |\alpha(x-y)+\beta(x'-y')| \\ &\neq \alpha|x-y|+\beta|x'-y'| \end{aligned}$$

Same reasoning for the following maps

2- $f(x,y)=(x^2,y^2)$

3- $f(x,y,z)=(|xy|,|z|)$

which are not linear.

Chapter 2

The Matrix

1 The matrix

1.1 Matrices concept

Définition 1.1
An matrix $A = (a_{ij})_{1\leq i\leq m; 1\leq j\leq n}$ is an table of the numbers with m rows and n colomns. The numbers a_{ij}that make up the matrix are called the elements or the coefficients of the matrix. This matrix is called of order (m, n). The set of matrix of order (m, n) with real coefficients is denoted $M_{\mathbb{R}}(m, n)$.

Example 1.1

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -5 \\ 8 & 0 & 2 \end{pmatrix}$$

$$A \in M_R(3,3) \text{ with } a_{22} = 2, a_{13} = 0, a_{32} = 0 \ldots.$$

Special cases

- A matrix in which all the elements are zero is called a zero matrix and it is written

$$\boldsymbol{A} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

- A matrix containts only one column is called a $(m, 1)$ matrix column.
- A matrix having the same number of rows and columns (m, m), is called a square matrix.

Definition 1.2
We call the unit matrix of order n, the square matrix whose diagonal elements are equal to 1 and the other elements are all zero. This matrix is denoted I_n.

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Example 1.2

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1 2 Transpose of a matrix

Definition 1.3

Let A be a matrix of m rows and n colomns. The transpose of the matrix A, is a matrix of n rows and m colomns denoted A^t whose rows are the columns of A and whose columns are the rows of A.

Example 1.3

Let the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 9 & -4 \end{pmatrix} \quad \text{then} \quad A^t = \begin{pmatrix} 1 & 2 \\ -2 & 9 \\ 0 & -4 \end{pmatrix}$$

1 3 Equality of two matrices

Definition 1 4

Let the matrix A and B have the same order. We say that the two matrices are equal, if all elements of A are equal to the corresponding elements of B.

Example 1 4

We give the matrix

$$A = \begin{pmatrix} x+1 & 3 \\ 3 & -2y+3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}$$

Let's determine the real numbers x and y so that the two previous matrices are equal.

$$A = B \Leftrightarrow \begin{cases} x+1=4 \\ -2y+3=5 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ y=-1 \end{cases}$$

1.4 Nilpotent matrix

Consider the square matrix A of order n. We say that A is nilpotent, if there exists an integer p such that $A^p = 0$

Example1.5

We give the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A.A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.5 Idempotent matrix

Consider the square matrix A of order n. We say that A is idempotent, if $A^2 = A$

Example1.6
We give the matrix

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} . \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{9} & \frac{3}{9} & \frac{3}{9} \\ \frac{3}{9} & \frac{3}{9} & \frac{3}{9} \\ \frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = A$$

2 Operations on matrices

2.1 The addition

Definition2.1
Let the matrix A and B have The same order (m, n) . The sum of the matrix A and B is the matrix C of order (m, n), each element of which is the sum of the two corresponding elements of A and B.

Example2.1

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 9 & -2 & 2 \\ -12 & 7 & -6 \end{pmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 1+9 & 2-2 & -3+2 \\ 0-12 & 5+7 & -4-6 \end{pmatrix} = \begin{pmatrix} 10 & 0 & -1 \\ -12 & 12 & -10 \end{pmatrix}$$

2.2 Multiplication by a scalar

Definition2.2
Let A be any matrix and α be a real or complex number. The product of A by α, is the matrix of the same order as A and each element of which is the product α by the corresponding element of the matrix A.

Example2.2

Let the matrix $A = \begin{pmatrix} x & 1 \\ y & z \end{pmatrix}$ and $\alpha \in \mathbb{C}$. Then $\alpha A = \begin{pmatrix} \alpha x & \alpha \\ \alpha y & \alpha z \end{pmatrix}$

3 Product of matrices

Definition3.1

Let be a matrix $A \in \mathcal{M}(m,n)$ and a matrix $B \in \mathcal{M}(n,p)$. We call the product of two matrices A and B, the matrix $\mathrm{C} \in \mathcal{M}(m,p)$ obtained by multiplying each row of A by each column of B.

If $A = (a_{ij}) \in \mathcal{M}(m,n)$ and $B = (b_{ij}) \in \mathcal{M}(n,p)$, then the product $C = AB \in \mathcal{M}(m,p)$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Example3.1

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ -1 & -2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 2 & 7 \end{pmatrix}$$

Then we have

$$C = AB = \begin{pmatrix} (2 \times 1) + (0 \times 0) & (2 \times 0) + (0 \times 2) & (-4 \times 2) + (7 \times 0) \\ (0 \times 1) + (3 \times 0) & (0 \times 0) + (3 \times 2) & (-4 \times 0) + (3 \times 7) \\ (-1 \times 1) + ((-2) \times 0) & (-1 \times 0) + (-2 \times 2) & (-4 \times -1) + (-2 \times 7) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -8 \\ 0 & 6 & 21 \\ -1 & -4 & -10 \end{pmatrix} \in M(3,3)$$

Remark

- The unit matrix I_n plays a similar role for the matrix product as the number 1 for the product of real numbers.

- For a square matrix A of order n, we have $A \times I_n = I_n \times A = A$.

Property

Assuming that the orders allow the product, we have

$$A(B + C) = AB + AC : \text{Left distributivity}$$

$$(A + B)C = AC + BC : \text{Right distributivity}$$

$$A(BC)) = (AB)C : \text{Associativity}$$

Definition3.2

Let A be a square matrix of order n. Let p be a non-zero natural number. We denote A^p the matrix defined as follow :

$$A^p = AAA \dots . A,$$

4 Inverse matrix

Property

Let A be a square matrix of order n. If there exists a square matrix B of the same order as A such that

$$AB = BA = I_n,$$

then the matrix B is unique. B is called the inverse of the matrix A and it is denoted A^{-1}

In this case, we say that A is invertible.

Example4.1
Let

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, B = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

Then we have

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$B = A^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

5 Determinant

Definition5.1
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We call the determinant of the matrix A, denoted $det(A)$, the real

$$det(A) = ad - bc$$

Definition5.2
Let A be the square matrix of order $n \geq 3$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

We call determinant of the matrix A with respect to the column j, the expression

$$det(A) = \sum_{k=1}^{k=n} (-1)^{k+j} a_{kj} det(A_{kj}),$$

where $\det(A_{kj})$is the determinant of the matrix from which we have removed the row k and the column j of A.

We call determinant of the matrix A with respect to the row i, the expression

$$det(A) = \sum_{k=1}^{k=n} (-1)^{i+k} a_{ik} det(A_{ik}),$$

where $\det(A_{ik})$ is the determinant of the matrix from which we have removed the row i and the column j of A.

Example5.1
Let’s find the determinant of the following matrix with respect to the first column.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -3 \\ 3 & 4 & 4 \end{pmatrix}$$

$$det(A) = 1 \times \begin{vmatrix} -1 & -3 \\ 4 & 4 \end{vmatrix} - 2 \times \begin{vmatrix} 0 & 2 \\ 4 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 0 & 2 \\ -1 & -3 \end{vmatrix} = 1 \times 8 - 2 \times (-8) + 3 \times 2 = 30$$

Let's find this determinant relative to the second row.

$$det(A) = -2 \times \begin{vmatrix} 0 & 2 \\ 4 & 4 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -2 \times (-8) - 1 \times (-2) + 3 \times 4 = 30$$

6 Calcul of an inverse matrix

Let A be the square matrix of order n with $det(A) \neq 0$, defined as follows :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The comatrix denoted by $Com(A)$, is a matrix of the same order as A and its elements (c_{ij}) are called cofacteurs of A given by :

$$c_{ij} = (-1)^{i+j} det(A_{ij}); i,j \in \{1,2,3,\dots,n\},$$

where $det(A_{ij})$is the determinant of the matrix from which we have removed the row i and the column j of A.

Theorem6.1

If $det(A) \neq 0$, then A is invertible and its inverse matrix given by :

$$A^{-1} = \frac{1}{det(A)} \left(Com(A)\right)^t$$

Example

Find the inverse of the following matrix :

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -3 \\ 3 & 4 & 4 \end{pmatrix}$$

Solution

We have $det(A) = 30$, therefore, A^{-1} exists :

$$A^{-1} = \frac{1}{det(A)} \left(Com(A)\right)^t$$

$$Com(A) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$c_{11} = (-1)^2 \begin{vmatrix} -1 & -3 \\ 4 & 4 \end{vmatrix} = 8 \, , c_{12} = (-1)^3 \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = -17, c_{13} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 11$$

$$c_{21} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 4 & 4 \end{vmatrix} = 8 \, , c_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \, , c_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -4$$

$$c_{31} = (-1)^4 \begin{vmatrix} 0 & 2 \\ -1 & -3 \end{vmatrix} = 2 \, , c_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7 \, , c_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1$$

$$C = \begin{pmatrix} 8 & -17 & 11 \\ 8 & -2 & -4 \\ 2 & 7 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{30} \begin{pmatrix} 8 & 8 & 2 \\ -17 & -2 & 7 \\ 11 & -4 & -1 \end{pmatrix}$$

Proof.

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -3 \\ 3 & 4 & 4 \end{pmatrix} . \frac{1}{30} \begin{pmatrix} 8 & 8 & 2 \\ -17 & -2 & 7 \\ 11 & -4 & -1 \end{pmatrix} = \frac{1}{30} \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

7 Matrix associated to a linear map

Definition7.1

Let E and F be two $K - vector\ spaces$ of finite, $dim(E) = n$ and $dim(F) = p, n, p \in \mathbb{N}^*$ and $f: E \to F$ be a linear map.

Let $B = \{e_1, e_2, e_3, \dots, e_n\}$ be a basis of E and $B' = \{f_1, f_2, f_3, \dots, f_n\}$ be a basis of F

The images by the map f of vectors $e_1, e_2, e_3, \dots, e_n$ are written in the basis B' as follows:

$$\begin{cases} f(e_1) = a_{11}.f_1 + a_{21}.f_2 + a_{31}.f_3 + \cdots + a_{p1}.f_p \\ f(e_2) = a_{12}.f_1 + a_{22}.f_2 + a_{32}.f_3 + \cdots + a_{p2}.f_p \\ \dots \\ \dots \\ f(e_n) = a_{1n}.f_1 + a_{2n}.f_2 + a_{3n}.f_3 + \cdots + a_{pn}.f_p \end{cases}$$

where $a_{ij} \in K, i \in \{1,2, \dots, p\}$ and $j \in \{1,2,3, \dots, n\}$

We call matrix associated to f relative to the bases B and B', the matrix denoted by $M(f, B, B')$ and defined by :

$$M(f, B, B') = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{p1} & a_{p2} & \dots & \dots & a_{pn} \end{pmatrix}$$

The column elements in $M(f, B, B')$ are the composents of the vector $f(e_j)$ in the basis B'.

The number of rows of $M(f,B,B')$ is equal to the dimension of F.

The number of columns of $M(f,B,B')$ is equal to the dimension of E.

Example7.1
We consider the linear map :

$$f:\mathbb{R}^3 \to \mathbb{R}^3$$

$$(x,y,z) \mapsto (y+z, x-2y)$$

Let $B = \{e_1,e_2,e_3\}$ be the standard basis of $\mathbb{R}^3$ and $B' = \{f_1,f_2\}$ the standard basis of $\mathbb{R}^2$

Let's write $M(f,B,B')$

We have

$$f(e_1) = f(1,0,0) = (0,1) = 0.f_1 + 1.f_2$$

$$f(e_2) = f(0,1,0) = (1,-2) = 1.f_1 - 2.f_2$$

$$f(e_3) = f(0,0,1) = (1,0) = 1.f_1 + 0.f_2$$

Then

$$M(f,B,B') = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

Example7.2
Consider the linear map :

$$f:\mathbb{R}^3 \to \mathbb{R}^3$$

$$(x,y,z) \mapsto (2x-y, 2y-z)$$

Let's write $M(f,B,B')$, where

$$B = \{U_1 = (1,-1,1), U_2 = (1,1,0), U_3 = (1,-1,0)\}$$

and

$$B' = \{V_1 = (1,1), V_2 = (1,2)\}$$

We have

$$f(U_1) = f(1,-1,1) = \alpha_1(1,1) + \alpha_2(1,2) = (3,-3)$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 3 \\ \alpha_1 + 2\alpha_2 = -3 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 9 \\ \alpha_2 = -6 \end{cases}$$

$$f(U_2) = f(1,1,0) = \beta_1(1,1) + \beta_2(1,2) = (1,2)$$

$$\Rightarrow \begin{cases} \beta_1+\beta_2=1 \\ \beta_1+2\beta_2=2 \end{cases} \Rightarrow \begin{cases} \beta_1=0 \\ \beta_2=1 \end{cases}$$

$$f(U_3)=f(1,-1,0)=(1,0)=\gamma_1(1,1)+\gamma_2(1,2)=(3,-2)$$

$$\Rightarrow \begin{cases} \gamma_1+\gamma_2=3 \\ \gamma_1+2\gamma_2=-2 \end{cases} \Rightarrow \begin{cases} \gamma_1=8 \\ \gamma_2=-5 \end{cases}$$

Then

$$M(f,B,B')=\begin{pmatrix} 9 & 0 & 8 \\ -6 & 1 & -5 \end{pmatrix}$$

Example7.3

Let f be a linear map defined by :

$$f:\mathbb{R}^2 \to \mathbb{R}^2$$

$$(x,y) \mapsto f(x,y)=\left(\tfrac{\sqrt{3}}{2}x-\tfrac{1}{2}y,\tfrac{1}{2}x+\tfrac{\sqrt{3}}{2}y\right)$$

Let’s write $M(f,B)$, where $B=\{U_1=(1,-2),U_2=(-2,1)\}$

$$f(U_1)=f(1,-2)=\alpha_1(1,-2)+\alpha_2(-2,1)$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}+1,\frac{1}{2}-\sqrt{3}\right)=\alpha_1(1,-2)+\alpha_2(-2,1)$$

$$\Rightarrow \begin{cases} \alpha_1-2\alpha_2=\dfrac{\sqrt{3}}{2}+1 \\ -2\alpha_1+\alpha_2=\dfrac{1}{2}-\sqrt{3} \end{cases} \Rightarrow \begin{cases} \alpha_1=\dfrac{\sqrt{3}}{2}-\dfrac{2}{3} \\ \alpha_2=-\dfrac{5}{6} \end{cases}$$

$$f(U_2)=f(-2,1)=\beta_1(1,-2)+\beta_2(-2,1)$$

$$\Rightarrow \begin{cases} \beta_1-2\beta_2=-\sqrt{3}-\dfrac{1}{2} \\ -2\beta_1+\beta_2=-1+\dfrac{\sqrt{3}}{2} \end{cases} \Rightarrow \begin{cases} \beta_1=\dfrac{5}{6} \\ \beta_2=\dfrac{1}{2}-\dfrac{\sqrt{3}}{6} \end{cases}$$

$$M(f,B)=\begin{pmatrix} \dfrac{\sqrt{3}}{2}-\dfrac{2}{3} & \dfrac{5}{6} \\ -\dfrac{5}{6} & \dfrac{1}{2}-\dfrac{\sqrt{3}}{6} \end{pmatrix}$$

8 Change of a basis and transition matrices

The matrix which represents a linear map was constructed by using a choise of bases in the starting space and the ending space.

Let E be a $K-vector\ space$ of finite dimension $n, n \in \mathbb{N}^*$,

$B = \{e_1, e_2, e_3, \dots, e_n\}$ and $B' = \{f_1, f_2, f_3, \dots, f_n\}$ be two bases of E.

We have

$$\begin{cases} f_1 = t_{11}.e_1 + t_{21}.e_2 + \cdots + t_{n1}.e_n \\ f_2 = t_{12}.e_1 + t_{22}.e_2 + \cdots + t_{n2}.e_n \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ f_n = t_{1n}.e_1 + t_{2n}.e_2 + \cdots + t_{nn}.e_n \end{cases}$$

where $i, j \in \{1,2,3, \dots, n\}$ and $t_{ij} \in K$.

We call transition matrix from the basis B to the basis B', the matrix P whose columns are the components of the vectors f_i in the basis B

$$P = \begin{pmatrix} t_{11} & t_{12} & . & . & t_{1n} \\ t_{21} & t_{22} & . & . & t_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ t_{n1} & t_{n2} & . & . & t_{nn} \end{pmatrix}$$

Example8.1
Let $B = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$

and $B' = \{f_1 = (2,1,-3); f_2 = (0,1,-1); f_3 = (1,0,3)\}$ be another basis of $\mathbb{R}^3$.

We have

$$\begin{cases} f_1 = 2e_1 + e_2 - 3e_3 \\ f_2 = 0e_1 + e_2 - e_3 \\ f_3 = e_1 + 0e_2 + 3e_3 \end{cases}$$

Then

$$P = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ -3 & -1 & 3 \end{pmatrix}$$

9 Rank of a matrix

Definition9.1
We call rank of matrix A, the maximum number of linearly independent column or row vectors of A. If $(R_1, R_2, \dots, R_n)$ are the row vectors of A or $(C_1, C_2, \dots, C_m)$ are column vectors of A such that :

$$A=\begin{pmatrix}R_1\\R_2\\.\\.\\R_n\end{pmatrix}\text{ or } A=(C_1\ C_2\ \dots C_m)$$

$(R_1, R_2, \dots, R_n)$ or $(C_1, C_2, \dots, C_m)$ are to be linearly independent

Theorem9.1
If A is a matrix of order (m,n), we have

$$rg_{column}(A)=rg_{row}(A)=rg(A)=rg(A^t)$$

Example9.1
Let A be the matrix associated to the linear map f such that

$$A=\begin{pmatrix}0&-1&0&3\\0&0&0&0\\2&0&1&0\end{pmatrix}$$

Find $rg(A)$.

$$A=\begin{pmatrix}0&-1&0&3\\0&0&0&0\\2&0&1&0\end{pmatrix}\begin{matrix}c_1\to c_1-2c_3\\c_4\to c_4+3c_2\\\to\end{matrix}\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}.$$

Then

$$rg(A)=2.$$

EXERCISES

Exercise1

Let the matrices A, B, C, D and E be defined as follows :

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} \,,\ B = \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix} \,,\ C = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{pmatrix}, D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

1- Calculate $A + B, 2A, 3B, -4A + 2B$.

2- Calculate AB , BA , AC^t , BC^t.

3- Verify $(A + B)C^t = AC^t + BC^t$, $(AB)^t = B^tA^t$, $(AC^t) = CA^t$.

4- Calculate DE and ED.

Exercise2

We consider tha matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

1- Find B such that $B = A - I_3$ and then calculate B^n for $n \in \mathbb{N}$

2- Calculate A^n for $n \in \mathbb{N}$

Exercise3

Let the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Show by recurrence that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Exercise4

Let the matrix

$$A = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$$

Calculate A^{3n} for $n \epsilon \mathbb{N}$

Exercise5

Let A and B be two square matrices of same order. Give a necessary and sufficient condition to have

$$(A + B)(A - B) = A^2 - B^2$$

Exercise6
We consider for each real number θ, the matrix

$$A(\theta) = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$$

1- Show that $A(\theta)A(\theta') = A(\theta + \theta')$.

2- Calculate $\left(A(\theta)\right)^n$ for $n\epsilon\mathbb{N}$.

Exercise7
Same questions as exercise6 with the matrix.

$$A(x) = \begin{pmatrix} chx & shx \\ shx & chx \end{pmatrix}$$

Exercise8
Let the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix} \text{ , } B = \begin{pmatrix} 3 & 0 \\ 5 & 2 \end{pmatrix}$$

Verified that $(AB)^{-1} = B^{-1}A^{-1}$.

Exercice9
Let A be an invertible square matrix such that

$$A^2 - 3A + I = 0$$

Show that

$$A^{-1} = 3I - A, \text{ where } I = \begin{pmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & 1 \end{pmatrix}$$

Exercise10
Let A be a square matrix such that $A^4 = 0$:
Show that

$$(I - A)^{-1} = A^3 + A^2 + A + I \text{ , where } I = \begin{pmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & 1 \end{pmatrix}$$

Exercise11
Let the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

1- Find two elementary matrices E_1 and E_2 such that $E_1E_2A = I_2$.

2- Write A^{-1} and A as the product of two elementary matrices

Exercise12
Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 0 \end{pmatrix}$$

Show that

$$A^3 - 2A^2 + A + 2I = 0$$

Deduce A^{-1}.

Exrecise13
Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{pmatrix}$$

Show that

$$A^3 - 26A^2 + 11A + 10I =$$

Deduce A^{-1}.

Exercise14
Calculate the associated matrix relative to the standards bases of each of the following linear maps :

1-

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x, y, z) \to (x + y, 2z)$$

2-

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x, y, z) \to (x + y + z, x + y, x)$$

3-

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x, y, z) \to (x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z)$$

Exercise15
Let $B_1 = \{(1,1,0); (0,1,0); (1,0,1)\}$ and $B_2 = \{(0,1); (1,0)\}$ be two bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Give in these bases, the associated matrix of the linear map :

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x,y,z) \to (x+y, y-z)$$

Exercise16

Let E be a vector space of dimension 4 provided with the basis $B = \{e_1, e_2, e_3, e_4\}$. We define the linear map f of E into itself by setting :

$$f(e_1) = e_1 + e_2 \ , \ f(e_2) = 2e_2 + e_3 \ , \ f(e_3) = 3e_3 + e_4 \ , \ f(e_4) = e_4$$

1- Give the associated matrix of f relative to the basis B.

2- Give the associated matrix of f starting from the basis B and arriving to the basis B' such that

$$B' = \{e_1 + e_2, 2e_2 + e_3, 3e_3 + e_4, e_4\}.$$

Exercise17

Let f be a linear map of E into itself. Suppose that the associated matrix of f in the basis $B = \{e_1, e_2, e_3\}$ is as follows :

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

Let the basis $B' = \{f_1, f_2, f_3\}$ such that

$$f_1 = e_2 + e_3 \ , \ f_2 = e_1 + e_3 \ , \ f_3 = e_1 + e_2$$

Find the matrix $M(f, B')$.

Exercise18

Let A be the associated matrix of the linear map $f: \mathbb{R}^4 \to \mathbb{R}^3$ relative to the standard bases.

$$A = \begin{pmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

1- Find $rg(A)$.

2- Find $Ker(f)$ and $Im(f)$, give a base for each of these vector subspace, specify the dimension

Exercise19

Let the linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$ whose associated matrix relative to the standard basis of $\mathbb{R}^3$ is given by :

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

1- Calculate $f(1,2,3)$ and determine $Kerf$

2- Prove that $Im\, f = \{X = (x, y, z) \in R^3 : f(X) = X\}$.

Exercise20
Calculate the rank of the following matrices :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 3 & 2 & 1 \\ -1 & -2 & 4 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 5 & 3 & 7 \\ 3 & 3 & 0 & -6 \end{pmatrix},\ D = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \\ 3 & 1 & 7 \\ 4 & 8 & 12 \\ 1 & 2 & 3 \end{pmatrix},\ F = \begin{pmatrix} -3 & -1 & 2 & 3 & 1 \\ -1 & 2 & 3 & 1 & -2 \\ 2 & 1 & 12 & -9 & 0 \end{pmatrix},\ G = \begin{pmatrix} 2 & -5 & 1 & 4 \\ 3 & 2 & 6 & 5 \\ -1 & 12 & 4 & -3 \\ 11 & -37 & 1 & 23 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix},\ J = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},\ K = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 4 \end{pmatrix}$$

Exercise21
Calculate the rank of the matrix

$$A = \begin{pmatrix} -2 & 2 \\ 3 & 3 \\ 4 & 4 \end{pmatrix}$$

Calculate the rank of the $A^t A$.

Exercise22
Discuss the rank of the following matrices :

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}, \alpha \in \mathbb{R} \quad ; \quad B = \begin{pmatrix} m-2 & 2 & -1 \\ 2 & m & 2 \\ m & 2m+2 & m+1 \end{pmatrix}, m \in \mathbb{R}$$

$$C = \begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix},\ a, b \in \mathbb{R}$$

Exercice23
If A is a square matrix such that $A^2 - A + I = 0$, show that A is invertible and find its inverse

Exercise24
If A is a square matrix of order n such that $(A - kI)^2 = 0$. Show that A is invertible if and only if $k \neq 0, k \in \mathbb{R}$.

SOLUTIONS

Exercise1

$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $E = (1 \quad 1 \quad 1)$

1 – Calculation of the following matrix $A + B, 2A, 4B, -4A + 2B$.

$$A + B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -6 & 1 \\ 0 & 9 & 2 \\ -3 & 6 & 5 \end{pmatrix}$$

$$2A = 2\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 10 & 0 \\ -6 & 6 & 4 \end{pmatrix}$$

$$4B = 4\begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -20 & 4 \\ 0 & 16 & 8 \\ 0 & 12 & 12 \end{pmatrix}$$

$$-4A + 2B = -4\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 4 & 0 \\ 0 & -20 & 0 \\ 12 & -12 & -8 \end{pmatrix} + \begin{pmatrix} 2 & -10 & 2 \\ 0 & 8 & 4 \\ 0 & 6 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -6 & 2 \\ 0 & -12 & 4 \\ 12 & -6 & -2 \end{pmatrix}$$

2 – Calculation of the following matrix AB, BA , AC^t,BC^t.

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -9 & -1 \\ 0 & 20 & 10 \\ -3 & 33 & 9 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -23 & 2 \\ -6 & 26 & 4 \\ -9 & 24 & 6 \end{pmatrix}$$

$$AC^t = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 5 \\ -9 & 11 \end{pmatrix}$$

$$BC^t = \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -6 & 12 \\ -9 & 15 \end{pmatrix}$$

3 – Check that $(A+B)C^t = AC^t + BC^t$, $(AB)^t = B^tA^t$, $AC^t = CA^t$

$$(A+B)C^t = \left(\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -6 & 1 \\ 0 & 9 & 2 \\ -3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ -6 & 17 \\ -18 & 26 \end{pmatrix}$$

$$AC^t + BC^t = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ -3 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -5 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 5 \\ -9 & 11 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ -6 & 12 \\ -9 & 15 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -6 & 17 \\ -18 & 26 \end{pmatrix}$$

So we have good

$$(A+B)C^t = AC^t + BC^t$$

$$(AB)^t = \left(\begin{pmatrix}1 & -1 & 0\\ 0 & 5 & 0\\ -3 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & -5 & 1\\ 0 & 4 & 2\\ 0 & 3 & 3\end{pmatrix}\right)^t = \begin{pmatrix}1 & -9 & -1\\ 0 & 20 & 10\\ -3 & 33 & 9\end{pmatrix}^t = \begin{pmatrix}1 & 0 & -3\\ -9 & 20 & 33\\ -1 & 10 & 9\end{pmatrix}$$

$$B^tA^t = \begin{pmatrix}1 & 0 & 0\\ -5 & 4 & 3\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & -3\\ -1 & 5 & 3\\ 0 & 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0 & -3\\ -9 & 20 & 33\\ -1 & 10 & 9\end{pmatrix}$$

$$(AB)^t = B^tA^t$$

$$(AC^t) = \left(\begin{pmatrix}1 & -1 & 0\\ 0 & 5 & 0\\ -3 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ -3 & 4\end{pmatrix}\right)^t = \begin{pmatrix}1 & -1\\ 0 & 5\\ -9 & 11\end{pmatrix}^t = \begin{pmatrix}1 & 0 & -9\\ -1 & 5 & 11\end{pmatrix}$$

$$CA^t = \begin{pmatrix}1 & 0 & -3\\ 0 & 1 & 4\end{pmatrix}\begin{pmatrix}1 & 0 & -3\\ -1 & 5 & 3\\ 0 & 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0 & -9\\ -1 & 5 & 11\end{pmatrix}$$

$$AC^t = CA^t$$

4 – We calculate :

$$DE = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\ ,\ ED = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = (1+1+1) = (3)$$

Exercise2

Let the following matrices :

$$A = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\ ,\ I_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

1 – We find the matrix B such as $B = A - I_3$ and then we calculate B^n for $n \in \mathbb{N}$

$$B = A - I_3 = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$B^2 = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$B^3 = B^2.B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ et } B^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \forall n \geq 3$$

1- We will calculate A^n for $n\epsilon\mathbb{N}$.

According to Newton’s binomial, we have

$$A^n = (B+I)^n = \sum_{k=0}^{n} C_n^k\, B^k I^{n-k} = \sum_{k=0}^{n} C_n^k \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{n-k}$$

$$= C_n^0 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n + C_n^1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{n-1}$$

$$+ C_n^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{n-2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{n(n-1)}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise3

Let the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We show by recurrence that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Set the property $p(n)$: $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$p(0)$: $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is true for $n = 0$

$p(1)$: $A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$ is true for $n = 1$

$p(2): A^2 = A.A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is true for $n = 2$

We assume that $p(n): A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is true and show that

$$p(n+1): A^{n+1} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix} \text{ is true ?}$$

$$p(n+1): A^{n+1} = A^n.A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$$

So $\forall n \in \mathbb{N}: p(n)$ is true.

Exercice4

Let the matrix

$$A = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$$

We calculate A^{3n} for $n \epsilon \mathbb{N}$.

$$A^2 = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & ac & 0 \\ 0 & 0 & ab \\ bc & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} 0 & ac & 0 \\ 0 & 0 & ab \\ bc & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} abc & 0 & 0 \\ 0 & abc & 0 \\ 0 & 0 & abc \end{pmatrix} = abc\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= abc.I$$

$$A^{3n} = (A^3)^n = (abc.I)^n = (abc)^n.I = (abc)^n\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} (abc)^n & 0 & 0 \\ 0 & (abc)^n & 0 \\ 0 & 0 & (abc)^n \end{pmatrix}$$

Exercise5

Let A and B be two square matrices of the same order. Let us find a necessary and sufficient condition to have

$$(A+B)(A-B) = A^2 - B^2$$

We have

$$(A+B)(A-B) = A^2 - B^2 \Leftrightarrow A^2 - AB + BA - B^2 = A^2 - B^2 \Leftrightarrow AB = BA$$

Exercise6

We consider for each real number θ, the matrix

$$A(\theta) = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$$

1 – We show that $A(\theta)A(\theta') = A(\theta + \theta')$.

$$A(\theta).A(\theta') = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}\begin{pmatrix} cos\,\theta' & -sin\,\theta' \\ sin\,\theta' & cos\,\theta' \end{pmatrix}$$

$$= \begin{pmatrix} cos\,\theta\, cos\,\theta' - sin\,\theta\, sin\,\theta' & -cos\,\theta\, sin\,\theta' - sin\,\theta\, cos\,\theta' \\ sin\,\theta\, cos\,\theta' + cos\,\theta\, sin\,\theta' & cos\,\theta\, cos\,\theta' - sin\,\theta\, sin\,\theta' \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta + \theta') & -sin(\theta + \theta') \\ sin(\theta + \theta') & cos(\theta + \theta') \end{pmatrix} = A(\theta + \theta')$$

2 -We calculate $\left(A(\theta)\right)^n$ for $n\epsilon\mathbb{N}$.

We have $A(\theta)A(\theta') = A(\theta + \theta')$, if we take $\theta = \theta'$, we will have

$$A(\theta)A(\theta) = \left(A(\theta)\right)^2 = A(\theta + \theta) = A(2\theta)$$

By recurrence, suppose that

$$\left(A(\theta)\right)^n = A(n\theta)$$

We show that

$$\left(A(\theta)\right)^{n+1} = A\left((n+1)\theta\right)$$

$$\left(A(\theta)\right)^{n+1} = \left(A(\theta)\right)^n A(\theta) = \begin{pmatrix} cos\,n\,\theta & -sin\,n\,\theta \\ sin\,n\,\theta & cos\,n\,\theta \end{pmatrix}\begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$$

$$= \begin{pmatrix} cos\,n\,\theta\, cos\,\theta - sin\,n\,\theta\, sin\,\theta & -cos\,n\,\theta\, sin\,\theta - sin\,n\,\theta\, cos\,\theta \\ sin\,n\,\theta\, cos\,\theta + cos\,n\,\theta\, sin\,\theta & cos\,n\,\theta\, cos\,\theta + sin\,n\,\theta\, sin\,\theta \end{pmatrix}$$

$$= \begin{pmatrix} cos(n+1)\,\theta & -sin(n+1)\,\theta \\ sin(n+1)\,\theta & cos(n+1)\,\theta \end{pmatrix} = A(n+1)\theta$$

So we have

$$\left(A(\theta)\right)^n = A(n\theta) = \begin{pmatrix} cosn\theta & -sinn\theta \\ sinn\theta & cosn\theta \end{pmatrix} \ , \ \forall n \in \mathbb{N}$$

Exercise7

Let the matrix

$$A(x) = \begin{pmatrix} chx & shx \\ shx & chx \end{pmatrix}$$

1 – We will show that

$$A(x).A(y) = A(x+y)$$

$$A(x)A(y) = \begin{pmatrix} chx & shx \\ shx & chx \end{pmatrix}\begin{pmatrix} chy & shy \\ shy & chy \end{pmatrix} = \begin{pmatrix} chxchy + shxshy & shxchy + shychx \\ shxchy + shychx & chxchy + shxshy \end{pmatrix}$$

$$= \begin{pmatrix} ch(x+y) & sh(x+y) \\ sh(x+y) & ch(x+y) \end{pmatrix} = A(x+y)$$

2 – We show by recurrence that

$$\left(A(x)\right)^n = A(nx)$$

In the equation

$A(x)A(y) = A(x+y)$, we pose $y = x$ we will have

$$\left(A(x)\right)^2 = A(2x)$$

Asume the property $\left(A(x)\right)^n = A(nx)$ is true and show that

$$\left(A(x)\right)^{n+1} = A((n+1)x) \text{ is true}$$

$$\left(A(x)\right)^{n+1} = (A(x))^n A(x) = \begin{pmatrix} chnx & shnx \\ shnx & chnx \end{pmatrix}\begin{pmatrix} chx & shx \\ shx & chx \end{pmatrix}$$

$$= \begin{pmatrix} chnxchx + shnxshx & shnxchx + chnxshx \\ shnxchx + chnxshx & chnxchx + shnxshx \end{pmatrix} = \begin{pmatrix} ch(n+1)x & sh(n+1)x \\ sh(n+1)x & ch(n+1)x \end{pmatrix}$$
$$= A(n+1)x$$

then

$$\left(A(x)\right)^n = A(nx), \forall n \in \mathbb{N}$$

Exercise8

We will check that $(AB)^{-1} = B^{-1}A^{-1}$

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{pmatrix} = \begin{pmatrix} 5 & -11 & -16 \\ 1 & 0 & -2 \\ 6 & -19 & -24 \end{pmatrix}$$

$$(AB)^{-1} = \begin{pmatrix} \frac{19}{9} & -\frac{20}{9} & -\frac{11}{9} \\ -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{19}{18} & -\frac{29}{18} & -\frac{11}{18} \end{pmatrix}$$

On the other hand, we have

$$B^{-1}A^{-1} = \frac{1}{18}\begin{pmatrix} 10 & -2 & -6 \\ -6 & 12 & 0 \\ 5 & -10 & -3 \end{pmatrix}\begin{pmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ -3 & 3 & 2 \end{pmatrix}$$
$$= \frac{1}{18}\begin{pmatrix} 38 & -40 & -22 \\ -12 & 24 & 6 \\ 19 & -29 & -11 \end{pmatrix} = \begin{pmatrix} \frac{19}{9} & -\frac{20}{9} & -\frac{11}{9} \\ -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{19}{18} & -\frac{29}{18} & -\frac{11}{18} \end{pmatrix}$$

Then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Exercise9

Let A be an invertible square matrix such that

$$A^2 - 3A + I = 0$$

Let's show that $A^{-1} = 3I - A$

Since A is invertible, then A^{-1} exists ($A^{-1}A = AA^{-1} = I$) and we have

$$A^2 - 3A + I = 0 \Leftrightarrow I = 3A - A^2 \Leftrightarrow A^{-1}I = A^{-1}(3A - A^2)$$

$$\Leftrightarrow A^{-1} = 3A^{-1}A - A^{-1}A^2$$

$$\Leftrightarrow A^{-1} = 3I - A$$

Exercise10

Let A be a square matrices such that $A^4 = 0$:

Show that

$$(I - A)^{-1} = A^3 + A^2 + A + I$$

Since $A^4 = 0$, we have

$$I = I - A^4 = (I - A)(I + A + A^2 + A^3)$$

$$\Leftrightarrow (I - A)^{-1}.I = (I - A)^{-1}(I - A)(I + A + A^2 + A^3).$$

$$\Leftrightarrow (I - A)^{-1} = I + A + A^2 + A^3$$

Exercice11

1 – We find two elementary matrices E_1 and E_2 such that $E_1E_2A = I_2$.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{l_2 \to l_2 - \frac{2}{3}l_1} \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{pmatrix} = E_1$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{l_2 \to \frac{1}{3}l_2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = E_2$$

$$E_1E_2A = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

2 – We write A^{-1} *and* A as the product of two elementary matrices

Since $E_1E_2A = I_2$ and E_1 , E_2 are two elementary matrices which also are invertibles.

Then we have

$$E_1E_2A = I_2 \Leftrightarrow A = {E_2}^{-1}{E_1}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{2}{3} & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 3\end{pmatrix}$$

$$A = {E_2}^{-1}{E_1}^{-1} \Leftrightarrow A^{-1} = E_1E_2 = \begin{pmatrix}1 & 0\\ -\dfrac{2}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \dfrac{1}{3}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\dfrac{2}{3} & \dfrac{1}{3}\end{pmatrix}$$

Exercise12

Let the matrix $A = \begin{pmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 2 & 2 & 0\end{pmatrix}$

We show that

$$A^3 - 2A^2 + A + 2I = 0$$

$$A^2 = \begin{pmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 2 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 2 & 2 & 0\end{pmatrix} = \begin{pmatrix}3 & 2 & 1\\ -4 & -1 & -2\\ 0 & 2 & 0\end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix}3 & 2 & 1\\ -4 & -1 & -2\\ 0 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 2 & 2 & 0\end{pmatrix} = \begin{pmatrix}3 & 4 & 1\\ -7 & -5 & -3\\ -2 & 2 & -2\end{pmatrix}$$

Then

$$A^3 - 2A^2 + A + 2I = \begin{pmatrix}3 & 4 & 1\\ -7 & -5 & -3\\ -2 & 2 & -2\end{pmatrix} + \begin{pmatrix}-6 & -4 & -2\\ 8 & 2 & 4\\ 0 & -4 & 0\end{pmatrix}$$
$$+ \begin{pmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 2 & 2 & 0\end{pmatrix} + \begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Déduction of A^{-1}

$$A^3 - 2A^2 + A + 2I = 0 \Leftrightarrow 2I = -A^3 + 2A^2 - A$$

$$\Leftrightarrow I = \tfrac{1}{2}(-A^3 + 2A^2 - A)$$

$$\Leftrightarrow A^{-1} = \tfrac{1}{2}(-A^2 + 2A - I)$$

$$\Leftrightarrow A^{-1} = \frac{1}{2}\left(\begin{pmatrix}-3 & -2 & -1\\ 4 & 1 & 2\\ 0 & -2 & 0\end{pmatrix} + \begin{pmatrix}2 & 0 & 2\\ -2 & 2 & -2\\ 4 & 4 & 0\end{pmatrix} + \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\right)$$

$$\Leftrightarrow A^{-1} = \begin{pmatrix} -1 & -1 & \frac{1}{2} \\ 1 & 1 & 0 \\ 2 & 1 & -\frac{1}{2} \end{pmatrix}$$

Exercice14

1-

$$f: \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \to (x + y, 2z)$$

Let B and C the standard bases respectively of $\mathbb{R}^3$ and $\mathbb{R}^2$.
We have

$$\begin{cases} f(1,0,0) = (1,0) \\ f(0,1,0) = (1,0) \\ f(0,0,1) = (0,2) \end{cases}$$

So the associted matrix of this linear map relative to the standard bases B and C is as follows :

$$M(f, B, C) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

2-

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \to (x + y + z, x + y, x)$$

We have

$$\begin{cases} f(1,0,0) = (1,1,1) \\ f(0,1,0) = (1,1,0) \\ f(0,0,1) = (1,0,0) \end{cases}$$

So the associted matrix of this linear map relative to the standard basis is as follows :

$$M(f, B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3-

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \to (x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z)$$

$$\begin{cases} f(1,0,0) = (1,4,7) \\ f(0,1,0) = (2,5,8) \\ f(0,0,1) = (3,6,9) \end{cases}$$

$$M(f, B) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Exercise15

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \to (x+y, y-z)$$

$$f(1,1,0) = (2,1) = \alpha(0,1) + \beta(1,0) \Leftrightarrow \begin{cases} \beta = 2 \\ \alpha = 1 \end{cases}$$

$$f(0,1,0) = (1,1) = \alpha(0,1) + \beta(1,0) \Leftrightarrow \begin{cases} \beta = 1 \\ \alpha = 1 \end{cases}$$

$$f(1,0,1) = (1,-1) = \alpha(0,1) + \beta(1,0) \Leftrightarrow \begin{cases} \beta = 1 \\ \alpha = -1 \end{cases}$$

So the associated matrix of this linear map relativeto the bases B_1 *and* B_2 is

$$M(f, B_1, B_2) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

Exercice16

1- We give the associted matrix of f relative to the basis B.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

2- We give the associated matrix of f starting from the basis B and arriving to the basis B'.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercice17

Let f be a linear map of E into itself. Suppose that the associated matrix of f relative to the basis $B = \{e_1, e_2, e_3\}$ is as follows :

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

Let the base $B' = \{f_1, f_2, f_3\}$ such that

$$f_1 = e_2 + e_3 \ , \ f_2 = e_1 + e_3 \ , \ f_3 = e_1 + e_2$$

We find the matrix $M(f, B')$.

Let $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ the transition matrix from B to base B'.

Then we have

$$M(f,B')=P^{-1}AP=\frac{1}{2}\begin{pmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{pmatrix}\begin{pmatrix}3&1&-1\\2&2&-2\\1&-1&1\end{pmatrix}\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{pmatrix}\begin{pmatrix}0&2&4\\0&0&4\\0&2&0\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}0&0&0\\0&4&0\\0&0&8\end{pmatrix}=\begin{pmatrix}0&0&0\\0&2&0\\0&0&4\end{pmatrix}$$

Exercice18

Let A be the associated matrix of the linear map f such that :

$$A=\begin{pmatrix}0&-1&0&3\\0&0&0&0\\2&0&1&0\end{pmatrix}$$

1- We find $rg(A)$.

$$A=\begin{pmatrix}0&-1&0&3\\0&0&0&0\\2&0&1&0\end{pmatrix}\begin{matrix}c_1\to c_1-2c_3\\c_4\to c_4+3c_2\\\to\end{matrix}\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}.$$

Then $rg(A)=2$

2- We find $Ker(f)$ and $Im(f)$.

Let f be the linear application whose associated matrix is A.

$$\begin{aligned}f:\mathbb{R}^4&\to\mathbb{R}^3\\(x,y,z,t)&\to(-y+3t,0,2x+z)\end{aligned}$$

$$Kerf=\{(x,y,z,t):f(x,y,z,t)=0\}$$

$$=\{(x,y,z,t):y=3t,z=-2x\}$$

$$=\{(x,3t,-2x,t):x,t\in\mathbb{R}\}$$

$$=\{x(1,0,-2,0)+t(0,3,0,1):x,t\in\mathbb{R}\}$$

So $B=\{(1,0,-2,0);(0,3,0,1)\}$ is a basis of $Ker(f)$ and $dim(Kerf)=2$

$$Im\,f=\{(X,Y,Z):f(x,y,z,t)=(X,Y,Z)\}$$

$$=\{(X,0,Z):X,Z\in\mathbb{R}\}$$

$$=\{(X,0,0)+(0,0,Z):X,Z\in\mathbb{R}\}$$

$$= \{X(1,0,0) + Z(0,0,1): X, Z \in \mathbb{R}\}$$

Then $B' = \{(1,0,0); (0,0,1)\}$ is a basis of $Im\, f$ and $dim(Im\, f) = 2$.

Exercice19

1- We calculate $f(1,2,3)$ and determine $Kerf$.

$$\begin{gathered} f: \mathbb{R}^3 \to \mathbb{R}^3 \\ (x, y, z) \to (x, x, x - y + z) \end{gathered}$$

So $f(1,2,3) = (1,1,2)$

$$Kerf = \{(x, y, z) \in R^3: f(x, y, z) = (0,0,0)\}$$

$$= \{(x, y, z) \in R^3: x = 0, y = z\}$$

2- We prove that $Im\, f = \{X = (x, y, z) \in R^3: f(X) = X\}$.

By definition, we have

$$Im\, f = \{(a, b, c) \in R^3: f(x, y, z) = (a, b, c)\}$$

$$= \{(a, b, c) \in R^3: a = x, b = x, c = x - y + z\}$$

$$= \{(a, a, c) \in R^3: a, c \in R\}$$

On the other hand, we have

$$\{(x, y, z) \in R^3: f(x, y, z) = (x, y, z)\} = \{(x, y, z) \in R^3: x = y\}$$

$$= \{(x, x, z) \in R^3: x, z \in R\}$$

Then we have

$$Im\, f = \{X = (x, y, z) \in R^3: f(X) = X\}.$$

Exercice20

Find the rank of the matrix A, B, C, D and E

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{matrix} l_2 \to l_2 - 2l_1 \\ l_3 \to l_3 - 3l_1 \\ \to \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & -5 & -8 \end{pmatrix} \begin{matrix} l_3 \to l_3 - 5l_2 \\ \to \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 12 \end{pmatrix}$$

$$rg(A) = 3$$

$$B=\begin{pmatrix}1&0&1&0\\-1&2&0&0\\1&3&2&1\\-1&-2&4&0\end{pmatrix}\begin{matrix}l_2\rightarrow l_2+l_1\\l_3\rightarrow l_3-l_1\\l_4\rightarrow l_4+l_1\\\rightarrow\end{matrix}\begin{pmatrix}1&0&1&0\\0&2&1&0\\0&3&1&1\\0&-2&5&0\end{pmatrix}$$

$$\begin{matrix}l_3\rightarrow 2l_3-3l_2\\l_4\rightarrow l_4+l_2\\\rightarrow\end{matrix}\begin{pmatrix}1&0&1&0\\0&2&1&0\\0&0&-1&2\\0&0&6&0\end{pmatrix}\begin{matrix}l_4\rightarrow l_4+6l_3\\\rightarrow\end{matrix}\begin{pmatrix}1&0&1&0\\0&2&1&0\\0&0&-1&0\\0&0&0&12\end{pmatrix}$$

$$rg(B)=4$$

$$C=\begin{pmatrix}1&1&0&-2\\0&1&-2&1\\0&5&3&7\\3&3&0&-6\end{pmatrix}\begin{matrix}l_4\rightarrow l_4-3l_1\\\rightarrow\end{matrix}\begin{pmatrix}1&1&0&-2\\0&1&-2&1\\0&5&3&7\\0&0&0&0\end{pmatrix}$$

$$\begin{matrix}l_3\rightarrow l_3-5l_2\\\rightarrow\end{matrix}\begin{pmatrix}1&1&0&-2\\0&1&-2&1\\0&0&13&2\\0&0&0&0\end{pmatrix}$$

$$rg(C)=3$$

$$D=\begin{pmatrix}1&3\\2&4\\2&4\end{pmatrix}\begin{matrix}l_2\rightarrow l_2-2l_1\\l_3\rightarrow l_3-2l_1\\\rightarrow\end{matrix}\begin{pmatrix}1&3\\0&-2\\0&-2\end{pmatrix}\begin{matrix}l_3\rightarrow l_3-l_2\\\rightarrow\end{matrix}\begin{pmatrix}1&3\\0&-2\\0&0\end{pmatrix}$$

$$rg(D)=2$$

$$E=\begin{pmatrix}2&4&6\\-1&-2&-3\\3&1&7\\4&8&12\\1&2&3\end{pmatrix}\begin{matrix}l_2\rightarrow 2l_2+l_1\\l_3\rightarrow 2l_3-3l_1\\l_4\rightarrow l_4-2l_1\\l_5\rightarrow 2l_5-l_1\end{matrix}\begin{pmatrix}2&4&6\\0&0&0\\0&-10&-4\\0&0&0\\0&0&0\end{pmatrix}$$

$$rg(E)=2$$

Exercice21

1- Calculate the rank of the matrix A

$$A=\begin{pmatrix}-2&2\\3&3\\4&4\end{pmatrix}\begin{matrix}l_2\rightarrow 2l_2+3l_1\\l_3\rightarrow l_3+2l_1\\\rightarrow\end{matrix}\begin{pmatrix}-2&2\\0&12\\0&8\end{pmatrix}\begin{matrix}l_3\rightarrow 3l_3-2l_2\\\rightarrow\end{matrix}\begin{pmatrix}-2&3\\0&12\\0&0\end{pmatrix}$$

So the rank of the matrix A is $rg(A)=2$

2- Calculate the matrix A^tA and his rank

$$A^tA=\begin{pmatrix}-2&3&4\\2&3&4\end{pmatrix}\begin{pmatrix}-2&2\\3&3\\4&4\end{pmatrix}=\begin{pmatrix}29&21\\21&29\end{pmatrix}$$

$$rg(A^tA)=2$$

Exercice22

Discuss the rank of the following matrix

$$A=\begin{pmatrix}\alpha&1&1\\1&\alpha&1\\1&1&\alpha\end{pmatrix},\alpha\in\mathbb{R}\quad;\quad B=\begin{pmatrix}m-2&2&-1\\2&m&2\\m&2m+2&m+1\end{pmatrix},m\in\mathbb{R}$$

$$C=\begin{pmatrix}a&0&b\\b&a&0\\0&b&a\end{pmatrix},\ a,b\in\mathbb{R}$$

Let the square matrix $A=\begin{pmatrix}\alpha&1&1\\1&\alpha&1\\1&1&\alpha\end{pmatrix}$

$\alpha\neq 0,\alpha\neq -1$

$$A=\begin{pmatrix}\alpha&1&1\\1&\alpha&1\\1&1&\alpha\end{pmatrix}\xrightarrow{\substack{l_2\to\alpha l_2-l_1\\ l_3\to\alpha l_3-l_1}}\begin{pmatrix}\alpha&1&1\\0&\alpha^2-1&\alpha-1\\0&\alpha-1&\alpha^2-1\end{pmatrix}$$

$$\xrightarrow{l_3\to(\alpha+1)l_3-l_2}\begin{pmatrix}\alpha&1&1\\0&\alpha^2-1&\alpha-1\\0&0&\alpha(\alpha-1)(\alpha+2)\end{pmatrix}$$

If $\alpha=1\Rightarrow rg(A)=1$

If $\alpha=-2\Rightarrow rg(A)=2$

If $\alpha\neq 1,\alpha\neq -2\Rightarrow rg(A)=3$

$\alpha=0$

$$A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\xrightarrow{l_3\to l_3-l_2}\begin{pmatrix}0&1&1\\1&0&1\\0&1&-1\end{pmatrix}\xrightarrow{l_3\to l_3+l_1}\begin{pmatrix}0&1&1\\1&0&1\\0&2&0\end{pmatrix}$$

$$rg(A)=3$$

$\alpha=-1$

$$A=\begin{pmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{pmatrix}\xrightarrow{\substack{l_2\to l_2+l_1\\ l_3\to l_3+l_1}}\begin{pmatrix}-1&1&1\\0&0&2\\0&2&0\end{pmatrix}$$

$$\xrightarrow{l_1\to 2l_1-l_2}\begin{pmatrix}-2&2&0\\0&0&2\\0&2&0\end{pmatrix}\xrightarrow{l_1\to l_1-l_3}\begin{pmatrix}-2&0&0\\0&0&2\\0&2&0\end{pmatrix}$$

$$rg(A)=3$$

Let the matrix $B=\begin{pmatrix}m-2&2&-1\\2&m&2\\2m&2m+2&m+1\end{pmatrix}$

$m=0$

$$B=\begin{pmatrix}-2&2&-1\\2&0&2\\0&2&1\end{pmatrix}\xrightarrow{l_2\to l_2+l_1}\begin{pmatrix}-2&2&-1\\0&2&1\\0&2&1\end{pmatrix}\xrightarrow{l_3\to l_3-l_2}\begin{pmatrix}-2&2&-1\\0&2&1\\0&0&0\end{pmatrix}$$

$$rg(B)=2$$

$m=1$

$$B=\begin{pmatrix}-1&2&-1\\2&1&2\\2&4&2\end{pmatrix}\xrightarrow[l_3\to l_3+2l_1]{l_2\to l_2+2l_1}\begin{pmatrix}-1&2&-1\\0&5&0\\0&8&0\end{pmatrix}\xrightarrow{l_3\to 5l_3-8l_2}\begin{pmatrix}-1&2&-1\\0&5&0\\0&0&0\end{pmatrix}$$

$$rg(B)=2$$

$m=2$

$$B=\begin{pmatrix}0&2&-1\\2&2&2\\4&6&3\end{pmatrix}\xrightarrow{l_2\to 2l_2-l_3}\begin{pmatrix}0&2&-1\\0&-2&1\\4&6&3\end{pmatrix}\xrightarrow{l_1\to l_1+l_2}\begin{pmatrix}0&0&0\\0&-2&1\\4&6&3\end{pmatrix}$$

$$rg(B)=2$$

If $m\neq 0, m\neq 1, m\neq 2,$ then $rg(B)=3$

Exercice23

If A is a square matrix such as $A^2-A+I=0$, show that A is invertible and find this inverse ?

$$A^2-A+I=0\Leftrightarrow I=A-A^2=A(I-A)$$

Then the matrix A is invertible and its inverse is as follows :

$$A^{-1}=I-A$$

Exercise 24

If A is a square matrix of order n such that $(A-kI)^2=0$. Show that A is invertible if and only if $k\neq 0, k\in\mathbb{R}$.

$$(A-kI)^2=(A-kI)(A-kI)=A^2-2kA+k^2I=0$$

$$\Leftrightarrow I=\frac{1}{k^2}A(2kI-A)=A\left[\frac{1}{k^2}(2kI-A)\right]$$

Then the matrix A is invertible if and only if $k\neq 0\ et\ k\epsilon\mathbb{R}$ and its inverse is as follow

$$A^{-1}=\frac{1}{k^2}(2kI-A)$$

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