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Abstract

In this thesis , we prove the existence and uniqueness of integrable solutions for boundary value problems for implicit fractional differential equation and inclusions with fractional derivative in the sense of Caputo and Caputo-Hardamard with non-local condition and integral boundary conditions. Our results are obtained by means of fixed point theorems (Schauder, Banach contraction principle, Nonlinear alternative of Leray-Schauder and fixed point theorem for contraction multivalued type maps due to Covitz and Nadler.)
Key words and phrases: Initial value problem, boundary value problem, existence of solutions , Caputo fractional derivative, Hardamard-Caputo fractional derivative, implicit fractional order differential equation, integrable solutions,convex, non convex, Banach space, fixed point ,non-local conditions , integral conditions, implicit fractional differential inclusions , fractional differential inclusions .

Résumé

Dans cette thèse, nous avons prouvé l'existence et l'unicité des solutions intégrales pour des problèmes aux limites concernant les équations et inclusions différentielles fractionnaires implicites avec des dérivées fractionnaires au sens de Caputo et Caputo-hardmard avec des conditions aux limites non locales et intégrales.
Pour cela, on a utilisé quelques théorèmes de point fixe tels que Banach , Schauder, Alternative non linèaire de Leray Schauder et le théorème de Covitz et Nadler.
Mots et phrases clés: problème à valeur initiale, problèmes aux limite, existence des solutions, dérivée fractionnaire de Caputo, dérivée fractionnaire de Hadamard-Caputo, équations différentielles fractionnaire implicite,solutions intégrables, convexe, non convexe, espace de Banach, point fixe, conditions non locales, Conditions intégrales, inclusions différentielles fractionnaires implicites, inclusions différentielles fractionnaires.

الملخص

في هذه الأطروحة ، أثبتنا وجود وتفرد الحلول والحلول القابلة للتكامل لمشاكل قيمة الحدود للمعادلات والاحتواءات التفاضليةالكسرية الضمنية ذات المشتقات الكسرية بمعنى كابوتو وكابوتوهاردمارد مع شروط ابتدائية و شروط غير المحلية و شروط الحدود المتكاملة . سيتم الحصول على نتائجنا عن طريق استخدام نظريات النقاط الثابتة (شودر ، مبدأ انكماش باناخ ، ليراي ـ شودر و ومبدا الإنكماش لكوفيتدز و نادل.)

الكلمات والعبارات المفتاحية : مشكلة القيمة الاولية، مشكلة القيم المحدودة ، وجود الحلول ،مشتقة الكسرية بمعنى كابوتو، مشتقة الكسرية بمعنى كابوتوهادمارد محدب وغير محدب ، الفضاء باناخ ، النقطة الثابتة ، شروط ابتدائية ، شروط غير محلية ، شروط متكاملة ، المعادلات التفاضلية الكسرية الضمنية ، الإحتواءات التفاضلية الكسرية والضمنية

Contents

Introduction

The story of the derivative of non-integer order is one of the most beautiful adventures of the human spirit in which several generations of mathematicians have engaged and physicists. It extends from the 17th century to the present day. The number of publications and scientific meetings in the recent period dedicated to him testifies to the importance of the problems that this concept has raised, both theoretical that applied. It can be said that it has become a full-fledged discipline. Specialist agree to trace the beginning of this story back to the the year 1695 when Leibniz, in a letter[77]to the Guillaume Hôspital, wanted to initiate a discussion on a possible theory of the non-integer derivation of a function. In its response, the Hospital questioned what if we put $n = \frac{1}{2}$ in the formulae $\dfrac{d^n y}{dx^n}$?!.Gottfried Leibniz's answer contained roughly this sentence : "...this would lead to a paradox from which, one day, we can draw useful consequences". It took until the 1990s to see the first " consequences useful.

The field of differential calculation and fractional integration has evolved significantly in recent years .Fractional order equations have become valuable tools in mathematical modeling of many phenomena in various fields of science and engineering in reality .There are many applications in elastic viscosity, electrochemistry, porous media, electromagnetic, control, porous media etc.In the monographs of Hilfer [67], Kilbas et al. [73], Podlubny [84], Momani et al. [79], Samko *et al.* [85], Delbosco and Rodino [45], Diethelm *et al.* [49],[50],[48],

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, including some applications and recent results, see the monographs of Kilbas et al. [73],Polubny [82], and the papers of Agarwal *et al.* [7], Momani *et al.*[79], Guerraiche et *al.* [59], and the references therein.

There are many researchers interested in the study of existence solutions and the uniqueness of solutions of initial and Boundary value problems of fractional differential equations (see Agarwal *et al.*[6], Ahmad *et al.*[11], Ahmad *et al.* [12], Benchohra *et al.*[18, 19, 20], Graef *et al.*[57],Guerraiche *et al.*[60], Houas *et al.*[66], Kilbas[73]).

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year ; see for instance [3] and the references therein. [25, 26, 27]

Nonlocal conditions were initiated by Byszewski [37] for evolution equations when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [38], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamics [31] and cellular systems.

The Caputo fractional derivative is very useful in many applied problems, because it saisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as the same data for boundary conditions.

The Caputo-Hadamard fractional derivative given by Jarad et al [70] is a modifi - cated Hadamard fractional derivative, but unlike the Hadamard fractional derivative, the Caputo-Hadamard fractional derivative of a constant is 0, and this was inherited from the Caputo's derivative. Recently, many researchers studied different fractional problems involving the Caputo Hadamrd derivative; see, for example, the papers of W. Shammack [86]and Y. Adjabi et al [5].

The previous study we relied on is the thesis[61]and the thesis[89]
In the thesis[89], the authors study the existence of solutions for many problems of implicit functional fractional differential equations with Caputo Fractional derivative.

In this thesis,we investigate the existence of solutions for many problems of implicit fractional differential equations and inclusions with Caputo and Caputo-Hadamard Fractional derivative, we use several fixed point theorems.
as follows:
In the first chapter, we give some notations and definitions concerned the fractional calculus and the set-valued maps also we recall some fixed point theorems. In the first section of this chapter we give some notations. The second section is devoted to the fractional calculus. In the third section we shall be concerned by the set-valued maps theory and definitions for Implicit deferential equation and inclusion. In the last section we recall some fixed point theorems

In the second chapter, we present our first main result. In the first section, we study the existence of integrable solutions for implicit fractional differential equations with the Caputo-Hadamard fractional derivative of order $\alpha \in (1, 2]$. Our results are based on Schauder's fixed point theorem and the Banach fixed point theorem. We give an example

to illustrate our main results.
In the second section , we study the existence of integrable solutions of the problem for implicit fractional differential equations with the Caputo- Hadamard fractional derivative order $\alpha \in (0, 1]$ with nonlocal condition . Our results are based on Schauder fixed point theorem and the Banach contraction principle. In the we give an example to illustrate our main results.

In the third chapter, in its first section , we study boundary value problems for implicit fractional differential inclusions with Caputo fractional derivative order $\alpha \in (0, 1]$ and nonlocal condition, In the convex case, our approach here is based upon the nonlinear alternative of Leray- Schauder.
In the nonconvex case, here our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. We give an example to illustrate our main results.
In the second section , we study problems for implicit fractional differential inclusion with Hadamard-Caputo fractional derivative order $\alpha \in (0, 1]$ with nonlocal condition. In the convexe case, our approach here is based fixed point theorem of Bohnnenblust-Karlin. In the the nonconvex case, here our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. We give an example to illustrate our main results

In the fourth chapter, we present our main result. In the first section, we study boundary value problems for fractional differential inclusion with nonlocal condition and Hadamard-Caputo fractional derivative of order $r \in (1, 2]$. In the convexe case, our approach here is based upon the nonlinear alternative of Leray- Schauder. In the the nonconvex case, here our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. We give an example to illustrate our main results.
In the second section, we study nonlinear boundary value problems for fractional differential inclusions with Integral Conditions and Hadamard-Caputo fractional derivative of order $r \in (1, 2]$. In the convex case, our approach here is based upon the nonlinear alternative of Leray- Schauder. In the nonconvex case, here our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. We give an example to illustrate our main results

Chapter 1

Preliminaries

1.1 Notations and definitions

Let $C(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

$L^1(J, \mathbb{R})$ denote the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

$AC(J, \mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous.
Let $\delta = t\dfrac{d}{dt}$, We set

$$AC^n_\delta(J, \mathbb{R}) = \{y : J \to \mathbb{R} : \delta^{n-1}h(t) \in AC(J, \mathbb{R})\},$$

And
$AC^1(J, \mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$ which are derivable and have a continuous first derivative.

Definition 1.1 *[44] A map $f : J \times \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{R}$ is said to be L^1 Carathéodory if*

(1) the map $t \longmapsto f(t, x, y)$ is measurable for each $(x, y) \in \mathbb{R} \times \mathbb{R}$

(2) the map $(x, y) \longmapsto f(t, x, y)$ continuous for almost all $t \in J$

(3) For each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R})$ such that

$$|f(t, x, y)| \leq h_q(t)$$

for $|x| \leq q$, $|y| \leq q$ and for a.e. $t \in J$

Definition 1.2 *([89]) An operator $T : E \longrightarrow E$ is called compact if the image of each bounded set $B \in E$ is relatively compact i,e($\overline{T(B)}$) is compact .T is called completetly continuous operator if is continuous and compact.*

Theorem 1.3 *(Kolmogorov compactness criterion [30]). Let $\Omega \subseteq L^p(J, \mathbb{R}), 1 \leq p \leq +\infty$.. If*

(i) Ω is bounded in $L^p(J, \mathbb{R})$ and

(ii) $u_h \to u$ as $h \to 0$, uniformly with respect to $u \in \Omega$, then Ω is relatively compact in $L^p(J, \mathbb{R})$, where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

1.2 Some properties of fractional calculus

1.2.1 Introduction

Gottfried Wilhelm Leibniz was the first mathematician who gave a sense to the expression $\frac{d^n f}{dx^n}$ which means the derivative of order n of the real function f, such that n is integer. In 1695, G.W.Leibniz and the French mathematician Marquis De L'Hospital discussed the possibility to generalize the concept of the derivative of order n, given by leibniz, to the case when n is non integer. This great idea whose appears easy to realize! Because like its known the following expression

$$\frac{d^\alpha}{dx^\alpha} e^{\lambda x} = \lambda^\alpha e^{\lambda x} \tag{1.1}$$

is valid when α is integer but it is not the case when α is not integer, this is called the Leibniz' paradox.
By defining and using his Gamma function Γ, which is a generalization of the product $1.2...n = n!$, Euler was able to introduce the following fractional derivative

$$\frac{d^\alpha}{dx^\alpha} x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha, \beta \in \mathbb{Q}. \tag{1.2}$$

Unfortunately, this don't resolve completely the Leibniz' paradox. Indeed, by using this fractional derivative, we have

$$\frac{d^\alpha}{dx^\alpha} e^x = \frac{d^\alpha}{dx^\alpha} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{k!\Gamma(1-\alpha+\beta)} x^{k-\alpha} \neq e^x \tag{1.3}$$

Motivating by Leibniz' paradox many researchers (Fourier, Liouville, Rieman, Grunwald, Letnikov,...) who have contributed with different ideas, which developed the sciense of Fractional Calculus. Nowadays, we distinguish different fractional integrals and derivatives.

1.2.2 Fractional integral and derivative of Riemann-Liouville

As we know, the integration of order n (n is integer) of the function f is given by

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt \tag{1.4}$$

by using the function Γ, we can give to this formula a sens when n is non integer.

Definition 1.4 *([73], [85]) Let $h \in L^1([a,b],\mathbb{R})$. The left sided fractional integral of Riemann-Liouville of order α is defined by*

$$(I_a^\alpha h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,$$

where $\alpha > 0$. When $a = 0$, we write

$$I^\alpha h(t) = h(t) * \varphi_\alpha(t)$$

where

$$\varphi_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \textit{for } t > 0 \\ 0 & \textit{for } t \leq 0, \end{cases}$$

and

$$\varphi_\alpha \to \delta(t) \ \textit{as}\ \alpha \to 0$$

where δ is the delta function.

Example 1.5 *([85], section 2.5) Let $h(t) = (t-a)^\beta$ where $t > a$ and $\beta > -1$. Then we have*

$$(I_a^\alpha h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^\beta ds,$$

We set $y = \dfrac{s-a}{t-a}$ and

$$\begin{aligned} \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^\beta ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 (t(1-y)-a(1-y))^{\alpha-1}y^\beta(t-a)^\beta(t-a)dy \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 ((1-y)(t-a))^{\alpha-1}y^\beta(t-a)^\beta(t-a)dy \\ &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-y)^{\alpha-1}y^\beta dy \end{aligned}$$

Since

$$B(\alpha,\beta+1) = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},\ \textit{(B is the Beta function, see the Annex)}$$

we have

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^\beta ds = \frac{\Gamma(\beta+1)(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

Example 1.6 *Let $h(t) = \exp(\lambda t)$ where $\lambda > 0$. Then we have*

$$\begin{aligned} I_a^\alpha \exp(\lambda t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \exp(\lambda s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sum_{k\geq 0} \frac{(\lambda s)^k}{k!} ds \\ &= \sum_{k\geq 0} \frac{\lambda^k}{k!\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} s^k ds \end{aligned}$$

Now, by using the Example1.5, we have

$$I_a^\alpha \exp(\lambda t) = \lambda^{-\alpha} \sum_{k\geq 0} \frac{(\lambda t)^{\alpha+k}}{\Gamma(\alpha+k+1)}$$

Definition 1.7 *([73], [85]) Let h be a function given on the interval $[a,b]$. The left-handed fractional derivative of Riemann-Liouville of order α, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Theorem 1.8 *([85], section 2.3) Let $h \in C([a,b], \mathbb{R})$. For all $\alpha > 0$ and $\beta > 0$, we have*

$$I^\alpha I^\beta h = I^{\alpha+\beta} h$$

Proof: We have

$$I^\alpha I^\beta h = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha-1} ds \int_a^s (s-x)^{\beta-1} h(x) dx$$

By Fubini theorem, we have

$$\begin{aligned} I^\alpha I^\beta h &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^s (t-s)^{\alpha-1} h(x) (s-x)^{\beta-1} ds dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(h(x) \int_x^t (t-s)^{\alpha-1} (s-x)^{\beta-1} ds \right) dx \end{aligned}$$

We set $s = x + z(t-x)$, we have

$$\begin{aligned} \int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1} ds &= \int_0^1 (t-x-z(t-x))^{\alpha-1} (z(t-x))^{\beta-1} (t-x) dz \\ &= (t-x)^{\alpha+\beta-1} \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz \\ &= (t-x)^{\alpha+\beta-1} \mathrm{B}(\alpha,\beta). \end{aligned}$$

Hence

$$\begin{aligned} I^\alpha I^\beta h &= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(x)(t-x)^{\alpha+\beta-1}dx \\ &= I^{\alpha+\beta}h \end{aligned}$$

□

Theorem 1.9 *([85], Theorem 2.4) Let h be a summate function. then we have*

$$D^\alpha_{a+}I^\alpha_a h = h$$

Proof: As in Theorem 1.8, we have

$$\begin{aligned} D^\alpha_{a+}I^\alpha_a h &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}ds\int_a^s (s-x)^{\alpha-1}h(x)dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t\int_a^s (t-s)^{n-\alpha-1}h(x)(s-x)^{\alpha-1}dsdx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t\left(h(x)\int_x^t (t-s)^{n-\alpha-1}(s-x)^{\alpha-1}ds\right)dx. \end{aligned}$$

We set $s = x + z(t-x)$, we have

$$\begin{aligned} \int_x^t (t-s)^{n-\alpha-1}(s-x)^{\alpha-1}ds &= \int_0^1 (t-x-z(t-x))^{n-\alpha-1}(z(t-x))^{\alpha-1}(t-x)dz \\ &= (t-x)^{n-1}\int_0^1 (1-z)^{n-\alpha-1}z^{\alpha-1}dz \\ &= (t-x)^{n-1}\mathrm{B}(n-\alpha,\alpha). \end{aligned}$$

Hence

$$\begin{aligned} D^\alpha_{a+}I^\alpha_a h &= \frac{B(n-\alpha,\alpha)}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t h(x)(t-x)^{n-1}dx \\ &= \frac{1}{\Gamma(n)}\left(\frac{d}{dt}\right)^n\int_a^t h(x)(t-x)^{n-1}dx. \end{aligned}$$

Now, by using the formule 1.4, we have

$$D^\alpha_{a+}I^\alpha_a h = h.$$

□

1.2.3 Fractional derivative of Caputo

Definition 1.10 *[73] Let h be a function given on the interval $[a,b]$. The left-sided fractional derivative of Caputo of order α, is defined by*

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

Here $n = [\alpha]+1$ and $\alpha > 0$.

Remark 1.11 *([73], Lemma 2.2) The fractional derivative of Riemann-Liouville and the fractional derivative of Caputo are connected with each other by the following relation:*

$$({}^cD^{\alpha}_{a+}h)(t) = D^{\alpha}_{a+}\left[h(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]$$

To prove this we use the following representation of the fractional derivative of Riemann-Liouville ([85], Theorem 2.2):

$$\begin{aligned}(D^{\alpha}_{a+}h)(t) &= \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds\\ &= \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + ({}^cD^{\alpha}_{a+}h)(t).\end{aligned}$$

or

$$({}^cD^{\alpha}_{a+}h)(t) = D^{\alpha}_{a+}\left[h(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right].$$

Example 1.12 *([73], Property 2.16) Let $h(t) = (t-a)^{\beta}$ where $t > a$ and $\beta > -1$, then we have*

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\frac{d^n}{ds^n}(s-a)^{\beta}ds.$$

Since

$$({}^cD^{\alpha}_{a+}h)(t) = I^{n-\alpha}_a\left(\frac{d^n}{dt^n}h(t)\right).$$

then

$$\begin{aligned}({}^cD^{\alpha}_{a+}h)(t) &= I^{n-\alpha}_a\left(\frac{d^n}{dt^n}(t-a)^{\beta}\right)\\ &= I^{n-\alpha}_a\left(\frac{\Gamma(\beta+1)(t-a)^{\beta-n}}{\Gamma(\beta-n+1)}\right)\\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}I^{n-\alpha}_a\left((t-a)^{\beta-n}\right).\end{aligned}$$

Now by using the result in Example 1.5, we have

$${}^cD^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}.$$

Proposition 1.13 *[86] Let $\alpha,\beta > 0$. Then we have*

(1) $I^{\alpha} : L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$, and if $f \in L^1(J,\mathbb{R})$, then

$$I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}f(t) = I^{\alpha+\beta}f(t)$$

(2) *if* $f \in L^p(J, \mathbb{R})$ $, 1 < p < \infty$ *, then* $\| I^\alpha f(t) \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \| f(t) \|_{L^p}$.

(3) *The fractional integration operator* I^α *is linear.*

(4) *The fractional order integral operator* I^α *maps* $L^1(J, \mathbb{R})$ *into itself.*

(5) *When* $\alpha = n$*,* I_0^α *is the* n*-fold integration.*

(6) *The Caputo and Riemann-Liouville fractional derivative are linear.*

(7) *The Caputo derivative of a constant is equal to zero.*

Lemma 1.14 *[86] Let* $\alpha \geqslant 0$*. Then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 ... + c_{n-1} t^{n-1}$$

$c_i \in \mathbb{R}$,$i = 0, 1, 2, 3, ..., n-1$, $n = [\alpha] + 1$.

Lemma 1.15 *[86] Let* $\alpha \geqslant 0$*. Then*

$$I^\alpha \, {}^cD^\alpha h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 ... + c_{n-1} t^{n-1} + h(t)$$

$c_i \in \mathbb{R}$,$i = 0, 1, 2, 3, ..., n-1$, $n = [\alpha] + 1$.

1.2.4 The Hadamard fractional integral and derivative

Definition 1.16 *([73],[85]) Let* h *be a real function defined on* $[a, +\infty)$*, such that* $a \geq 0$*. The Hadamard fractional integral of order* α *of* h *is defined by*

$$({}^H I^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds, \ \alpha > 0,$$

provided the integral exists.

Definition 1.17 *([73], [85]) Let* h *be a real function defined on* $[a, +\infty)$*, such that* $a \geq 0$*. The* α *Hadamard fractional-order derivative of* h *is defined by*

$$({}^H D^\alpha h)(t) \ = \ \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s} ds.$$

Here $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α *and* $\log(.) = \log_e(.)$.

Example 1.18 *([73]) Let* $h(t) = \left(\log \frac{t}{a}\right)^\beta$ *where* $t > a > 0$ *and* $\beta > -1$*. Then we have*

$$({}^H I^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{t}{a}\right)^\beta \frac{ds}{s},$$

We set, $\left(\log\frac{s}{a}\right)=y\left(\log\frac{t}{a}\right)$, *we have*

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^{\beta}\frac{ds}{s} &= \frac{1}{\Gamma(\alpha)}\int_0^1(\log t(1-y)-\log a(1-y))^{\alpha-1}\\ &\quad y^{\beta}(\log t-\log a)^{\beta}(\log t-\log a)dy\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1((1-y)(\log t-\log a))^{\alpha-1}y^{\beta}\\ &\quad(\log t-\log a)^{\beta}(\log t-\log a)dy\\ &= \frac{(\log t-\log a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1(1-y)^{\alpha-1}y^{\beta}dy\\ &= \frac{\Gamma(\beta+1)\left(\log\frac{t}{a}\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.\end{aligned}$$

Proposition 1.19 *[86] Let* $\alpha,\beta>0$. *Then we have*

(1) ${}^{H}I^{\alpha}: L^1(J,\mathbb{R})\to L^1(J,\mathbb{R})$, *and if* $f\in L^1(J,\mathbb{R})$, *then*

$${}^{H}I^{\alpha}\,{}^{H}I^{\beta}f(t)={}^{H}I^{\beta}\,{}^{H}I^{\alpha}f(t)={}^{H}I^{\alpha+\beta}f(t)$$

(2) If $f\in L^p(J,\mathbb{R})$,$1<p<\infty$, *then* $\|\,{}^{H}I^{\alpha}f(t)\,\|_{L^p}\leq\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|\,f(t)\,\|_{L^p}$.

(3) The fractional order integral operator ${}^{H}I^{\alpha}$ *is linear.*

(4) The fractional order integral operator ${}^{H}I^{\alpha}$ *maps* $L^1(J,\mathbb{R})$ *into itself.*

(5) When $\alpha=n$, ${}^{H}I_0^{\alpha}$ *is the n-fold integration.*

1.2.5 The Caputo-Hadamard fractional derivative

Definition 1.20 *[5] Let* $r\geq 0$ *and* $n-1<r\leq n$, *where* $n=[r]+1$, *and* $h\in AC_{\delta}^{n}[1,+\infty)$. *The Caputo-Hadamard fractional derivative of order r is defined by*

$$\begin{aligned}({}_{c}^{H}D^{r}h)(t) &= \frac{1}{\Gamma(n-r)}\int_1^t\left(\log\frac{t}{s}\right)^{n-r-1}\delta^{n}h(s)\frac{ds}{s}\\ &= {}^{H}I^{n-r}(\delta^{n}h)(t).\end{aligned}$$

Example 1.21 *We consider the function f defined by:*

$$f:t\to\left(\log\frac{t}{a}\right)^{\beta}.$$

We have

$${}_{H}^{C}D_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta}=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^{n}\left(\log\frac{s}{a}\right)^{\beta}\frac{ds}{s}.$$

If we take $\beta = \dfrac{3}{2}$ *,*$\alpha = \dfrac{1}{2}$*, et* $n = 1$*, so*

$$\delta^n \left(\log \frac{s}{a}\right)^{\beta} = \delta^1 \left(\log \frac{s}{a}\right)^{\frac{3}{2}} = \frac{3}{2}\left(\log \frac{s}{a}\right)^{\frac{1}{2}}$$

and

$$ {}_{H}^{C}D_a^{\frac{1}{2}} \left(\log \frac{t}{a}\right)^{\frac{3}{2}} = \frac{\frac{3}{2}}{\Gamma\left(\frac{1}{2}\right)} \int_a^t \left(\log \frac{t}{s}\right)^{1-\frac{1}{2}-1} \left(\log \frac{s}{a}\right)^{\frac{1}{2}} \frac{ds}{s}.$$

We set $\mu = \dfrac{\log \frac{s}{a}}{\log \frac{t}{a}}$,

$$\left(\log \frac{t}{a}\right) d\mu = \frac{ds}{s}.$$

we obtain

$$\begin{aligned} {}_{H}^{C}D_a^{\frac{1}{2}} \left(\log \frac{t}{a}\right)^{\frac{3}{2}} &= \frac{\frac{3}{2}\left(\log \frac{t}{a}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 \mu^{\frac{1}{2}} d\mu, \\ &= \frac{3}{2\sqrt{\pi}}\left(\log \frac{t}{a}\right). \end{aligned}$$

Definition 1.22 *[73] The Hadamard fractional derivative of order* $r > 0$ *applied to the function* $h \in AC_{\delta}^{n}([1, +\infty), \mathbb{R})$ *is defined as*

$$({}^{H}D_1^{q}h)(t) = \delta^n({}^{H}I_1^{n-r}h)(t),$$

where $n - 1 < r < n,\ n = [r] + 1,$ *and* $[r]$ *is the integer part of* r*.*

Definition 1.23 *[70] For a given function* $h \in AC_{\delta}^{n}([a, b], \mathbb{R})$*, such that* $0 < a < b$*, the Caputo-Hadamard fractional derivative of order* $r > 0$ *is defined as follows:*

$${}_{c}^{H}D^{r}y(t) = {}^{H}D^{r}\left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}\left(\log \frac{s}{a}\right)^k\right](t),$$

where $Re(\alpha) \geq 0$ *and* $n = [Re(\alpha)] + 1$*.*

Lemma 1.24 *[70] Let* $y \in AC_{\delta}^{n}([a, b], \mathbb{R})$ *or* $C_{\delta}^{n}([a, b], \mathbb{R})$ *and* $\alpha \in \mathbb{C}$*. Then*

$${}^{H}I^{r}({}^{Hc}D^{r}y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}\left(\log \frac{t}{a}\right)^k.$$

1.3 Multi-valued analysis

For any Banach space $(X, \|\cdot\|)$, we set

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},$$

$$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},$$

$$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\},$$

$$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}.$$

Definition 1.25 *A multi-valued map(also called set valued map)$F : X \mapsto Y$ is an application which associate with any $x \in X$ a subset $F(x)$ which belongs to $\mathcal{P}(Y)$, where X and Y are two sets.*

The multi-valued map has a great important in many fields of mathematics, as examples

- **Quasi variational problems** (as example, see [78]): in this kind of problems the multi-valued map is called the variational selection S and it is depends with the own solution of the problem, as example, we give the following quasi variational problem:
  find u,
  $$u \in S(u) \quad \text{and } f(u, w) \leq 0, \quad \forall w \in S(u).$$

- **Differential inclusions:** in this type of problems we make different conditions on the multi-valued map in order to find the solution. As example we give the following differential inclusions:
  $$y'(t) \in F(t, y(t)), \text{ for almost all } t \in J. \tag{1.5}$$
  Where F is a multi-valued map.

- **Ill posed problems:** In this well known problems, the set valued map allows us to get the unique solution of the problem without restriction of the map.

For more details on multi-valued maps see for example the book of Aubin and Frankowska [16].

1.3.1 Definitions

Definition 1.26 *[16] The domain of a multivalued map F is the subset of elements $x \in X$ such that $F(x)$ is not empty:*

$$Dom(F) = \{x \in X | F(x) \neq \emptyset\}$$

Definition 1.27 *[16] Let X and Y be metric spaces. A multi-valued map F from X to Y is characterized by its graph $Graph(F)$, the subset of the product space $X \times Y$, defined by*

$$Graph(F) = \{(x, y) \in X \times Y | y \in F(x)\}.$$

Definition 1.28 *[19] Let X, Y be nonempty sets and $F : X \mapsto P(Y)$. Then*

(1) The single-valued operator $f : X \mapsto Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.

(2) The set of selections of F is defined by

$$S_{F,y} = \{v \in L^1(J, \mathrm{IR}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Definition 1.29 *[16] A multi-valued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.*

Definition 1.30 *[19] A multi-valued map $F : J \to P_{cl}(\mathrm{IR})$ is said to be measurable if for every $y \in \mathrm{IR}$, the function:*

$$t \to d(y, F(t)) = inf\{|y - z| : z \in F(t)\}$$

is measurable.

1.3.2 Continuity of multi-valued maps

Let X be a topological space and let $F : X \mapsto \mathcal{P}(X)$ be a multi-valued map.

Definition 1.31 *[59]*

(1) F is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty subset of X, and for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subset N$.

(2) F is called lower semi-continuous on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty subset of X, and for each open set N such that $N \cap F(x_0) \neq \emptyset$, there exists an open neighborhood N_0 of x_0 such that

$$x \in N_0 \Rightarrow F(x) \cap N \neq \emptyset.$$

Now we give the equivalent definitions of semi continuity in metric spaces X and Y.

Remark 1.32 *([16], P 39) There is no relation between the upper semi continuity and the lower semi continuity.*

Proposition 1.33 *([16], Proposition (1.4.8)) The graph of an upper semi-continuous multi-valued maps $F : X \to Y$ with closed domain and closed values is closed. The converse is true if we assume that Y is compact.*

Definition 1.34 *A multi-valued map operator F is called completely continuous if $F(B)$ is relatively compact for every $B \in P_B(X)$.*

Definition 1.35 *[19] A multi-valued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:*

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$.

(2) $u \to F(t,u)$ is upper semi-continuous for almost all $t \in J$.

Definition 1.36 *[51] A multi-valued maps $F : J\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to Carathéodory if*

(1) $t \to F(t,u,v)$is measurable for each $u, v \in \mathbb{R}$.

(2) $u \to F(t,u,v)$ is upper semi-continuous for almost all $t \in J$.
Further a Carathéodory function is called L^1-carathéodory,

(3) For each $\rho > 0$,there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that
$\|F(t,u,v)\| = \sup\{|v|, v \in F(t,u,v)\} < \phi_\rho(t)$,*for all* $|v|$,$|u| < \rho$.

Definition 1.37 *([39], P 132) Let (X,d) be a metric space induced from the normed space $(X, |\cdot|)$. The Hausdorff-Pompeiu metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ is given by:*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

Definition 1.38 *[54] A multi-valued operator $N : X \to P_{cl}(X)$ is called*

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in X.$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

1.4 Some fixed point theorems

Theorem 1.39 *[55](Banach fixed point theorem) Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.40 *[55] Let X be a Banach space and C a nonempty closed convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to P_{cp,c}(C)$ is a upper semi-continuous compact map. Then either*

(1) T has fixed points in $\overline{U}$, or

(2) There exist $u \in \partial U$ and $\lambda \in]0,1[$ with $u \in \lambda T(u)$.

Theorem 1.41 *[29](Bohnenblust-Karlin 1950). Let X be a Banach space and $K \in P_{cl,cv}$ and suppose that the operator $G : K \to P_{cl,cv}(k)$ is upper semi-continuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Now we give the theorem of Covitz and Nadler concerning the multi-valued contraction.

Theorem 1.42 *[43] Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 1.43 *[47](Schauder fixed point theorem) Let E be a Banach space and Q be a convex subset of E and $T : Q \to Q$ be a compact and continuous map. Then T has at least one fixed point in Q.*

Chapter 2

Integrable solutions for Implicit Fractional Differential Equations

[1]

2.1 Integrable solutions for implicit frac diff equa

In this chapter, we study the existence of integrable solutions of the boundary value problem for implicit fractional order differential equation

$$ {}^H_cD^{\alpha}y(t) = f(t, y(t), {}^H_cD^{\alpha}y(t)), \text{ for a.e. } t \in J = [1, T], \quad 1 < \alpha \leq 2, \tag{2.1}$$

$$ y(1) = y_1, \quad y(T) = y_T, \tag{2.2}$$

where ${}^H_cD^{\alpha}$ is the Caputo-Hadamard fractional derivative, and $f : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_1, y_T \in \mathbb{R}$.

We shall present two existence results for the problem (2.1)-(2.2).In the first section. the first result is based on Schauder's fixed point Theorem 1.43 and second one on the Banach contraction principle theorem 1.39. An example is presented in the third section.

2.1.1 Existence of solutions

Let us start by defining what we mean by a solution of the problem (2.1)-(2.2).

Definition 2.1 *A function $y \in L^1([1, T], \mathbb{R})$ is said to be a solution of (2.1)-(2.2) if y satisfies (2.1) and(2.2).*

For the existence of solutions for the problem (2.1)-(2.2), we need the following auxiliary lemma.

[1]A.Zahed,S. Hamani and J. Henderson, Integrable Solutions for Caputo-Hadamard Implicit Fractional Order Differential Equations, *Advances in Dynamical Systems and Applications.* **15** (2020), No.1,17 -28 .

Lemma 2.2 *For $1 < \alpha \leq 2$, the solution of the BVP (2.1)-(2.2) can be expressed by the integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s} + y_1 \\ &+ \frac{y_T - y_1 - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}}{\log(T)} \log(t), \end{aligned} \tag{2.3}$$

where x is the solution of the functional integral equation

$$\begin{aligned} x(t) &= f\Big(t, \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s} \\ &+ y_1 + \frac{y_T - y_1 - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}}{\log(T)} \log(t), x(t)\Big), \end{aligned} \tag{2.4}$$

Proof. Let ${}^H_c D^{\alpha} y(t) = x(s)$ in equation (2.1). Then

$$x(t) = f(t, y(t), x(t)). \tag{2.5}$$

Applying the Caputo-Hadamard fractional integral of order α to both sides, and by using Lemma 2.2, we find that y satisfies (2.2). Then Lemma 1.24 implies that

$$y(t) = c_1 + c_2 \log(t) + {}^H I^{\alpha} x(t). \tag{2.6}$$

The boundary conditions (2.2) imply that

$$c_1 = y_1$$

and

$$y(T) = \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} + y_1 + c_2 (\log T).$$

Hence

$$c_2 = \frac{y_T - y_1 - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}}{(\log T)}.$$

Finally, we obtain (2.3).

□

Let us introduce a list of assumptions.

(H1) $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$, and is continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.

(H2) There exist a positive function $a \in L^1([1,T], \mathbb{R})$ and constants $b_i > 0$, $i = 1, 2$, such that : $|f(t, u_1, u_2)| \leq a(t) + b_1|u_1| + b_2|u_2|$ for all $(u_1, u_2) \in \mathbb{R}^2$ and $t \in J$

Our first result is based on the Schauder fixed point theorem

Theorem 2.3 *Assume the assumptions (H1) and (H2) are satisfied. If*

$$\left(2b_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + b_2\right) < 1, \tag{2.7}$$

then the problem (2.1)-(2.2) has at least one solution on J.

Proof. Transform the problem (2.1)-(2.2) into a fixed point problem. Consider the operator

$$H : L^1(J, \mathbb{R}) \to: L^1(J, \mathbb{R})$$

defined by

$$\begin{aligned} (Hx)(t) \quad ;= \quad & f\Bigg(t, \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s} + y_1 \\ + \quad & \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}}{\log(T)} \log(t), x(t)\Bigg). \end{aligned} \tag{2.8}$$

The operator H is well defined. Indeed, for each $x \in L^1([1,T], \mathbb{R})$, from assumptions (H1) and (H2), for

$$r = \frac{\|a\|_{L^1} + 2b_1|T-1|(|y_T| + |y_1|)}{1 - \left(2b_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + b_2\right)},$$

Consider now the set, with r as above,

$$B_r = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L^1} < r\}.$$

Clearly B_r is nonempty, bounded, convex and closed. we have

$$\begin{aligned}
\|Hx\|_{L^1} &= \int_1^T |(Hx)(t)|dt \\
&= \int_1^T |f(t, \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+ y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t), x(t))|dt \\
&\le \int_1^T |a(t)|dt \\
&+ b_1\int_1^T |\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+ y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t)|dt \\
&+ b_2\int_1^T |x(t)|dt \\
&\le \|a\|_{L^1} + 2b_1|T-1|(|y_T| + |y_1|) + 2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|x\|_{L^1} + b_2\|x\|_{L^1} \\
&\le \|a\|_{L^1} + 2b_1|T-1|(|y_T| + |y_1|) + \left(2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + b_2\right) r \\
&\le r.
\end{aligned} \tag{2.9}$$

Then $HB_r \subset B_r$.
Assumption (H1) implies that H is continuous.

Now, we show that H is compact; that is, HB_r is relatively compact. Clearly HB_r is bounded in $L^1([1,T],\mathbb{R})$); in particular, Condition (i) of the Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \to (Hx)$ as $h \to 0$ in $L^1([1,T],\mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$, then we have

$$\begin{aligned}
\|(Hx)_h - (Hx)\|_{L^1} &= \int_1^T |(Hx)_h - (Hx)|dt = \int_1^T |\frac{1}{h}\int_t^{t+h}(Hx)(s)ds - (Hx)(t)|dt \\
&\leq \int_1^T (\frac{1}{h}\int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds)dt \\
&\leq \int_1^T \frac{1}{h}\int_t^{t+h} |f(s, \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + y_1 \\
&+ \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t), x(s)) \\
&- f(t, \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + y_1 \\
&+ \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t), x(t))|dsdt.
\end{aligned} \tag{2.10}$$

Since $x \in B_r \subset L^1([1,T], \mathbb{R})$ and assumption (H2) that implies $a \in L^1([1,T], \mathbb{R})$, we have

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h} \Big| f(s, \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t), x(s)) \\
&-f(t, \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}\log(t), x(t))\Big| ds \\
&\leq \frac{1}{h}\int_t^{t+h} \Big| |a(t)| + 2b_1(|y_T| + |y_1|) + 2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}|x(t)| + b_2|x(t)| \\
&-|a(s)| + 2b_1(|y_T| + |y_1|) + 2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}|x(s)| + b_2|x(s)|\Big| ds \\
\leq\ & \frac{1}{h}\int_t^{t+h} |a(t) - a(s)|ds + \left(2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + b_2\right)\frac{1}{h}\int_t^{t+h}|x(t) - x(s)|ds \to 0,
\end{aligned}$$

as $h \to 0$.
Hence

$$(Hx)_h \to (Hx) \text{ uniformly as } h \to 0.$$

Then by the Kolmogorov compactness criterion, $H(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem the BVP(2.1)-(2.2) has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 2.4 *Assume that the (H1) and the following condition hold.*
(H3) There exist constants $k_1 > 0$ *and* $k_2 > 0$ *such that,*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

and

$$\left(2Tk_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + k_2\right) < 1. \tag{2.11}$$

Then the problem (2.1)-(2.2) has a unique solution $y \in L^1(J, \mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H, defined by (2.8), has a fixed point. Let x,$y \in L^1(J, \mathbb{R})$ and $t \in J$. Then we have,

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= \Big| f(t, \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + y_1 \\
&+ \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)} \log(t), x(t)) \\
&- f(t, \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s} + y_1 \\
&+ \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} y(s)\frac{ds}{s}}{\log(T)} \log(t), y(t))\Big| \\
&\leq 2k_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \int_1^T |x(s) - y(s)|ds + k_2|x(t) - y(t)|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(Hx) - (Hy)\|_{L^1} &\leq 2Tk_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|x - y\|_{L^1} + k_2\|x - y\|_{L^1} \\
&\leq \left(2Tk_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + k_2\right) \|x - y\|_{L^1}.
\end{aligned}$$

Consequently by (2.11), H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a unique fixed point which is a solution of the problem (2.1)-(2.2). □

2.1.2 An example

Consider the boundary value problem:

$$ {}^H_cD^{\frac{3}{2}}y(t) = \frac{1}{4e^{t+5}(1+|y(t)|+|{}^H_cD^{\frac{3}{2}}y(t)|)}, \text{ for a.e. } t \in J = [1,e], \quad 1 < \alpha \leq 2, \quad (2.12) $$

$$ y(1) = 0, \; y(e) = 1. \quad (2.13) $$

Set

$$ f(t,u,v) = \frac{1}{4e^{t+5}(1+u+v)}, t \in J \times [0,\infty) \times [0,\infty), $$

and let $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $t \in J$. Then we have

$$ \begin{aligned} |f(t,u_1,v_1) - f(t,u_2,v_2)| &= \left|\frac{1}{4e^{t+5}}\left(\frac{1}{(1+u_1+v_1)} - \frac{1}{(1+u_2+v_2)}\right)\right| \\ &\leq \left|\frac{1}{4e^{t+5}}\left(\frac{|u_1-u_2|+|v_1-v_2|}{(1+u_1+v_1)(1+u_2+v_2)}\right)\right| \\ &\leq \frac{1}{4e^{t+5}}(|u_1-u_2|+|v_1-v_2|) \\ &\leq \frac{1}{4e^5}|u_1-u_2| + \frac{1}{4e^5}|v_1-v_2|. \end{aligned} $$

Hence the condition (H3) holds with $k_1 = k_2 = \frac{1}{4e^5}$. We shall check that condition (2.11) is satisfied with $T = e$. Indeed

$$ 2k_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + k_1 = \frac{1}{2e^5\Gamma(\alpha+1)} + \frac{1}{4e^5} < 1. $$

Then by Theorem 2.4, the problem (2.12)-(2.13) has a unique integrable solution on $[1,e]$.

2.2 Integrable Solutions for implicit fract diff equa with nonlocal condition

[2]

In this section,we study the existence of integrable solutions of Initial values problem for implicit fractional order differential equation

$$ {}^H_cD^\alpha y(t) = f(t, y(t), {}^H_cD^\alpha y(t)), \text{ for a.e. } t \in J = [1,T], \quad 0 < \alpha \leq 1, \quad (2.14) $$

$$ \sum_{k=1}^m a_k y(t_k) = y_1; \quad (2.15) $$

[2] A.Zahed ,S. Hamani and J. Henderson Integrable Solutions for Caputo-Hadamard Implicit Fractional Order Differential Equations with Nonlocal Condition,(*submitted*)

where $f:[1,T]\times \mathbb{R}\times \mathbb{R}\to \mathbb{R}$ is a given function, ${}^{H}_{c}D^{\alpha}$ is the Hadamard-Caputo fractional derivative and $1<t_1<t_2<...t_m<T$, $k=1,2..,m$ we shall present two existence results for the problem (2.14)-(2.15).In the first section, the first result is based on Schauder's fixed point theorem 1.43 and second one on the Banach contraction principle theorem 1.39. An Example is given in the third section to demonstrate the application of our main results.

2.2.1 Existence of solutions

Definition 2.5 *A function $y\in L^1([1,T],\mathbb{R})$ is said to be a solution of (2.14)-(2.15) if y satisfies (2.14) and(2.15).*

Let us start by defining what we mean by an integrable solution of the nonlocal problem(2.14)-(2.15)

We assume that $\sum_{k=1}^{m}a_k\neq 0$,Set

$$a=\frac{1}{\sum_{k=1}^{m}a_k}.$$

For the existence of solutions for the nonlocal problem (2.14)-(2.15) we need the following auxiliary lemma.

Lemma 2.6 *For $0<\alpha\leq 1$, the solution of the problem (2.14)-(2.15) can be expressed by the integral equation*

$$y(t)=ay_1-\frac{a\sum_{k=1}^{m}a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},\quad (2.16)$$

where x is the solution of the functional integral equation

$$\begin{aligned} x(t) &= f(t,ay_1-\frac{a\sum_{k=1}^{m}a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}\\ &+ \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},x(t)). \end{aligned}\quad (2.17)$$

Proof. Let ${}^{C}_{H}D^{\alpha}y(t)=x(t)$ in equation (1.5)

$$x(t)=f(t,y(t),x(t))\quad (2.18)$$

and

$$\begin{aligned} y(t) &= c_1+{}^{H}I^{\alpha}x(t)\\ &= c_1+\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}. \end{aligned}\quad (2.19)$$

Let $t=t_k$ in (2.19), we obtain

$$y(t_k)=c_1+\frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t_k}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}$$

and

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k y(1) + \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}. \tag{2.20}$$

Substitute (2.15)into (2.20)

$$y_1 = \sum_{k=1}^{m} a_k y(1) + \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}$$

and

$$c_1 = a(y_1 - \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}). \tag{2.21}$$

Theorem 2.7 *Assume that the assumptions(H1)-(H2) are satisfied. If*

$$\left(\frac{2b_1(\log T)^{\alpha}}{\Gamma(\alpha+1)} + b_2\right) < 1, \tag{2.22}$$

then the problem(2.14)-(2.15) has at least one solution.

Proof. Transform the nonlocal problem(2.14)-(2.15) into a fixed point problem.Consider the operator

$$H : L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$$

defined by :

$$(Hx)(t) = f\left(t, ay_1 - \frac{a\sum_{k=1}^{m} a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, x(t)\right). \tag{2.23}$$

Let

$$r = \frac{|ay_1||T-1| + \|a\|_{L^1}}{1 - \left(\dfrac{2b_1(\ logT)^{\alpha}}{\Gamma(\alpha+1)} + b_1\right)}$$

and consider the set

$$B_r = \{x \in L(J, \mathbb{R}) : \|x\|_{L^1} \leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that$HB_r \subset B_r$,indeed, for each $x \in B_r$ from(2.22) and (2.23) we get

$$\begin{aligned}
\|Hx\|_{L^1} &= \int_1^T |Hx(t)|dt \\
&= \int_1^T |f(t, ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, x(t))|dt \\
&\leq \int_1^t [|a(t)| + b_1|ay_1 - a\sum_{k=1}^m a_k \frac{1}{\Gamma(\alpha)} \int_1^{s_k} \left(\log \frac{s_k}{\tau}\right)^{\alpha-1} |x(s)|\frac{d\tau}{\tau})ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\alpha-1} |x(s)|\frac{d\tau}{\tau})ds| + b_2|x(t)|]dt \qquad (2.24) \\
&\leq |ay_1|b_1|T-1| + \|a\|_{L^1} + \frac{b_1 a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\|x\|_{L^1} \\
&+ \frac{b_1(\log T)^\alpha}{\Gamma(\alpha)}\|x\|_{L^1} b_2\|x\|_{L^1} \\
&\leq |ay_1||T-1| + \|a\|_{L^1} + \frac{2b_1(\log T)^\alpha}{\Gamma(\alpha+1)}\|x\|_{L^1} + b_2\|x\|_{L^1} \\
&\leq |ay_1||T-1| + \|a\|_{L^1} + \left(\frac{2b_1(\log T)^\alpha}{\Gamma(\alpha+1)} + b_1\right)\|x\|_{L^1} \\
&\leq r.
\end{aligned}$$

Then $H(B_r) \subset B_r$.
Assumption (H1) implies that H is continuous. Now, we will show that$H$ is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J, \mathbb{R})$ i.e condition (i) of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \to (Hx)$ in $L^1(J, \mathbb{R})$ for each $x \in B_r$
Let $x \in B_r$.

$$\begin{aligned}
\|(Hx)_h - (Hx)\|_{L^1} &= \int_1^T |(Hx)_h(t) - (Hx)(t)|dt = \int_1^T \left|\frac{1}{h}\int_t^{t+h} (Hx)_h(s)ds - (Hx)(t)\right| dt \\
&\leq \int_1^T \left(\frac{1}{h}\int_t^{t+h} |(Hx)_h(s)ds - (Hx)(t)|ds\right) dt \\
&\leq \int_1^T \left(\frac{1}{h}\int_t^{t+h} |{}^H I^\alpha x(s) - {}^H I^\alpha x(t)|ds\right) dt \\
&\leq \int_1^T \frac{1}{h}\int_t^{t+h} |f(t, ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{s_k} \left(\log \frac{s}{\tau}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\alpha)} \int_s^1 \left(\log \frac{s}{\tau}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau}, x(t)) \\
&- f(t, ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, x(t))|dsdt.
\end{aligned}$$

Since $x \in B_r \subset L^1(J, \mathbb{R})$ and by assumption (H2) we have

$$\frac{1}{h}\int_t^{t+h} \Big| f(t, ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{s_k} \left(\log\frac{s}{\tau}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau}, x(s))$$
$$-f(t, ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, x(t))\Big| ds$$
$$\leq \frac{1}{h}\int_t^{t+h} \Big| |ay_1| + a(t) + \frac{2b_1(\log T)^\alpha}{\Gamma(\alpha+1)} x(t) + b_2 x(t)$$
$$-|ay_1| - a(s) - \frac{2b_1(\log T)^\alpha}{\Gamma(\alpha+1)} x(s) - b_2 x(s)) \Big|$$
$$\leq \frac{1}{h}\int_t^{t+h} |a(t) - a(s)| ds + \left(2b_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + b_2\right)\frac{1}{h}\int_t^{t+h} |x(t) - x(s)| ds \to 0,$$

$$\text{as } h \to 0, t \in J.$$

Hence

$$(Hx)_h \to (Hx) \text{ uniformly as } h \to 0.$$

Then by Kolmogorov compactness criterion, $H(B_r)$ is relatively compact. As a consequence of Schauder fixed point theorem the problem (2.14)-(2.15) has at least one solution in B_r.

The following result is based on the Banach contraction principle.

Theorem 2.8 *Assume that (H1) and the following condition holds:*

(H3) There exist constants $k_1, k_2 > 0$ *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}$$

If

$$\frac{2k_1(\log T)^\alpha}{\Gamma(\alpha+1)} + k_2 < 1, \tag{2.25}$$

then the problem(2.14)-(2.15)has a unique solution $y \in L^1(J, \mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by(2.23) has a fixed point.Let $x, y \in L^1(J, \mathbb{R})$, and $t \in J$,then we have,

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= |f(t, ay_1 - a\textstyle\sum_{k=1}^m a_k^H I^\alpha x(t)|_k +^H I^\alpha x(t), x(t)) \\
&- f(t, ay_1 - a\textstyle\sum_{k=1}^m a_k^H I^\alpha y(t)|_k +^H I^\alpha y(t), y(t))]| \\
&\leq k_1 \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log\frac{t_k}{s}\right)^{\alpha-1} \frac{ds}{s}|x(s) - y(s)| \\
&+ \frac{k_1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}|x(s) - y(s)| + k_2|x(s) - y(s)| \\
&\leq \frac{a\sum_{k=1}^m a_k k_1(\log T)^\alpha}{\Gamma(\alpha+1)}|x(s) - y(s)| \\
&+ \frac{k_1(\log T)^\alpha}{\Gamma(\alpha+1)}|x(s) - y(s)| + k_2|x(s) - y(s)|.
\end{aligned}$$

$$\begin{aligned}
\|(Hx) - (Hy)\|_{L^1} &\leq \frac{a\sum_{k=1}^m a_k k_1(\log T)^\alpha}{\Gamma(\alpha+1)}\|x - y\|_{L^1} + \frac{k_1(\log T)^\alpha}{\Gamma(\alpha+1)}\|x - y\|_{L^1} + k_2\|x - y\|_{L^1} \\
&\leq \left(\frac{2k_1(\log T)^\alpha}{\Gamma(\alpha+1)} + k_2\right)\|x - y\|_{L^1}.
\end{aligned}$$

Consequently by (4.31) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem(2.14)-(2.15)

□

2.2.2 An example

Let us consider the following fractional nonlocal problem,

$${}^C_H D^\alpha y(t) = \frac{1}{(e^{t+6})(1 + y(t) +^C_H D^\alpha y(t))}, \text{ for a.e. } t \in J = [1, e],\ 0 < \alpha \leq 1 \tag{2.26}$$

$$\sum_{k=1}^{m} a_k y(t_k) = e, \tag{2.27}$$

where $a_k \in \mathbb{R}$, $1 < t_1 < t_2 < t_3 < t_4 < ... < e$

Set $f(t, y, z) = \dfrac{1}{(e^{t+6})(1 + y + z)}$.

Let $y, z \in [0, +\infty)$ and $t \in J$, Then we have

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)| &\leq \left| \frac{1}{(e^{t+6})} \left(\frac{1}{(1 + y_1 + z_1)} - \frac{1}{(1 + y_2 + z_2)} \right) \right| \\ &\leq \frac{|y_1 - y_2| + |z_1 - z_2|}{(e^{t+6})(1 + y_1 + z_1)(1 + y_2 + z_2)} \\ &\leq \frac{1}{e^{t+6}}(|y_1 - y_2| + |z_1 - z_2|) \\ &\leq \frac{1}{e^6}|y_1 - y_2| + \frac{1}{e^6}|z_1 - z_2|. \end{aligned}$$

Hence (H3) holds with $k_1 = k_2 = \dfrac{1}{e^6}$,We shall check that condition (2.25) is satisfied. Indeed

$$\frac{2k_1(\log T)^\alpha}{\Gamma(\alpha + 1)} + k_2 = \frac{2}{e^6\Gamma(\alpha + 1)} + \frac{1}{e^6} < 1.$$

Then by Theorem2.8, the nonlocal problem (2.26)-(2.27) has a unique integrable solution on $[1, e]$.

Chapter 3

Implicit Fractional Differential Inclusions

[1]

3.1 Implicit fractional differential inclusions with Caputo frac derivative and nonlocal conditions

In this section, we study the existence of solution of nonlocal problems for the following differential inclusion,

$$^cD^\alpha y(t) \in F(t, y(t), ^cD^\alpha y(t)), \text{ for a.e. } t \in J = [0, T],\ 0 < \alpha \leq 1, \tag{3.1}$$

$$\sum_{k=1}^{m} a_k y(t_k) = y_0, \tag{3.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}, y_0 \in \mathbb{R}$ $a_k \in \mathbb{R}$, and $0 < t_1 < t_2 < ... < t_m < T,\ k = 1, 2, ..., m$.
We present an existence result for the problem (3.1)-(3.2) when the right hand side is convex valued by using fixed point theorem (nonlinear alternative of Leray Schauder 1.40. The second results are given for non-convex valued right hand sides, which are based upon a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler 1.42. An example is given to demonstrate the application of our main results

3.1.1 The convex case

Let us start by defining what we mean by a solution of the problem (3.1)-(3.2)

[1]A.Zahed and S.Hamani , Implicit Fractional Differential inclusion with Caputo Fractional derivative and nonlocal conditions , *submitted*.

Definition 3.1 *A function $y \in L^1([0,T],\mathbb{R})$ such that ${}^cD^\alpha y(t)$ is measurable is said to be a solution of (3.1)-(3.2) if there exists a function $x \in L^1([0,T],\mathbb{R})$ with $x(t) \in F(t, y(t), {}^cD^\alpha y(t))$ for a.e $t \in [0,T]$ such that ${}^cD^\alpha y(t) = x(t)$ and the function y satisfies conditions (3.2).*

Remark 3.2 *We assume that $\sum_{k=1}^m a_k \neq 0$ and*

$$a = \frac{1}{\sum_{k=1}^m a_k}.$$

For the existence of solutions for the nonlocal problem (3.1)-(3.2), we need the following auxiliary lemma.

Lemma 3.3 *The nonlocal problem(3.1)-(3.2) can be expressed by the integral equation*

$$y(t) = ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \tag{3.3}$$

where x is the solution of the integral equation

$$x(t) \in F(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, x(t)). \tag{3.4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$ in equation (3.1)

$$x(t) \in F(t, y(t), x(t)) \tag{3.5}$$

and

$$\begin{aligned} y(t)) &= y(0) + I^\alpha x(t) \\ &= y(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned} \tag{3.6}$$

Let $t = t_k$ in (3.6),we obtain

$$y(t_k) = y(0) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

and

$$\sum_{k=1}^m a_k y(t_k) = \sum_{k=1}^m a_k y(0) + \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \tag{3.7}$$

Substitute from (3.2) into (3.7), we have

$$y_0 = \sum_{k=1}^m a_k y(0) + \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

and

$$y(0) = a(y_0 - \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds). \tag{3.8}$$

Substitute (3.8) into (3.6) and (3.5),we obtain (3.3) and (3.7)

Theorem 3.4 *Assume the following hypotheses are valid :*

(B1) $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(B2) There exist $p \in L^1(J, \mathbb{R}^+)$ *continuous and nondecreasing such that*

$$\|F(t, u_1, u_2)\|_P = \sup\{|v| : v \in F(t, u_1, u_2)\} \leq p(t)(1 + |u_1| + |u_2|) \text{ for } t \in J \text{ and each } u_1, u_2 \in \mathbb{R}.$$

(B3) There exist $l_1, l_2 \in L^1([0;T];\mathbb{R})$, *such that*

$$H_d(F(t, x, y), F(t, \bar{x}, \bar{y})) < l_1(t)|x - \bar{x}| + l_2(t)|y - \bar{y}| \text{ for every } x, \bar{x}, y, \bar{y} \in \mathbb{R}.$$

Then, the problem (3.1)-(3.2) has at least one solution on J.

Remark 3.5 *Note that for an* L^1- *Carathéodory multifunction* $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *the set* $S^1_{F,y}$ *is not empty*

Proof. Transform the problem (3.1)-(3.2) into a fixed point problem. Consider the multi-valued operator,

$$N : L^1(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$$

$$N(x) = \left\{ h \in L^1(J, \mathbb{R}) \begin{array}{rcl} h(t) & = & ay_0 - a\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ & + & \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \;\; v \in S^1_{F,y} \end{array} \right\}$$

Clearly, from Lemma3.3 , the fixed points of N are solutions to (3.1)-(3.2) . We shall show that N satisfies the assumptions of nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.
Let $r > 0$ and consider the bounded set

$$B_r = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L^1} \leq r\}.$$

Step 1:$N(x)$*is convex for each* $y \in B_r$
Indeed, if h_1 , h_2 belong to $N(y)$ then there exist v_1, v_2 such that for each $t \in J$ we have,for $i = 1, 2$

$$\begin{array}{rcl} h_i(t) & = & ay_0 - a\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds \\ & + & \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds \end{array}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$ we have

$$\begin{aligned}(dh_1 + (1-d)h_2)(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}(dv_1 + (1-d)v_2)(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(dv_1 + (1-d)v_2)(s)ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have.

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2: $N(B_r)$ *is relatively compact.*

(a) $N(B_r)$ *is Bounded.* Let $y \in B_r$, for each $h \in N(x)$ and $t \in J$, we have by (B2),

$$\begin{aligned}h(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds.\end{aligned}$$

By (B2), we have, for each $t \in J$

$$\begin{aligned}
\|h\|_{L^1} &= \int_0^T |h(t)|dt \\
&= \int_0^T \left| ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right| \\
&\leq |ay_0|T + \int_0^T \left(a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\
&+ \int_0^T \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\
&\leq |ay_0|T + \int_0^T \frac{a\sum_{k=1}^m a_k (t_k)^\alpha}{\Gamma(\alpha+1)} |v(s)|ds + \int_0^T \frac{T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\
&\leq |ay_0|T + \int_0^T \frac{2T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\
&\leq |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T [|p(t)| \\
&+ \left| ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right| |p(t)| \\
&+ |x(t)||p(t)|]dt \\
&\leq |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} + \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)} |ay_0| \|p\|_{L^1} \\
&+ \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} \\
&\leq |ay_0|T + \left(\frac{2T^\alpha + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^\alpha}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] \|x\|_{L^1} \right) \|p\|_{L^1}.
\end{aligned}$$

Thus,

$$\|h\|_{L^1} \leq |ay_0|T + \left(\frac{2T^\alpha + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^\alpha}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] r \right) \|p\|_{L^1} \leq r.$$

Then the above inequalities show that

$$\|N(x)\| = \sup\{\|h\|_{L^1} : h \in N(x)\} \leq r,$$

which shows that $N(B_r) \subset B_r$ is bounded, then $N(B_r)$ is bounded.

(b) $N(x)_r \to N(y)$, in $L^1(J, \mathbb{R})$ for each $y \in B_r$.

Let $y \in N(y)$, then we have

$$\begin{aligned}\|h_\tau - h\|_{L^1} &= \int_0^T |h_\tau(t) - h(t)|dt \\ &= \int_0^T \left|\frac{1}{\tau}\int_t^{t+\tau} h(s)ds - h(t)\right| dt \\ &\leq \int_0^T \left(\frac{1}{\tau}\int_t^{t+\tau} |h(s) - h(t)|ds\right) dt\end{aligned}$$

$$\leq \int_0^T \left|\frac{1}{\tau}\int_t^{t+\tau}\left(a\sum_{k=1}^m a_k \int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)ds - a\sum_{k=1}^m a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right| dt$$
$$+\int_0^T \left|\frac{1}{\tau}\int_t^{t+\tau}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right| dt \to 0 \text{if} \tau \to 0.$$

Because:

$$\frac{1}{\tau}\int_t^{t+\tau}\left(a\sum_{k=1}^m a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)ds = a\sum_{k=1}^m a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \text{ if } \tau \to 0$$

and

$$\frac{1}{\tau}\int_t^{t+\tau}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\right)ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \text{ if } \tau \to 0$$

by Proposition1.13 it follows that $I^\alpha v \in L^1(J, \mathbb{R})$
we have also :

$$\begin{aligned}\lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau} I^\alpha v(s)ds &= D^1(I^1 I^\alpha v(s)) \\ &= D^1(I^{\alpha+1}v(s)) \\ &= I^\alpha v(s).\end{aligned} \tag{3.9}$$

Hence

$$N(y)_\tau \to N(y) \text{ uniformaly } \tau \to 0.$$

As a consequence of (a) and (b) together with the Kolmogorov compactness criterion, we can conclude that $N(B_r)$ is relatively compact.

Step 3: *Nhas a closed graph.*
Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$ We need to show that $h_n \in N(y_n)$ means that there exists $v_n \in S^1_{F,y}$, such that, for each $t \in J$

$$\begin{aligned} h_n(t) &= ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \end{aligned}$$

We must show that there exists $v_* \in S^1_{F,y}$ such that, for each $t \in J$

$$\begin{aligned} h_*(t) &= ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds \end{aligned}$$

Since $F(t,.,.)$ is upper semi-continuous, then for every $\epsilon > 0$ there exists $n_0(\epsilon) > 0 >$ such that, for every $n > n_0$,we have
$v_n(t) \in F(t, y(t), x(t)) \subset F(t, y_*(t), x_*(t)) + \epsilon B(0,1)$. a.e $t \in J$
Since $F(.,.,.)$ has compact values, then there exists a subsequence $v_{n_m}(.)$ such that $v_{n_m}(.) \to v_*$ as $m \to \infty$,and $v_*(t) \in F(t, y_*(t), x_*(t))$ a.e.$t \in J$
For every $w \in F(t, y_*(t), x_*(t)))$,we have

$$|v_{n_m} - v_*| \le |v_{n_m} - w| + |w - v_*|.$$

Then

$$|v_{n_m} - v_*| \le d(v_{n_m}, F(t, y_*(t), x_*(t)).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_* , and it follows that

$$\begin{aligned} |v_{n_m} - v_*| &\le H_d(F(t, y_{n_m}(t), x_{n_m}(t)), F(t, y_*(t), x_*(t)))) \\ &\le l_1(t)|y_{n_m}(t) - y_*(t)| + l_2(t)|x_{n_m}(t) - x_*(t)|. \end{aligned}$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\le a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v_{n_m} - v_*|ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_{n_m} - v_*|ds \\ &\le \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T (l_1(t)|y(t) - \bar{y}(t)| + l_2(t)|x(t) - \bar{x}(t)|)ds \\ &\le (\frac{4T^{2\alpha}}{\Gamma(2\alpha+1)}\|l_1\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)}\|l_2\|_{L^1})\|x - \bar{x}\|_{L^1}. \end{aligned}$$

Hence,

$$\|h_n - h_*\|_{L^1} \to 0, \text{as} \quad m \to \infty.$$

Step 4: *A priori bounds on solutions.*
Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$, Then there exists $v \in S^1_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} h(t) &= ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds. \end{aligned}$$

By (B2), we have, for each $t \in J$

$$\begin{aligned} \|h\|_{L^1} &= \int_0^T |h(t)|dt \\ &= \int_0^T \left| ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right| \\ &\leq |ay_0|T + \int_0^T \left(a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\ &+ \int_0^T \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\ &\leq |ay_0|T + \int_0^T \frac{a\sum_{k=1}^m a_k (t_k)^\alpha}{\Gamma(\alpha+1)} |v(s)|ds + \int_0^T \frac{T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\ &\leq |ay_0|T + \int_0^T \frac{2T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\ &\leq |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T [|p(t)| \\ &+ \left| ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right| |p(t)| \\ &+ |x(t)||p(t)|]dt \\ &\leq |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} + \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)} |ay_0| \|p\|_{L^1} \\ &+ \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} \\ &\leq |ay_0|T + \left(\frac{2T^\alpha + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^\alpha}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] \|x\|_{L^1} \right) \|p\|_{L^1}; = r. \end{aligned}$$

Let $U = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L^1} < r+1\}$. The operator $N : \bar{U} \to \mathcal{P}(L^1(J, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U,there is no $x \in \partial U$ such that $x \in \lambda N(x)$.For some $\lambda \in (0,1]$, as a consequence of the nonlinear alternative of Leray-Shauder, we deduce that N has a fixed point $x \in \bar{U}$ which is a solution of the problem (3.1)-(3.2).This completes the proof. □

3.1.2 The non-convex case

We present the result for the problem (3.1)-(3.2) with a non-convex valued right hand side. Our considerations are based on the fixed point result in theorem of Covitz and

Nadler(1.42)

Theorem 3.6 *Assume(B3) and the following hypotheses hold:*

(B5) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(X)$ *has the property that,*
$F(.,u,v) : J \to \mathcal{P}_{cp}(X)$ *is measurable for each* $u, v \in \mathbb{R}$.

If

$$\left(\frac{2T^{\alpha}}{\Gamma(\alpha+1)}\right)(\|l_1\|_{L^1} + \|l_2\|_{L^1}) < 1, \tag{3.10}$$

then the problem (3.1)-(3.2) has at least one solution on J.

Remark 3.7 *By (B5), we can see that* $S^1_{F,y}$ *is nonempty for each* $y \in L^1(J,\mathbb{R})$, *so* F *has a measurable selection (see [41], Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of in theorem of Covitz and Nadler 1.42. The proof will be given in two steps.

Step1: $N(x) \in \mathcal{P}_{cl}(L^1(J,\mathbb{R}))$ *for each* $x \in L^1(J,\mathbb{R})$.
Indeed, let $(h_n)_{n\geq 0} \subset N(x)$ be such that $h_n \to \widetilde{h}$ in $L^1(J,\mathbb{R})$,then $\widetilde{h}$ in $L^1(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that for each $t \in J$.

$$\begin{aligned} h_n(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds. \end{aligned}$$

Using the fact that F has compact values and from (B3) we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazurs theorem implies that v_n converges strongly to v and hence $v \in S^1_{F,y}$ Then for each $t \in J$

$$\begin{aligned} h_n(t) \to \widetilde{h}(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \end{aligned}$$

So , $\widetilde{h} \in N(x)$.

Step2: *There exists* $\gamma < 1$ *such that* $H_d(N(x), N(\bar{x})) < \gamma\|x-\bar{x}\|_{L^1}$ *for each* $x, \bar{x} \in L^1(J,\mathbb{R})$. Let $x, \bar{x} \in L^1(J,\mathbb{R})$ and $h_1 \in N(x)$,Then there exists $v_1 \in F(t, y(t), x(t))$ such that for each $t \in J$,

$$\begin{aligned} h_1(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds. \end{aligned}$$

From (B3) it follows that

$$H_d(F(t,y(t),x(t)),F(t,\bar{y}(t),\bar{x}(t)) \leq l(t)|y(t)-\bar{y}(t)| + l(t)|x(t)-\bar{x}(t)|$$

Hence, there exists $w \in F(t,\bar{y}(t),\bar{x}(t))$ such that

$$|v_1(t)-w| \leq l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|, t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \leq l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t,\bar{y}(t),\bar{x}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V .So $v_2 \in F(t,\bar{y}(t),\bar{x}(t))$,and for each $t \in J$

$$|v_1(t)-v_2(t)| \leq l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|, t \in J.$$

Let us define for each $v_2 \in J$,

$$\begin{aligned} h_2(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds. \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t)-h_2(t)| &\leq a\textstyle\sum_{k=1}^{m} a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1-v_2|ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1-v_2|ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}|v_1-v_2| \\ &+ \frac{T^\alpha}{\Gamma(\alpha+1)}|v_1-v_2| \\ &\leq \frac{2T^\alpha}{\Gamma(\alpha+1)}|v_1-v_2| \\ &\leq \left(\frac{2T^\alpha}{\Gamma(\alpha+1)}\right)(l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|). \end{aligned}$$

Thus,

$$\|h_1 - h_2\|_{L^1} \leq \left(\frac{2T^\alpha}{\Gamma(\alpha+1)}\right) \int_0^t (l_1(t)|y(t) - \bar{y}(t)| + l_2(t)|x(t) - \bar{x}(t)|)dt.$$

Hence

$$\|h_1 - h_2\|_{L^1} \leq \left(\frac{2T^\alpha}{\Gamma(\alpha+1)}\right) (\|l_1\|_{L^1}\|x - \bar{x}\|_{L^1} + \|l_2\|_{L^1}\|x - \bar{x}\|_{L^1}).$$

For an analogous relation, obtained by interchanging the roles of x and $\bar{x}$, it follows that

$$H_d(N(x), N(\bar{x})) \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right) (\|l_1\|_{L^1} + \|l_2\|_{L^1})\|x - \bar{x}\|_{L^1}.$$

So by (3.10),N is a contraction and thus, by theorem of Covitz and Nadler1.42 , N has a fixed point x which is solution to (3.1)-(3.2) . The proof is complete.. □

3.1.3 An example

we give an example to illustrate our main result. We apply Theorem 3.4 to the the following fractional differential inclusion

$$^cD^\alpha y(t) \in F(t, y(t), ^cD^\alpha y(t)), \text{ for a.e. } t \in J = [0,1],\ 0 < \alpha \leq 1 \tag{3.11}$$

$$\sum_1^m a_k y(t_k) = 1, \tag{3.12}$$

where

$$F(t, y(t), ^cD^\alpha y(t)) = \{v \in \mathbb{R} : f_1(t, y(t), ^cD^\alpha y(t)) \leq v \leq f_2(t, y(t), ^cD^\alpha y(t))\}$$

and $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [0,1]$, $f_1(t,\cdot,\cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y(t), ^cD^\alpha y(t)) > \mu_1\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [0,1]$, $f_2(t,\cdot,\cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y(t), ^cD^\alpha y(t)) < \mu_2\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in L^1([0,1], \mathbb{R}^+)$ such that

$$\begin{aligned}\|F(t, u_1, u_2)\|_P &= \sup\{|v| : v \in F(t, u_1, u_2)\} \\ &= \max(|f_1(t, y(t), x(t))|, |f_2(t, y(t), x(t))|) \\ &\leq p(t)(1 + |x| + |y|) \end{aligned},$$

for $t \in J$ and each $x, y \in \mathbb{R}$.

It is clear that$F$ is compact and convex-valued, and it is upper semi-continuous.

Since all the conditions of Theorem 3.4 are satisfied, problem (3.11)-(3.12) has at least one solution y on $[0,1]$.

3.2 Implicit fractional differential inclusion with Hadamard-Caputo fract derivative and nonlocal condition

[2] We are concerned with the existence of solution of Nonlocal problems for a implicit fractional differential inclusion,

$$ {}_H^C D^\alpha y(t) \in F(t, y(t), {}_H^C D^\alpha y(t)), \text{ for a.e. } t \in J = [1, T], \ 0 < \alpha \leq 1 \tag{3.13} $$

$$ \sum_{k=1}^{m} a_k y(t_k) = y_1, \tag{3.14} $$

where ${}_H^C D^\alpha$ is the the Caputo-Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.,$y_1 \in \mathbb{R}$ $a_k \in \mathbb{R}$ and $1 < t_1 < t_2 < ... < t_m < T, \ k = 1, 2, ..., m$.
We present an existence result for the problem (3.13)-(3.14)when the right hand side is convex valued by using fixed point theorem of Bohnnenblust-Karlin. The second results are given i for non-convex valued right hand sides, which are based upon a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler 1.42 . An example is given to demonstrate the application of our main results.

3.2.1 The convex case

Definition 3.8 *A function $y \in L^1([1, T], \mathbb{R})$ such that ${}_H^C D^\alpha y(t)$ is measurable is said to be a solution of (3.13)-(3.14) if there exists a function $x \in L^1([1, T], \mathbb{R})$ with $x(t) \in F(t, y(t), {}_H^C D^\alpha y(t))$ for a.e $t \in [1, T]$ such that ${}_H^C D^\alpha y(t) = x(t)$ and the function y satisfies conditions (3.14).*

We assume that $\sum_{k=1}^{m} a_k \neq 0$ and we set

$$ a = \frac{1}{\sum_{k=1}^{m} a_k}. $$

For the existence of solutions for the nonlocal problem (3.13)-(3.14) we need the following auxiliary lemma.

Lemma 3.9 *The solutions of nonlocal problem(3.13)-(3.14) can be expressed by the integral equation*

$$ y(t) = a y_1 - \frac{a \sum_{k=1}^{m} a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s} \right)^{\alpha - 1} x(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s} \right)^{\alpha - 1} x(s) \frac{ds}{s}, \tag{3.15} $$

[2]A.Zahed , S. Hamani and J.R. Graef , Caputo-Hadamard implicit Fractional Differential inclusion with nonlocal condition, *Archivum Mathematicum (Brno)***57** ,(2021) No 5, 285297.

where x is the solution of the integral equation

$$\begin{aligned} x(t) &\in F(t, ay_1 - \frac{a\sum_{k=1}^{m} a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, x(t)). \end{aligned} \tag{3.16}$$

Proof. Let ${}^{C}_{H}D^{\alpha}y(t) = x(t)$ in equation (1.5)

$$x(t) \in F(t, y(t), x(t)) \tag{3.17}$$

and

$$\begin{aligned} y(t) &= c_1 +^{H} I^{\alpha}x(t) \\ &= c_1 + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}. \end{aligned} \tag{3.18}$$

Let $t = t_k$ in (3.18),we obtain

$$y(t_k) = c_1 + \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}$$

and

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k y(1) + \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \tag{3.19}$$

Substitute from (3.14)into (3.19)

$$y_1 = \sum_{k=1}^{m} a_k y(1) + \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}$$

and

$$c_1 = a(y_1 - \sum_{k=1}^{m} a_k \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}) \tag{3.20}$$

Substitute from (3.19) into (3.18) and (3.17) ,we obtain (3.15) and (3.16).
Let $t = t_k$ in ,we obtain(3.15)

$$\begin{aligned} y(t_k) = ay_1 - \frac{a\sum_{k=1}^{m} a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \\ = ay_1 + (1 - a\sum_{k=1}^{m} a_k)\frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}. \end{aligned}$$

Then, we have

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k a y_1 + \sum_{k=1}^{m} a_k (1 - a\sum_{k=1}^{m} a_k)\frac{1}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} = y_1$$

This complete the proof.
Let us introduce the following assumptions:

(B6) There exist a positive function $a_1 \in L^1(J)$ and constants,$b_i > 0$; $i = 1, 2$

$$\|F(t, u_1, u_2)\|_{\mathcal{P}} = \sup\{|f| : f \in F(t, u_1, u_2)\} \leq |a_1(t)| + b_1|u_1| + b_2|u_2|.$$

(B7) There exist constants $l_1, l_2 > 0$ such that

$$H_d(F(t, x, z), F(t, \bar{x}, \bar{z})) < l_1|x - \bar{x}| + l_2|z - \bar{z}|$$

for every $x, \bar{x}, z, \bar{z} \in \mathbb{R}$.

(B8) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ has the property that $F(., u_1, u_2) : J \to \mathcal{P}_{cp}$ is measurable, and integrable bounded for each $u_1, u_2 \in \mathbb{R}$.

Our first result is based of Bohnenblust-Karlin fixed point theorem.

Theorem 3.10 *Assume that the assumptions(B1)-(B6)-(B7)are satisfied. If*

$$\left(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \right) < 1, \tag{3.21}$$

then the problem(3.13)-(3.14) has at least one solution

Remark 3.11 *Note that for an L^1- Carathéodory multifunction $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ the set $S^1_{F,y}$ is not empty*

Proof. Transform the problem (3.13)-(3.14) into a fixed point problem. Consider the multi-valued operator,

$$N : L^1(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$$

$$(Nx)(t) = \left\{ h \in L^1(J, \mathbb{R}) \quad \begin{array}{rcl} h(t) & = & ay_1 - \dfrac{a \sum_{k=1}^m a_k}{\Gamma(\alpha)} \displaystyle\int_1^{t_k} \left(\log \frac{t}{s} \right)^{\alpha - 1} v(s) \frac{ds}{s} \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s} \right)^{\alpha - 1} v(s) \frac{ds}{s} \quad v \in S^1_{F,x} \end{array} \right\} \tag{3.22}$$

where $v \in S^1_{F,x}$ Clearly, from theorem(1.41)the fixed points of N are solutions to(3.13)-(3.14).We shall show that N satisfies the assumptions of Bohnenblust-Karlin fixed point theorem.

Let

$$r \geq \frac{ay_1||T - 1| + \dfrac{2(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \|a_1\|_{L^1} + \dfrac{2b_1|ay_1|(\log T)^{\alpha}}{\Gamma(\alpha + 1)}}{1 - \left(\dfrac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha + 1)} + \dfrac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \right)}$$

and consider the bounded set

$$B_r = \{x \in L^1(J;\mathbb{R}), \|x\|_{L^1} \leq r\}.$$

The proof will be given in several steps.

Step 1: $N(x)$ *is convex for each* $y \in L^1(J,\mathbb{R})$
Indeed, if h_1 , h_2 belong to $N(y)$ then there exist $v_1, v_2 \in S^1_{F,y}$ such that for each $t \in J$ we have,for $i = 1,2$

$$\begin{aligned} h_i(t) &= ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v_i(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} v_i(s)\frac{ds}{s} \end{aligned}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$ we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} (dv_1 + (1-d)v_2)(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} (dv_1 + (1-d)v_2)(s)\frac{ds}{s}, \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have.

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2: $N(B_r)$ *is relatively compact.*

(a) $N(B_r)$ *is Bounded*

Let $y \in B_r$ for each $h \in N(x)$ and $t \in J$, we have by (B6),

$$
\begin{aligned}
\|h\|_{L^1} &= \int_1^T |h(t)|dt \\
&= \int_1^T \left| ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} \right| dt \\
&\leq |ay_1||T-1| + 2\int_1^T \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} \right| dt \\
&\leq |ay_1||T-1| + 2\int_1^T \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)|\frac{ds}{s} \right) dt \\
&\leq |ay_1||T-1| + 2\int_1^T \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |a_1(t)|\frac{ds}{s} \right) dt \\
&+ 2b_1 \int_1^T \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |ay_1| \right) dt \\
&+ 4b_1 \int_1^T \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |x(s)|\frac{ds}{s}| \right) dt \\
&+ 2b_2 \int_1^T \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |x(s)|\frac{ds}{s} \right) dt \\
&\leq |ay_1||T-1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}\|a_1\|_{L^1} + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}\|x\|_{L^1} \\
&+ \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)}\|x\|_{L^1} \\
&\leq |ay_1||T-1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}\|a_1\|_{L^1} + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} \\
&+ \left(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)} \right) r \leq r
\end{aligned}
$$

Then the above inequalities show that

$$\|N(x)\| = \sup\{\|h\|_{L^1} : h \in N(x)\},$$

which proves that $N(B_r) \subset B_r$ and B_r is bounded,then $N(B_r)$ is bounded.

- (b)$(Nx)_c \to (Nx)$,in $L^1(J, \mathbb{R})$ for each $x \in B_r$

Let $x \in B_r$ and $h \in N(x)$ then we have

$$\begin{aligned}
&\|h_c - h\|_{L^1} \\
&= \int_1^T |h_c(t) - h(t)|dt \\
&= \int_1^T \left| \frac{1}{c} \int_t^{t+c} h(s)ds - h(t) \right| dt \\
&= \int_1^T \left(\frac{1}{c} \int_t^{t+c} |h(s) - h(t)|ds \right) dt
\end{aligned}$$

Since $a_1,\ x \in L^1(J, \mathrm{I\!R})$ We have

$$\begin{aligned}
& \frac{1}{c}\int_t^{t+c} \Bigg| \left(ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{s_k} \left(\log\frac{s}{\tau}\right)^{\alpha-1} v(s)\frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\alpha-1} v(s)\frac{d\tau}{\tau}ds \right) \\
- & \left(ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} \right) \Bigg| dt \\
\leq & \frac{1}{c}\int_t^{t+c} (|ay_1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}|a_1(s)| + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}|x(s)| \\
+ & \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)}|x(s)|) - (|ay_1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}|a_1(t)| + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}|x(t)| \\
+ & \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)}|x(t)|)dt \\
\leq & \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}\frac{1}{c}\int_t^{t+c} |a_1(s) - a_1(t)|dt \\
+ & \left(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)} \right) \frac{1}{c}\int_t^{t+c} |x(s) - x(t)|dt \to 0.
\end{aligned}$$

Hence

$$(Nx)_c \to (Nx) \text{ uniformly as } c \to 0.$$

As a consequence of (a) and (b)together with the Kolmogorov compactness compactness criterion, we can conclude that $N(B_r)$ is relatively compact

Step 3: *Nhas a closed graph.*
Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$ We need to show that $h_* \in N(x_*)$.Now $h_n \in N(x_n)$ implies there exists $v_n \in S^1_{F,x_n}$ such that, for each $t \in J$

$$\begin{aligned}
h_n(t) &= ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}
\end{aligned}$$

We show that there exists $v_* \in S^1_{F,x_*}$ such that for each $t \in J$

$$\begin{aligned} h_*(t) &= ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s)\frac{ds}{s} \quad v_* \in S^1_{F,x_*}. \end{aligned}$$

Since $F(t,.,.)$ is upper semi-continuous,for every ,$\epsilon > 0$,there exist $n_0(x)$ such that for every $n \geq n_0$, we have $v_n \in F(t,y(t),x(t)) \subset F(t,y_*(t),x_*(t)) + \epsilon B(0,1)$ a,e $t \in J$ since F has compact values, there exists a subsequence $v_{n_m}(.)$ such that

$$v_{n_m}(.) \to v_* \text{ as } m \to \infty$$

$$v_* \in F(t,y_*(t),x_*(t)) \text{ as } t \in J$$

; For every $w(t) \in F(t,y_*(t),x_*(t))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w(t)| + |w(t) - v_*|$$

and so

$$|v_{n_m} - v_*| \leq d(v_{n_m}(t), F(t,y_*(t),x_*(t)))$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_* it follows that

$$\begin{aligned} |v_{n_m} - v_*| &\leq H_d(F(t,y_{n_m}(t),x_{n_m}(t)), F(t,y_*(t),x_*(t))) \\ &\leq l_1|y_{n_m} - y_*| + l_2|x_{n_m} - x_*| \\ &\leq l_1|I^\alpha(x_* - x_{n_m})|_{t=t_k} + I^\alpha(x_{n_m} - x_*)| + l_2|x_{n_m} - x_*| \\ &\leq 2l_1|I^\alpha(x_{n_m} - x_*)| + l_2|x_{n_m} - x_*|. \end{aligned}$$

Therefore,

$$|h_{n_m}(t) - h_*(t)| \leq \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_{n_m} - v_*|\frac{ds}{s}$$

$$\|h_{n_m} - h_*\|_{L^1} \leq \left(\frac{4l_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2l_2(\log T)^\alpha}{\Gamma(\alpha+1)}\right) \|x_{n_m} - x_*\|_{L^1}$$

Then

$$\|h_{n_m} - h_*\|_{L^1} \to 0 \text{ as } m \to 0.$$

Therefore, we deduce from Bohnenblust-Karlin fixed point theorem that N has a fixed point x in $B_r \subset L^1(J, \mathbb{R})$ which is a solution of the nonlocal problem(3.13)-(3.14). This completes the proof. □

3.2.2 The non-convex case

We present now a result for the problem (3.13)-(3.14) with a non-convex valued right hand side.Our considerations are based on the fixed point result in the theorem for contraction multi-valued maps given by Covitz-Nadler 1.42

Theorem 3.12 *Assume that the assumptions(B7)and(B8)are satisfied.If:*

$$\left(\frac{2(l_1+l_2)(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right) < 1, \tag{3.23}$$

then the problem (3.13)-(3.14)has at least one solution $y \in L^1(J, \mathbb{R})$.

Proof : For each $y \in L^1(J, \mathbb{R})$ the set $S^1_{F,y}$is nonempty since, by $(B8)$ F has a measurable selection (see([41],Theorem III.6)).We shall show that N given by(3.22) satisfies the assumptions of Covitz and Nadler fixed point theorem. The proof will be given in two steps.

Step 1: $N(x) \in P_cl(L^1(J.\mathbb{R}))$ *for all* $x \in L^1(J, \mathbb{R})$.
Let $(h_n)_{n\leq 0}) \in N(x)$ be such that $h_n \to h \in L^1(J.\mathbb{R})$.Then there exists $v_n \in S^1_{F,y}$ such that, for each $t \in J$

$$h_n(t) = ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}.$$

From (B8) and the fact that F has compact values, we may pass to a subsequence if necessary to obtain that v_n converges to v in $L^1(J.\mathbb{R})$ and hence $v \in S^1_{F,y}$.Thus, for each $t \in J$

$$h_n(t) \to \tilde{h}(t) = ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}$$

so $\tilde{h} \in N(x)$.

Step2: *There exists* $\gamma < 1$ *such that* $H_d(N(y), N(\bar{y})) < \gamma\|y - \bar{y}\|_\infty$ *for each* $y, \bar{y} \in C(J, \mathbb{R})$.

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$,then there exists $v_1 \in F(t, y(t), x(t))$ such that for each $t \in J$

$$h_1(t) = ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s}.$$

From (B7) it follows that

$$H_d(F(t, y(t), x(t)), F(t, \bar{y}(t), \bar{x}(t)) \leq l_1|y(t) - \bar{y}(t)| + l_2|x(t) - \bar{x}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t), \bar{x}(t))$ such that

$$|v_1(t) - w| \leq l_1|y(t) - \bar{y}(t)| + l_2|x(t) - \bar{x}(t)|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l_1|y(t) - \bar{y}(t)| + l_2|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t), \bar{x}(t)))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V, so $v_2 \in F(t, \bar{y}(t), \bar{x}(t))$,and for each $t \in J$

$$|v_1(t) - v_2(t)| \leq l_1|y(t) - \bar{y}(t)| + l_2|x(t) - \bar{x}(t)|, \quad t \in J$$

Let us define

$$h_2(t) = ay_1 - \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} v_2(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_2(s)\frac{ds}{s}.$$

Then for each $t \in J$,we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha)-1} |v_1(t) - v_2(t)|\frac{ds}{s} \\
&+ \frac{a\sum_{k=1}^m a_k}{\Gamma(\alpha)} \int_1^{t_k} \left(\log \frac{t}{s}\right)^{\alpha-1} |v_2(t) - v_1(t)|\frac{ds}{s} \\
&\leq \frac{2(\log(T))^\alpha}{\Gamma(\alpha+1)}|v_1(t) - v_2(t)|. \\
&+ \left(\frac{2(\log(T))^\alpha}{\Gamma(\alpha+1)}\right)(l_1(t)|y(t) - \bar{y}(t)| + l_2(t)|x(t) - \bar{x}(t)|)
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_{L^1} \leq \left(\frac{(l_1 + l_2)2(\log(T))^\alpha}{\Gamma(\alpha+1)}\right)\|x - \bar{x}\|_{L^1}.$$

For an analogous relation, obtained by interchanging the roles of x and $\bar{x}$ it follows that

$$H_d(N(x), N(\bar{x})) \leq \left(\frac{(l_1 + l_2)2(\log(T))^\alpha}{\Gamma(\alpha+1)}\right)\|x - \bar{x}\|_{L^1}.$$

So by (4.31),N is a contraction and thus, by theorem 1.42 , N has a fixed point x which is the solution to (3.13)-(3.14).The proof is complete.. □

3.2.3 An example

We apply Theorem 3.10 to the following implicit fractional differential inclusion

$${}^C_H D^\alpha y(t) \in F(t, y(t), {}^C_H D^\alpha y(t)), \text{ for a.e. } t \in J = [1, e],\ 0 < \alpha \leq 1 \tag{3.24}$$

$$\sum_{1}^{m} a_k y(t_k) = 1, \tag{3.25}$$

where

$$F(t, y(t), {}^{C}_{H}D^{\alpha}y(t)) = \{v \in \mathbb{R} : f_1(t, y(t), {}^{C}_{H}D^{\alpha}y(t)) \leq v \leq f_2(t, y(t), {}^{C}_{H}D^{\alpha}y(t))\},$$

and $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1, e]$, $f_1(t, \cdot, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y(t), {}^{c}D^{\alpha}y(t)) > \mu_1\}$ is open for each $\mu_1 \in \mathbb{R}$), and assume that for each $t \in [1, e]$, $f_2(t, \cdot, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y(t), {}^{c}D^{\alpha}y(t)) < \mu_2\}$ is open for each $\mu_2 \in \mathbb{R}$). Assume that there are $a \in L^1([1, e], \mathbb{R}^+)$ such that

$$\max(f_1(t, y(t), x(t)), f_2(t, y(t), x(t))) \leq \frac{t}{9} + \frac{1}{16}|y(t)| + \frac{1}{16}|x(t)|, t \in J.$$

We have $T = e$,$a(t) = \frac{t}{9}$, $b_1 = b_2 = \frac{1}{16}$ It is easy to see that

$$\left(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right) = \left(\frac{1}{4\Gamma(2\alpha+1)} + \frac{1}{8\Gamma(\alpha+1)}\right) < 1.$$

Since all the conditions of Theorem 3.10 are satisfied, problem (3.24)-(3.25) has at least one solution y on $[1, e]$.

Chapter 4

BVP for Fractional Differential Inclusions with Hadamard-Caputo Fractional Derivative

4.1 BVP for Hadamard-caputo fractional differential inclusion with nonlocal condition

[1] In this chapter, we study the existence of solution for boundary values problem fractional differential inclusion,

$$ {}^H_c D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \quad 1 < r \leq 2, \tag{4.1} $$

$$ y(1) = y_1,\ y(T) = g(y), \tag{4.2} $$

where ${}^H_c D^r$ is the Caputo-Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $g : C(J, \mathbb{R}) \to \mathbb{R}$ a continuous function.

In this section, we present two existence results for the problem (4.1)-(4.2) when the right hand side is convex as well as non-convex valued. The first result relies on the nonlinear alternative of Leray Schauder type, while the second one is based upon a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler.

4.1.1 The convex case

Let us start by defining what we mean by a solution of the problem (4.1)-(4.2).

Definition 4.1 *A function $y \in AC^2_\delta([1, T], \mathbb{R})$ is said to be a solution of (4.1)-(4.2) if there exists a function $v \in L^1([1, T], \mathbb{R})$ with $v(t) \in F(t, y(t))$ for a.e $t \in [1, T]$ such that ${}^H_c D^r y(t) = v(t)$ and the function y satisfies conditions (4.2).*

[1] **A.Zahed** and S. Hamani, Boundary Value Problems for Hadamard-Caputo Fractional Differential Inclusions with Nonlocal Conditions, *PanAmerican Mathematical Journal* **29** ,(2019) No 4, 15-28..

For the existence of solutions for the problem (4.1)-(4.2), we prove the lemma :

Lemma 4.2 *Let* $h : [1,T] \to \mathbb{R}$ *be a continuous function. A function* y *is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}}{\log(T)}\log(t) \end{aligned} \tag{4.3}$$

if and only if y *is a solution of the fractional boundary value problem*

$${}^H_c D^r y(t) = h(s) \text{ for a.e. } t \in [1,T], 1 < r \leq 2 \tag{4.4}$$

$$y(1) = y_1; y(T) = g(y). \tag{4.5}$$

Proof. Applying the Caputo- Hadamard fractional integral of order r to both sides of (4.4) , and by using Lemma (1.24) , we find Assume y satisfies (4.5). Then Lemma 1.24 implies that

$$y(t) = c_1 + c_2\log(t) +^H I^r h(t). \tag{4.6}$$

The boundary condition (4.5) implies that

$$c_1 = y_1$$

and

$$y(T) = \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s} + y_1 + c_2(\log T).$$

Hence,

$$c_2 = \frac{g(y) - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}}{(\log T)}.$$

Finally, we obtain the solution (4.3)
Inversely, it is clear that if y satisfies equation (4.3), then equations (4.4)-(4.5) hold. □

Theorem 4.3 *Assume the following hypothesis hold :*

(H́1) $F : J \times \mathbb{R} \to P_{cp,c}(X)$ *is a Carathéodory multi-valued map.*

(H́2) There exist $p \in L^1(J, \mathbb{R}^+)$ *and* $\psi : [0;\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$\|F(t,u)\|_P \leq p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

(H́3) There exists $l \in L^1([1;T];\mathbb{R})$ *, such that*

$$H_d(F(t,u), F(t,\bar{u})) < l(t)|u - \bar{u}| \text{ for every } u, \bar{u} \in \mathbb{R}$$

(H́4) There exist a constant $p^* > 0$ *and* $\psi^* : [0;\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$|g(y)| \leq p^*\psi^*(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

(H́5) There exists a number $M > 0$*, such that*

$$\frac{M}{\frac{2\|p\|_{L^1}(T-1)(\log(T))^r}{\Gamma(r+1)}\psi(M) + 2|y_1| + p^*\psi^*(M)} > 1 \tag{4.7}$$

Then the problem (4.1)-(4.2) has at least one solution on J

Proof. Transform the problem (4.1)-(4.2) into a fixed point problem. Consider the multi-valued operator,

$$N(y) = \left\{ h \in C(J, \mathbb{R}) \quad \begin{array}{rcl} h(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} v(s) \frac{ds}{s} + y_1 \\ & + & \frac{g(y) - y_1 - \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{T}{s} \right)^{r-1} v(s) \frac{ds}{s}}{\log(T)} \log(t) \quad v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 4.2 , the fixed points of N are solutions to (4.1)-(4.2) . We shall show that N satisfies the assumptions of nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each y .*
Indeed, if h_1 , h_2 belong to $N(y)$ then there exist v_1,v_2 such that for each $t \in J$ we have,for $i = 1, 2$

$$\begin{array}{rcl} h_i(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} v_i(s) \frac{ds}{s} + y_1 \\ & + & \frac{g(y) - y_1 - \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} v_i(s) \frac{ds}{s}}{\log(T)} \log(t). \end{array}$$

Let $0 \le d \le 1$. Then, for each $t \in J$ we have

$$\begin{array}{rcl} (dh_1 + (1-d)h_2)(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} (dv_1 + (1-d)v_2)(s) \frac{ds}{s} + y_1 \\ & + & \frac{g(y) - y_1 - \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} (dv_1 + (1-d)v_2)(s) \frac{ds}{s}}{\log(T)} \log(t). \end{array}$$

Since $S_{F,y}$ is convex (because F has convex values), we have;

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in $C(J, \mathbb{R})$.*

Let B_{μ_*}=$\{y \in C(J, \mathbb{R}) : \|y\|_\infty \le \mu_*\}$ be a bounded set in $C(J, \mathbb{R})$ and $y \in B_{\mu_*}$ Then for each $h \in N(y)$ there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s}}{\log(T)} \log(t). \end{aligned}$$

By (H́2), we have, for each $t \in J$

$$\begin{aligned} |h(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s} + |y_1| \\ &+ \frac{|g(y)| + |y_1| + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s}}{\log(T)} \log(t) \\ &\leq \frac{2}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v(s)| \frac{ds}{s} + 2|y_1| + |g(y)| \\ &\leq \frac{2(\log(T))^r}{\Gamma(r+1)} \int_1^T p(t)\psi(|y(s)|) ds + 2|y_1| + p^*\psi^*(|y(s)|). \end{aligned}$$

Thus

$$\|h\|_\infty \leq \frac{2(\log(T))^r \psi(\mu_*)}{\Gamma(r+1)} \|p\|_{L^1} + 2|y_1| + p^*\psi^*(\mu_*) := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$*
Let $t_1, t_2 \in J$, $t_1 < t_2$ and let B_{μ_*} be bounded set of $C(J, \mathbb{R})$ as in Step 2. Let $y \in B_{\mu_*}$ and $h \in N(y)$. Then

$$\begin{aligned} |h(t_2) - h(t_1)| &= \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{v(s)}{s} ds \right. \\ &+ \left. \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{v(s)}{s} ds \right| \\ &\leq \frac{p(t)\psi(|y(s)|)}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} \\ &+ \frac{p(t)\psi(|y(s)|)}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\ &\leq \frac{\|p\|_{L^1}\psi(\mu_*)}{\Gamma(r)} \int_1^{t_1} \left[\left(\log(\frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} \\ &+ \frac{\|p\|_{L^1}\psi(\mu_*)}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \end{aligned}$$

As $t_1 \to t_2$, the right side of the above inequality tens to zero. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : C(J; \mathbb{R}) \to P(C(J; \mathbb{R}))$ is completely continuous.

Step 4: *Nhas a closed graph.*
Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$ We need to show that $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in J$

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \dfrac{1}{\Gamma(r)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s}}{\log(T)} \log(t). \end{aligned}$$

We must show that there exists $v_* \in S_{F,y^*}$ such that, for each

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_*(s) \frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \dfrac{1}{\Gamma(r)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_*(s) \frac{ds}{s}}{\log(T)} \log(t). \end{aligned}$$

Since $F(t, .)$ is upper semi-continuous, then for every $\epsilon > 0$ there exists a natural number $n_0(\epsilon)$ such that, for every $n > n_0$,we have
$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0, 1)$. a.e $t \in J$
Since $F(., .)$ has compact values, then there exists a subsequence $v_{n_m}(.)$ such that $v_{n_m}(.) \to v_*$ as $m \to \infty$,and $v_*(t) \in F(t, y_*(t))$ a.e.$t \in J$
For every $w \in F(t, y_*(t))$,we have

$$|v_{n_m} - v_*| \le |v_{n_m} - w| + |w - v_*|.$$

Then

$$|v_{n_m} - v_*| \le d(v_{n_m}, F(t, y_*(t))),$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_* , and it follows that

$$|v_{n_m} - v_*| \le H_d(F(t, y_n(t)), F(t, y_*(t))) \le l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned}|h_n(t)-h_*(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{|v_n-v_*|}{s}ds\\ &+ \frac{\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}\frac{|v_n-v_*|}{s}ds}{\log(T)}\log(t)\\ &\leq \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{|l(s)}{s}ds\|y_{n_m}-y_*(t)\|_\infty\\ &+ \frac{\log(t)}{\Gamma(r)\log(T)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{|l(s)|}{s}ds\|y_{n_m}-y_*\|_\infty\\ &\leq \frac{\|l\|_{L^1}}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{1}{s}ds\|y_{n_m}-y_*(t)\|_\infty\\ &+ \frac{\|l\|_{L^1}}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{1}{s}ds\|y_{n_m}-y_*\|_\infty.\end{aligned}$$

Hence

$$\|h_n-h_*\|_\infty \leq \frac{2\|l\|_{L^1}(\log T)^r}{\Gamma(r+1)}\|y_{n_m}-y_*\|_\infty \to 0.$$

as $m\to\infty$.

Step 5:*A priori bounds on solutions.*
Let y be such that $y\in\lambda N(y)$ with $\lambda\in(0,1]$ Then there exists $v\in S_{F,y}$ such that, for each $t\in J$,

$$\begin{aligned}h(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}+y_1\\ &+ \frac{g(y)-y_1\Gamma(r)\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}}{\log(T)}\log(t).\end{aligned}$$

This implies by ('H2) that, for each $t\in J$, we have

$$\begin{aligned}|h(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}|v(s)|\frac{ds)}{s}+|y_1|\\ &+ \frac{|g(y)|+|y_1|\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}|v(s)|\frac{ds}{s}}{\log(T)}\log(t)\\ &\leq \frac{2}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}|v(s)|\frac{ds}{s}+2|y_1|+|g(y)|\\ &\leq \frac{2(\log(T))^r}{\Gamma(r+1)}\int_1^T p(t)\psi(|y(s)|)ds+2|y_1|+p^*\psi^*(|y(s)|)\\ &\leq \frac{2\|p\|_{L^1}(\log(T))^r}{\Gamma(r+1)}\int_1^T\psi(\|y\|_\infty)ds+2|y_1|+p^*\psi^*(\|l\|_\infty)\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{\frac{2\|p\|_{L^1}(T-1)(\log(T))^r}{\Gamma(r+1)}\psi(\|y\|_\infty)+2|y_1|+p^*\psi^*(\|y\|_\infty)}<1.$$

Then by condition (4.7), there exists $M>0$ such that $\|y\|_\infty \neq M$. Let $U=\{y\in C(J,\mathbb{R}):\|y\|_\infty<M\}$.The operator $N:\bar{U}\to\mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N(y)$.for some $\lambda\in(0,1]$.As a consequence of the nonlinear alternative of Leray-Shauder, we deduce that N has a fixed point $y\in\bar{U}$ which is a solution of the problem (4.1)-(4.2).This completes the proof. □

4.1.2 The non-convex case .

We present the result for the problem (4.1)-(4.2) with a non-convex valued right hand side. Our considerations are based on the fixed point result in the theorem of Covitz and Nadler 1.42

Theorem 4.4 *Assume($\acute{H}3$) and the following hypotheses hold:*

($\acute{H}6$) $F:J\times\mathbb{R}\to\mathcal{P}_{cp}(X)$ has the property that,
$F(.,u):J\to\mathcal{P}_{cp}(X)$ is measurable for each $u\in\mathbb{R}$.

($\acute{H}7$) there exists $k^*>0$ with such that
$|g(u)-g(\bar{u})|\leq k^*|u-\bar{u}|$ for $u,\bar{u}\in C(J,\mathbb{R})$.

If

$$\frac{2(\log(T))^r}{\Gamma(r+1)}\|l\|_{L^1}+k^*<1, \tag{4.8}$$

then the problem (4.1)-(4.2) has at least one solution on J.

Remark 4.5 *By ($\acute{H}6$), we can see that $S_{F,y}$ is nonempty for each $y\in C(J,\mathbb{R})$, so F has a measurable selection (see [41], Theorem III.6)*

Proof. We shall show that N satisfies the assumptions of the theorem of Covitz and Nadler1.42. The proof will be given in two steps.

Step1: *$N(y)\in\mathcal{P}_{cl}(C(J;\mathbb{R}))$ for each $y\in C(J,\mathbb{R})$*
Indeed, let $(y_n)_{n\geq 0}\subset N(y)$ be such that $y_n\to\bar{y}$ in $C(J,\mathbb{R})$,then $\bar{y}$ in $C(J,\mathbb{R})$ and there exists $v_n\in S_{F,y}$ such that for each $t\in J$.

$$\begin{aligned} y_n(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}+y_1 \\ &+ \frac{g(y)-y_1-\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}}{\log(T)}\log(t). \end{aligned}$$

Using the fact that F has compact values and from (H́3) we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazurs theorem implies that v_n converges strongly to v and hence$v \in S_{F,y}$ Then for each $t \in J$

$$\begin{aligned} y_n(t) \to \bar{y}(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}}{\log(T)}\log(t). \end{aligned}$$

So,$\bar{y} \in N(y)$

Step2: *There exists $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) < \gamma\|y - \bar{y}\|_\infty$ for each $y, \bar{y} \in C(J,\mathbb{R})$*

Let $y, \bar{y} \in C(J,\mathbb{R})$ and $h_1 \in N(y)$,then there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} y_1(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_1(s)\frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_1(s)\frac{ds}{s}}{\log(T)}\log(t). \end{aligned}$$

From (H́3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y}(t)) \le l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, y(t))$ such that

$$|v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|, t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V . so $V_2 \in F(t, \bar{y}(t))$,and for each $t \in J$

$$|v_1(t) - v_2(t)| \le l(t)|y(t) - \bar{y}(t)|, t \in J$$

Let us define for each $v_2 \in J$

$$\begin{aligned} y_2(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_2(s)\frac{ds}{s} + y_1 \\ &+ \frac{g(y) - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_2(s)\frac{ds}{s}}{\log(T)}\log(t) \end{aligned}$$

Then for each $t \in J$

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_1(t) - v_2(t)| \frac{ds}{s} \\
&+ \frac{\log(t)}{\log(T)} \left[|g(y) - g(\bar{y})| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{t}{s}\right)^{r-1} |v_1(t) - v_2(t)| \frac{ds}{s} \right] \\
&\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \\
&+ k^*|y - \bar{y}| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \\
&\leq \left[2\frac{(\log(T))^r}{\Gamma(r+1)} \int_1^T l(s) ds + k^* \right] \|y - \bar{y}\|_\infty
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \left[\frac{2(\log(T))^r}{\Gamma(r+1)} \|l\|_{L^1} + k^* \right] \|y - \bar{y}\|_\infty$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$ it follows that

$$H_d(N(y), N(\bar{y})) \leq \left[\frac{2(\log(T))^r}{\Gamma(r+1)} \|l\|_{L^1} + k^* \right] \|y - \bar{y}\|_\infty$$

So by (4.8),N is a contraction and thus, by theorem of Covitz and Nadler 1.42 , N has a fixed point y which is solution to (4.1)-(4.2) . The proof is complete. □

4.1.3 An example

We take:

$$ {}^H_c D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, e], \quad 1 < r \leq 2, \tag{4.9}$$

$$y(1) = 0, \; y(e) = \sum_{i=1}^{n} c_i y(t_i), \tag{4.10}$$

where $0 < t_1 < t_2 < ... < t_n < 1, c_i, i = 1, .., n$ are given positives constants with $\sum_{i=1}^n c_i < \frac{4}{5}$. We set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\}$$

where $f_1, f_2 : [1, e] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1, e]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1, e]$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C([1, e], \mathbb{R}^+)$ and $\psi : [0, \infty) \mapsto (0, \infty)$ continuous and nondecreasing such that

$$\begin{aligned}
\|F(t, u)\|_{\mathcal{P}} &= \sup\{|v|, v(t) \in F(t, y)\} \\
&= \max(|f_1(t, y)|, |f_2(t, y)|) \leq p(t), \quad \text{for each } t \in [1, e], y \in \mathbb{R}.
\end{aligned}$$

Assume that there is $\psi^* : [0, \infty) \mapsto (0, \infty)$ continuous and nondecreasing such that

$$|g(u)| \leq \frac{4}{5}\psi^*(|u|), \quad \text{for each } u \in \mathbb{R}.$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous.
Finally we assume that there exists a number $M > 0$ such that

$$\frac{M}{\dfrac{2(e-1)\psi(M)}{\Gamma(r+1)}\|p\|_{L^1} + \dfrac{4}{5}\psi^*(M)} > 1. \tag{4.11}$$

Since all the conditions of Theorem 4.3 are satisfied, problem (4.9)-(4.10) has at least one solution y on $[1, e]$.

4.2 BVP for fractional differential inclusions and Caputo-Hadamard fractional derivative with integral conditions

[2]

4.2.1 Introduction

For $r \in (1, 2]$ in this section, we are concerned with the existence of solution of boundary value problems for a fractional differential inclusion,

$${}^{H}_{c}D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \tag{4.12}$$

$$y(1) - y'(1) = \int_1^T g(s, y(s))ds, \tag{4.13}$$

$$y(T) + y'(T) = \int_1^T h(s, y(s))ds, \tag{4.14}$$

where ${}^{H}_{c}D^r$ is the Caputo-Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $g, h : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In this Section we shall present two existence results for the problem (4.12)-(4.14) when the right hand side is convex, as well as non-convex, valued. The first result relies on the nonlinear alternative of Leray Schauder type, while the second one is based upon a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler.

[2] **A.Zahed** , S. Hamani and J. Henderson, Boundary Value Problems for Caputo-Hadamard Fractional Differential Inclusions with Integral Conditions *Moroccan J. of Pure and Appl. Anal.***6** (2020), No. ,62-75.

4.2.2 The convex case

Definition 4.6 *A function $y \in AC_\delta^2([1,T],\mathbb{R})$ is said to be a solution of (4.12)-(4.14) if there exists a function $v \in L^1([1,T],\mathbb{R})$ with $v(t) \in F(t,y(t))$ for a.e. $t \in [1,T]$ such that ${}_c^H D^r y(t) = v(t)$, and the function y satisfies conditions (4.13)-(4.14).*

Lemma 4.7 *Let $1 < r \leq 2$, and let $h, \rho_1, \rho_2 : [1,+\infty) \to \mathbb{R}$ be continuous functions. A function y is a solution of the fractional integral equation*

$$y(t) = P(t) + \int_1^T G(t,s)h(s)\frac{ds}{s} \tag{4.15}$$

if and only if y is a solution of the fractional boundary value problem

$$ {}_c^H D^r y(t) = h(t) \text{ for a.e. } t \in [1,T], \tag{4.16}$$

$$y(1) - y'(1) = \int_1^T \rho_1(s)ds, \tag{4.17}$$

$$y(T) + y'(T) = \int_1^T \rho_2(s)ds, \tag{4.18}$$

where

$$P(t) = \frac{(\frac{1}{T} - \log(T) + \log(t))}{T^*}\int_1^T \rho_1(s)ds - \frac{(1+\log(t))}{T^*}\int_1^T \rho_2(s)ds, \tag{4.19}$$

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-1} + \frac{1+\log(t)}{T^*\Gamma(r)}\left(\log\frac{T}{s}\right)^{r-1} \\ -\frac{(1+\log(t))(r-1)}{TT^*\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-2}, & 0 < s < t, \\ \frac{1+\log(t)}{T^*\Gamma(r)}\left(\log\frac{T}{s}\right)^{r-1} + \frac{1+\log(t)(r-1)}{TT^*\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-2}, & t < s < T, \end{cases} \tag{4.20}$$

and

$$T^* = \frac{1 - T - T\log(T)}{T}.$$

Proof. Assume y is a solution of (4.16)-(4.18). Applying the Caputo-Hadamard fractional integral of order r to both sides of (4.16), and by using Lemma 1.24, we find

$$y(t) = c_1 + c_2\log(t) + {}^H I^r h(t) \tag{4.21}$$

From (4.17) and (4.18), we get

$$c_1 - c_2 = \int_1^T \rho_1(s)ds \tag{4.22}$$

and

$$\begin{aligned} & c_1 + c_2(\log(T) - \frac{1}{T}) + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s} \\ & + \frac{(r-1)}{T\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s} = \int_1^T \rho_2(s)ds. \end{aligned} \tag{4.23}$$

Equations (4.22)and (4.23) give

$$c_2 = \frac{1}{(-1+\frac{1}{T}-\log(T))}\Big(\int_1^T \rho_1(s)ds - \int_1^T \rho_2(s)ds + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}$$

$$-\frac{(r-1)}{T\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s}\Big) \tag{4.24}$$

and

$$\begin{aligned} c_1 = & \int_1^T \rho_1(s)ds + \frac{1}{(-1+\frac{1}{T}-\log(T))}\Big(\int_1^T \rho_1(s)ds - \int_1^T \rho_2(s)ds \\ & + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s} - \frac{(r-1)}{T\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s}\Big). \end{aligned} \tag{4.25}$$

From (4.21), (4.24) and (4.25), and using the fact that $\int_1^T = \int_1^t + \int_t^T$, we get

$$P(t) = \frac{(\frac{1}{T} - \log(T) + \log(t))}{T^*} \int_1^T \rho_1(s)ds - \frac{(1+\log(t))}{T^*} \int_1^T \rho_2(s) \tag{4.26}$$

and

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(r)} \left(\log \frac{t}{s}\right)^{r-1} + \frac{1+\log(t)}{T^*\Gamma(r)} \left(\log \frac{T}{s}\right)^{r-1} \\ -\frac{(1+\log(t))(r-1)}{TT^*\Gamma(r)} \left(\log \frac{t}{s}\right)^{r-2}, \quad 0 < s < t, \\ \frac{1+\log(t)}{T^*\Gamma(r)} \left(\log \frac{T}{s}\right)^{r-1} + \frac{1+\log(t)(r-1)}{TT^*\Gamma(r)} \left(\log \frac{t}{s}\right)^{r-2}, \quad t < s < T, \end{cases} \tag{4.27}$$

where

$$T^* = \frac{1 - T - T\log(T)}{T}$$

Finally, we obtain that y is a solution of (4.15).

Conversely, it is clear that if y satisfies equation (4.15), then equations (4.16)-(4.18) are satisfied by y.

□

Theorem 4.8 *Assume (H́3)and the following hypotheses hold:*

(H́8) There exist $\phi_f \in L(J, \mathbb{R}^+)$ and $\psi : [0; \infty) \to (0, \infty)$ continuous and nondecreasing such that $\|F(t, u)\|_P \leq \phi_f(t)\psi(|u|)$ for $t \in J$ and each $u \in \mathbb{R}$.

(H́9) There exist $\phi_g \in L(J, \mathbb{R}^+)$ and $\psi^* : [0; \infty) \to (0, \infty)$ continuous and nondecreasing such that $|g(t, u)| \leq \phi_g(t)\psi^*(|u|)$ for all $t \in J$ and each $u \in \mathbb{R}$.

(H́10) There exist $\phi_h \in L(J, \mathbb{R}^+)$ and $\overline{\psi} : [0; \infty) \to (0, \infty)$ continuous and nondecreasing such that $|h(t, u)| \leq \phi_h(t)\overline{\psi}(|u|)$ for all $t \in J$ and each $u \in \mathbb{R}$.

(H́11) There exists a number $M > 0$, such that

$$\frac{M}{A\psi^*(M) + B\overline{\psi}(M) + CG^*\psi(M)} > 1, \tag{4.28}$$

where

$$G^* = \|G\|_L,$$

and

$$A = \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|} \int_1^T \phi_g(s)ds,$$

$$B = \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|} \int_1^T \phi_h(s)ds,$$

and

$$C = \int_1^T \phi_f(s)ds.$$

Then the problem (4.12)-(4.14) has at least one solution on J.

Proof. Transform the problem (4.12)-(4.14) into a fixed point problem. Consider the multi-valued operator,

$$N(y) = \left\{ h \in C(J, \mathbb{R}) : h(t) = P_y(t) + \int_1^T G(t, s)v(s)\frac{ds}{s}, v \in S_{F,y} \right\}, \tag{4.29}$$

where

$$P_y(t) = \frac{(\frac{1}{T} - \log(T) + \log(t))}{T^*} \int_1^T g(s, y(s))ds - \frac{(1 + \log(t))}{T^*} \int_1^T h(s, y(s))ds \tag{4.30}$$

and the function $G(t, s)$ is given by (4.20). Clearly, from Lemma 4.7, the fixed points of N are solutions to (4.12)-(4.14). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder1.41.The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each y.*
Indeed, if h_1 , h_2 belong to $N(y)$ then there exist v_1,v_2 such that for each $t \in J$, we have,for $i = 1, 2$,

$$h_i(t) = P_y(t) + \int_1^T G(t,s)v_i(s)\frac{ds}{s}.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$ we have

$$(dh_1 + (1-d)h_2)(t) = P_y(t) + \int_1^T G(t,s)(dv_1(s) + (1-d)v_2(s))\frac{ds}{s}.$$

Since $S_{F,y}$ is convex (because F has convex values), we have. $dh_1 + (1-d)h_2 \in N(y)$.

Step 2: *N maps bounded sets into bounded sets in $C(J, \mathbb{R})$.*
Let B_{η^*}:=$\{y \in C(J, \mathbb{R}) : \|y\|_\infty \leq \eta^*\}$ be a bounded set in $C(J, \mathbb{R})$ and $y \in B_{\eta^*}$. Then for each $h \in N(y)$ and $t \in J$, we have from (H́8)-(H́10),

$$\begin{aligned}
|h(t)| &\leq \frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\int_1^T |g(s,y(s))|ds \\
&+ \frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\int_1^T |h(s,y(s))|ds \\
&+ \int_1^T G(t,s)|v(s)|\frac{ds}{s} \\
&\leq \frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\psi^*(\|y\|_\infty)\int_1^T \phi_g(s)ds \\
&+ \frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\overline{\psi}(\|y\|_\infty)\int_1^T \phi_h(s)ds \\
&+ \psi(\|y\|_\infty)\int_1^T \phi_f(s)G^*ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|h\|_\infty &\leq \frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\psi^*(\eta^*)\int_1^T \phi_g(s)ds \\
&+\frac{1+\log(T)}{|-1+\dfrac{1}{T}-\log(T)|}\overline{\psi}(\eta^*)\int_1^T \phi_h(s)ds \\
&+\psi(\eta^*)\int_1^T \phi_f(s)dsG^* \\
&:= l.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*
Let $t_1, t_2 \in J$, $t_1 < t_2$ and let B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in Step 2. Let $y \in B_{\eta^*}$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq \frac{\log(t_2)-\log(t_1)}{|-1+\frac{1}{T}-\log(T)|}\int_1^T |g(s,y(s))|ds \\
qs+ & \frac{\log(t_2)-\log(t_1)}{|-1+\frac{1}{T}-\log(T)|}\int_1^T |h(s,y(s))|ds \\
+ & \int_1^T |G(t_2,s)-G(t_1,s)||v(s)|\frac{ds}{s} \\
\leq & \frac{\log(t_2)-\log(t_1)}{|-1+\frac{1}{T}-\log(T)|}\psi^*(\eta^*)\int_1^T \phi_g(s)ds \\
+ & \frac{\log(t_2)-\log(t_1)}{|-1+\frac{1}{T}-\log(T)|}\overline{\psi}(\eta^*)\int_1^T \phi_h(s)ds \\
+ & \psi(\eta^*)\int_1^T \phi_f(s)ds\int_1^T |G(t_2,s)-G(t_1,s)|\frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : C(J;\mathbb{R}) \to \mathcal{P}(C(J;\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*
Let $y_n \to y_*$, from (H2)-(H4) and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now, $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in J$,

$$h_n(t) = P_{y_n}(t) + \int_1^T G(t,s)v_n(s)\frac{ds}{s}.$$

We must show that there exists $v_* \in S_{F,y}$, such that

$$h_*(t) = P_{y_*}(t) + \int_1^T G(t,s)v_*(s)\frac{ds}{s}.$$

Since $F(t,.)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n > n_0$, we have $v_n(t) \in F(t,y_n(t)) \subset F(t,y_*(t)) + \epsilon B(0,1)$ a.e. $t \in J$.

Since $F(.,.)$ has compact values, then there exists a subsequence $v_{n_m}(.)$ such that $v_{n_m}(.) \to v_*$, as $m \to \infty$, and $v_*(t) \in F(t,y_*(t))$ a.e. $t \in J.$. For every $w \in F(t,y_*(t))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w| + |w - v_*|.$$

Then

$$|v_{n_m} - v_*| \leq d(v_{n_m}, F(t, y_*(t))).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_*, and it follows that

$$|v_{n_m} - v_*| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n(t) - y_*(t)\|_\infty.$$

Therefore,

$$\begin{aligned} |h_{n_m}(t) - h_*(t)| &\leq \int_1^T |g(s, y_{n_m}(s)) - g(s, y_*(s))|ds \\ &+ \int_1^T |h(s, y_{n_m}(s)) - h(s, y_*(s))|ds \\ &+ \int_1^T G(t,s)|v_{n_m}(s) - v_*(s))|ds. \end{aligned}$$

Since

$$\begin{aligned} \int_1^T G(t,s)|v_{n_m}(s) - v_*(s))|ds &\leq \int_1^T G(t,s)\frac{|l(s)|}{s}ds\|y_{n_m}(t) - y_*(t)\|_\infty \\ &\leq G^*\|l\|_\infty\|y_{n_m}(t) - y_*(t)\|_\infty \end{aligned}$$

and g and h are continuous, $\|h_{n_m}(t) - h_*(t)\|_\infty \to 0$ as $m \to \infty$.

Step 5: *A priori bounds on solutions.*
Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0, 1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$h(t) = P_y(t) + \int_1^T G(t,s)v(s)\frac{ds}{s}.$$

This implies by (H́8)-(H́10) that, for each $t \in J$, we have

$$\begin{aligned} |h(t)| &\leq \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|}\int_1^T |g(s, y(s))|ds \\ &+ \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|}\int_1^T |h(s, y(s))|ds \\ &+ \int_1^T G(t,s)|v(s)|\frac{ds}{s} \\ &\leq \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|}\int_1^T \phi_g(s)ds\psi^*(\|y\|_\infty) \\ &+ \frac{1 + \log(T)}{|-1 + \frac{1}{T} - \log(T)|}\int_1^T \phi_h(s)ds\overline{\psi}(\|y\|_\infty) \\ &+ \int_1^T \phi_f(s)dsG^*\psi(\|y\|_\infty). \end{aligned}$$

Thus,

$$\frac{\|y\|_\infty}{A\psi^*(\|y\|_\infty)+B\overline{\psi}(\|y\|_\infty)+CG^*\psi(\|y\|_\infty)}\leq 1.$$

Then by condition (4.28), there exists $M>0$ such that $\|y\|_\infty\neq M$. Let $U=\{y\in C(J,\mathbb{R}):\|y\|_\infty<M\}$. The operator $N:\bar{U}\to\mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N(y)$, for some $\lambda\in(0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that N has a fixed point $y\in\bar{U}$ which is a solution of the problem (4.12)-(4.14). This completes the proof. □

4.2.3 The nonconvex case

We present now a result for the problem (4.12)-(4.14)in the case of nonconvex values.

Theorem 4.9 *Assume (Ή3)- (Ή6)-(Ή7)and the following hypotheses hold:*

(Ή8) There exists $k^{**}>0$ such that

$$|h(t,u)-h(t,\bar{u})|\leq k^{**}|u-\bar{u}| \text{ for each } t\in J \text{ and all } u,\bar{u}\in C(J,\mathbb{R}).$$

If

$$\frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^*+\frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^{**}+kG^*<1, \tag{4.31}$$

where $k=\|l\|_{L^1}$, then the problem (4.12)-(4.14) has at least one solution on J.

Remark 4.10 *By (Ή6), we can see that $S_{F,y}$ is nonempty for each $y\in C(J,\mathbb{R})$, so F has a measurable selection (see [41], Theorem III.6)*

Proof. We shall show that N satisfies the assumptions of theorem of Covitz and Nadler (1.42). The proof will be given in two steps.

Step 1: $N(y)\in\mathcal{P}_{cl}(C(J;\mathbb{R}))$ *for each* $y\in C(J,\mathbb{R})$.
Indeed, let $(y_n)_{n\geq 0}\subset N(y)$ be such that $y_n\to\bar{y}$ in $C(J,\mathbb{R})$. Then $\bar{y}$ in $C(J,\mathbb{R})$ and there exists $v_n\in S_{F,y}$ such that for each $t\in J$,

$$h_n(t)=P_y(t)+\int_1^T G(t,s)v_n(s)\frac{ds}{s}.$$

Using the fact that F has compact values and from (H5) we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazurs theorem implies that v_n converges strongly to v and hence $v\in S_{F,y}$. Then for each $t\in J$

$$h_n(t)\to\tilde{h}(t)=P_y(t)+\int_1^T G(t,s)v(s)\frac{ds}{s}.$$

So, $\tilde{h} \in N(y)$.

Step 2: *There exists $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty$ for each $y, \bar{y} \in C(J, \mathbb{R})$.*

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$. Then there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$,

$$h_1(t) = P_y(t) + \int_1^T G(t,s)v_1(s)\frac{ds}{s}.$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y}(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, y(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for F, so that $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, t \in J.$$

Let us define for each $t \in J$,

$$h_2(t) = P_{\bar{y}}(t) + \int_1^T G(t,s)v_2(s)\frac{ds}{s},$$

where

$$P_{\bar{y}}(t) = \frac{(\frac{1}{T} - \log(T) + \log(t))}{-1 + \frac{1}{T} - \log(T)} \int_1^T g(s, \bar{y}(s))ds - \frac{(1 + \log(t))}{-1 + \frac{1}{T} - \log(T)} \int_1^T h(s, \bar{y}(s))ds.$$

Then for each $t \in J$

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}\int_1^T |g(s,y(s)) - g(s,\bar{y}(s))|ds \\
&+ \frac{(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}\int_1^T |h(s,y(s)) - h(s,\bar{y}(s))|ds \\
&+ \int_1^T G(t,s)|v_1(s) - v_2(s)|\frac{ds}{s} \\
&\leq \frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^*\|y(s) - \bar{y}(s)\|_\infty \\
&+ \frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^{**}\|y(s) - \bar{y}(s)\|_\infty \\
&+ kG^*\|y(s) - \bar{y}(s)\|_\infty \\
&\leq \left[\frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^* + \frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^{**} + kG^*\right] \\
&\times \|y(s) - \bar{y}(s)\|_\infty.
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \left[\frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^* + \frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^{**} + kG^*\right]\|y(s) - \bar{y}(s)\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$\begin{aligned}
H_d(N(y), N(\bar{y})) &\leq \left[\frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^* + \frac{(T-1)(1+\log(T))}{|-1+\frac{1}{T}-\log(T)|}k^{**} + kG^*\right] \\
&\times\|y - \bar{y}\|_\infty.
\end{aligned}$$

So by (4.31), N is a contraction and thus, by theorem of Covitz and Nadler , N has a fixed point $y$which is solution to (4.12)-(4.14). The proof is complete. □

4.2.4 An example

We apply Theorem 4.8 to the the following problem ,

$$ {}_c^H D^\alpha y(t) \in F(t,y), \text{ a.e. } t \in J = [1,e], \quad 1 < \alpha \leq 2, \tag{4.32}$$

satisfying the boundary conditions,

$$y(1) - y'(1) = 0, \tag{4.33}$$

$$y(e) + y'(e) = 0. \tag{4.34}$$

Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

where $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$. We assume that for each $t \in J$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu_1\}$ is open for each $\mu_1 \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu_2\}$ is open for each $\mu_2 \in \mathbb{R}$). Assume that there are $p \in C(J, \mathbb{R})$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, y)|, \ |f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in J, \text{ and all } y \in \mathbb{R}.$$

From (4.20), G is given by

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-1} + \frac{1+\log(t)}{T^*\Gamma(r)}\left(\log\frac{e}{s}\right)^{r-1} \\ -\frac{(1+\log(t))(r-1)}{eT^*\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-2}, \quad 0 < s < t, \\ \frac{1+\log(t)}{T^*\Gamma(r)}\left(\log\frac{e}{s}\right)^{r-1} + \frac{1+\log(t)(r-1)}{eT^*\Gamma(r)}\left(\log\frac{t}{s}\right)^{r-2}, \quad t < s < T, \end{cases} \tag{4.35}$$

where

$$T^* = \frac{1-2e}{e}.$$

Assume that there exists a constant $M > 0$ such that

$$\frac{M}{\int_1^e p(s)dsG^*\psi(M)} < 1.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [46]). Since all the conditions of Theorem 4.8 are satisfied, the boudary value problem (4.32)-(4.34) has at least one solution y defined on J.

Annex

Definition 4.11 *The Mellin transform of a function* $\varphi(t)$ *of a real variable* $t \in \mathbb{R}^+ = (0, \infty)$ *is defined by*

$$(M\varphi)(p) = M[\varphi(t)](p) = \varphi^*(s) = \int_0^\infty t^{s-1}\varphi(t)dt \quad s \in \mathbb{C}$$

Definition 4.12 *[73] The Euler Gamma function* $\Gamma(z)$ *is defined by the so-called Euler integral of the second kind*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad re(z) > 0$$

This integral is convegent for all complex number $z \in \mathbb{C}$ $re(z) > 0$. *It follows that the Gamma function is the Mellin transform of the exponential function.*

Property 1 *From the definition of the Gamma function, we can find*

- $$\Gamma(n) = (n-1)!$$
- $$\Gamma(z+1) = z\Gamma(z) \quad rez > 0$$

Definition 4.13 *[85] The Beta function* $B(z, w)$ *is defined by the so-called Euler integral of the first order*

$$B(z, w) = \int_0^1 x^{z-1}(1-x)^{w-1}dx, \quad re(w) > 0$$

It is connected with the Gamma function by the relation

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

Theorem 4.14 (Mazur)*[87] Let* $\{x_n\}$ *be a weakly convergent sequence to* x *in a Banach space* E. *then, there is a sequence of convex combination of elements of* $\{x_n\}$ *which converges strongly to* x.

Theorem 4.15 *(Arzela-Ascoli theorem)[75] Let A be a subset of $C(J;E)$;A is relatively compact in $C(J;E)$ if and only if the following conditions are met:*

- *(a) The set A is bounded ie :*
  $\exists k > 0 : \|f(x)\| \leq k, \forall x \in J$ *and* $\forall f \in A$.
- *(b) Set A is equicontinuous ie :*
  $\forall \epsilon > 0, \exists \delta > 0 : |t_1 - t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\| \leq \epsilon$ *for all* $t_1, t_2 \in J$ *and all* $f \in A$.
- *(c) For all $x \in J$: set $\{f(x), f \in A\} \subset E$ is relatively compact.*

Theorem 4.16 *(Fubini's theorem)[53]*
Suppose A, B are complete measure spaces.$|f(x,y)|d(x,y)$ is $A \times B$.measurable. If

$$\int_{A\times B} |f(x,y)|d(x,y) < \infty$$

where the integral is taken with respect to a product measure on the space over$A \times B$,then

$$\int_A \Big(\int_B |f(x,y)|dy \Big) dx = \int_A \Big(\int_B |f(x,y)|dx \Big) dy = \int_{A\times B} |f(x,y)|d(x,y)$$

Conclusion:

In this thesis, we studied the existence of integrable solutions for fractional implicit differential equations with Caputo-Hadamard fractional derivative of order $\alpha \in (1, 2]$ and the problem for fractional implicit differential equations with the Caputo- Hadamard fractional derivative order $\alpha \in (0, 1]$ with nonlocal condition .
Our results are based on Banach contraction principle and Schauder fixed point theorems which are given in chapter 2.

In chapter 3, we gave sufficient conditions for the existence of solution for boundary value problems for implicit fractional differential inclusions with Caputo fractional derivative order $\alpha \in (0, 1]$ and nonlocal condition and boundary value problems for implicit fractional differential inclusion of order $\alpha \in (0, 1]$ with nonlocal condition.

In the fourth chapter, we studied the existence solutions for boundary value problems for Hadamard-Caputo fractional differential inclusions of order $r \in (1, 2]$ with nonlocal and integral condition.
Our results are based on the nonlinear alternative of Leray- Schauder for the convex case. In the nonconvex case, our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

In the future, we will study the boundary value problem for implicit fractional differential equations and inclusions by the Monch's fixed point theorem combined with technique of measure of noncompacteness of Kuratowski and physical interpretation of this problems.

Bibliography

[1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, Implicit Fractional Differential and Integral Equations: Existence and Stability, De Gruyter, Berlin, 2018.

[2] S. Abbas, M. Benchohra and G.M. N'Guérékata, Topics in Fractional Differential Equation, Springer, New York, 2012

[3] S. Abbas, M. Benchohra and G.M. N'Guérékata, dvanced Fractional Differential and Integral Equations, Nova Science Publishers, New York, 2015

[4] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, Implicit Fractional Differential and Integral Equations: Existence and Stability, De Gruyter, Berlin, 2018.

[5] Y. Adjabi, F. Jarad, D. Beleanu and T. Abdeljawad, On cauchy problems with Caputo Hadamard fractional derivatives, *J. Comput. Anal.*, **21**, (4)(2016), 661681.

[6] R.P Agarwal, M. Benchohra and S. Hamani, A survey on existence results for Boundary value problems for nonlinear fractional differential equations and inclusions, *Acta Applicandae Mathematics.* Vol. 109 **3** (2010), 973-1033.

[7] R.P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential inclusions, *Adv.Stud. Contemp. Math.* **16**, No. 2 (2008), 181-196.

[8] R.P. Agarwal, M. Meehan and D. O'Regan, Fixed point theory and applications, *Camb. Trac. in Math* **141** Cambridge University Press, Cambridge, UK, (2001).

[9] B. Ahmad, Existence of solutions for fractional equations of order $q \in (2,3]$ with anti-periodic conditions, *J. Appl. Math. Compt.* **24** (2011), 822-825.

[10] B. Ahmad and J.J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Larray Schauder degree theory, *Topol. Meth. Nonlinear Anal.* **35** (2010), 295-304.

[11] B. Ahmad, S.K. Ntouyas and A. Alsaedi, New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions, *Bound. Value Probl.* **275**(2013), 1-14.

[12] B. Ahmad and S.K. Ntouyas, Initial value problems for hybrid Hadamard fractional equations, *Electron. J. Diff. Equa* **2014** (2014), No. 161, p. 1-8.

[13] B. Ahmad and V. Otero Espiner, Existence of solutions for fractional inclusions with anti-periodic boundary conditions, *Bound. Value Probl.* **11** (2009): Art ID 625347.

[14] J.P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.

[15] J.P. Aubin and A. Cellina, Differential Inclusions, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[16] J.P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.

[17] L.C. Becker, T. A. Burton and I. K. Purnaras, An Inversion of a Fractional Differential Equation and Fixed point, *Nonl. Dyn. and Syst. Theory* , **15**(3) (2015) 242252.

[18] M. Benchohra, S. Djebali and S. Hamani, Boundary value problems of differential inclusions with Riemann-Liouville fractional derivative, *Nonl. Oscill.* Vol. 14 **1** (2011), 7-20.

[19] M. Benchohra and S. Hamani, Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative, *Topol. Meth. Nonl. Anal.*, **32**(1) (2008), 115-130.

[20] M. Benchohra and S. Hamani, Boundary value problems for differential inclusions with fractional order, *Diss. Math. Diff. Inclusions. Control and Optimization.*,**28** (2008), 147-164.

[21] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, *Appl. Anal.***87** (7) (2008), 851-863.

[22] M. Benchohra, S. Hamani and S.K. Ntouyas, boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[23] M. Benchohra, S. Hamani and S.K. Ntouyas, boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonl. Anal.* **71** (2009), 2391-2396.

[24] M. Benchohra and S. Hamani, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonl. Anal.* **71**(2009), 2391–2396.

[25] M. Benchohra and M.S. Souid, Integrable Solutions for Implicit Fractional Order Differential Equations. *Trans. J. of Math. and Mech 6.* (2014), **2**, 101-107.

[26] M. Benchohra and M. S. Souid, L1-Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition, *Filomat* **30**, No 6 (2016), 14851492.

[27] M. Benchohra and M. S. Souid, Integrable Solutions For Implicit Fractional Order Functional Differential Equations with Infinite Delay, *Arch. Math.* (BRNO) **51** (2015), 67-76 .

[28] W. Benhamida, S. Hamani and J. Henderson, Boundary value problems for Caputo-Hadamard fractional differential equations, *Adv. in the They. of Nonl. Anal. and its Appl.*2 (2018) No. 3, 138-145.

[29] H.F. Bohnenblust and S. Karlin, On a theorem of ville. Contribution to the theory of games.155-160, Annals of Mathematics Studies, no. 24. Priceton University Press, Princeton. N. G. 1950

[30] H. Brezis, Analyse fonctionnelle : Théorie et applications, Masson, Paris, 1987.

[31] K. W. Blayneh, Analysis of age structured host-parasitoid model, *Far East J. Dyn. Syst.* **4** (2002), 125-145.

[32] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Composition of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[33] P. L. Butzer, A. A. Kilbas and J. J. Trujillo; Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.

[34] P. L. Butzer, A. A. Kilbas and J. J. Trujillo; Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.

[35] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type, *Math. Nachr.* **189** (1998), 23-31.

[36] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

[37] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem,*J. Math. Anal. Appl.* **162** (1991), 494-505.

[38] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem.*Selected problems of mathematics,* 25-30, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, 6, Cracow Univ. Technol., Krakw, 1995

[39] C. Berge,Espaces topologiques: Fonctions multivoques, Dunod, Paris, 1959.

[40] A. Cabada and G. Wang, Positive soution of nonlinear fractional diffrential equations with integral boundary value conditions, *J. Math. Anal. Appl* **389** (2012), 403-411.

[41] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[42] T. Chen and W. Liu, An anti-periodic boundary value problem for the fractional differential equations with a p-Laplacian operator, *Appl. Math. Lett.* **25** (11) (2012), 1671-1675.

[43] H. Covitz and S.B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.

[44] R. F Curtain and A.J. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic press,1977.

[45] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609-625.

[46] K. Deimling, Multivalued Differential Equations, Walter De Gruyter, Berlin-New York, 1992.

[47] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.

[48] K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms* **16** (1997), 231-253.

[49] K. Diethelm and N.J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[50] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.

[51] A.M.A. EL-Sayed, F. M. Nesreen and EL-Haddade, Existence of integrable solutions for a functional integral inclusion, *Diff. Equa. Cont. Proce* **3** (2017),1-14.

[52] A.M.A. El-Sayed and H.H.G. Hashem, Integrable and continuous solutions of a nonlinear quadratic integral equation, *Electron.J. Qual. Theory Differ. Equ.* **25** (2008) , 1-10.

[53] G. Fubini, "Sugli integrali multipli", Rom. Acc. L. Rend. (5), 16 (1)(1907): 608-614, JFM 38.0343.02 Reprinted in G. Fubini, Opere scelte, 2, Cremonese,(1958) pp. 243-249

[54] M. Frigon, Théorèmes d'existence de solution d'inclusions différentielles, Topological methods in differential equations and inclusions, NATO ASI Series, Ser. C: Math. and Phys. Sci. , Kluwer, Dordrecht.

[55] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[56] V. Gupta and J. Dabas, Existence Results for a Fractional Integro-Differential Equation with Nonlocal Boundary Conditions and Fractional Impulsive Conditions *Nonl. Dyn. Syst. Theory,* **15**(4) (2015) 370382

[57] J R. Graef, N. Guerraiche and S. Hamani, Boundary value problems for fractional differential inclusions with Hadamrd type derivatives in Banach spaces, *Stud. Univ. Babes-Bolyai. Math,* **62** (2017), No. 4, 427-438.

[58] J R. Graef, N. Guerraiche and S. Hamani, Initial value problem of fractional functional differential inclusions with Hadamard type derivative, *Surv. Math. Appl.*, Vol **13**, p27-40, 2018.

[59] N. Guerraiche, S. Hamani and J. Henderson, Boundary value Problems for Differential Inclusions with Integral and Anti-periodic Conditions, *Comm. Appl. Nonl. Anals.* **23** (2016), No. 3, 33 - 46.

[60] N. Guerraiche, S. Hamani and J. Henderson, Nonlinear boundary value Problems for Hadamard fractional differential inclusions with integral boundary conditions, *Adv. Dyn. Syst. Appl.* **12** (2017), No. 2, 107-121.

[61] N.Guerraiche, Thèses de doctorat(LMD), Les Inclusions Differentielles Fractionnaires, Université de Mostaganem - Abdelhamid Ibn Badis (2019).

[62] D. Guo, V. Lakshmikantham and X. Liu, Nonlinear integral equations in abstract spaces, Mathematics and its Applications, **373**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[63] J. Hadamard, Essai sur l'etude des fonctions donnees par leur development de Taylor, *J.Mat. Pure Appl.* Ser. **8** (1892), 101-186.

[64] S. Hamani, M. Benchohra and J. R. Graef, Existence results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions, *Electron. J. Diff. Equ.*, Vol. 10(2010), **20**, p. 1-16.

[65] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, *Rheologica Acta,* **45**(2006), no. 5, 765-772.

[66] M. Houas and Z. Dahmani, On existence of solutions for fractional differential equations with nonlocal multi-point boundary conditions. *Loba. J. of Math.* **37**, (2016), no. 2, 120-127.

[67] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[68] Sh. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, Theory **I**, Kluwer, Dordrecht, 1997.

[69] D. H. Hyers, G. Isac and T. M. Rassias,Topics in Nonlinear Analysis and Applications, World Scientific,1997

[70] F. Jarad, T. Abdeljawad and D. Beleanu, Caputo-type modification of the Hadamard fractional derivative, *Adv. in Diff. Equa.* 2012, 142, (2012).

[71] E. R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation,*Electron. J. Qual. Theory Differ. Equ.* (2007), No. 3, p. 11.

[72] A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Equa.* **41** (2005), 84-89.

[73] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.

[74] A. Lasota and Z. Opial, An application of the Kakutani-ky Fan theorem in the theory of ordinary differential equation, *Bull. Accd. Pol. Sci., Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.

[75] J.K. Hale and S. V. Lunel, Introduction to functional differential equations, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[76] V. Lakshmikantham and S. Leela, Nonlinear differential equations in abstract spaces, International Series in Mathematics: Theory, Methods and Applications, **2**, Pergamon Press, Oxford, UK, 1981.

[77] G. W. Leibniz," letter from hanover, Germany to Johann G.F.A. L'Hospital, September 30, 1695 " , Leibniz Mathematische Schriften, Olms-Verlag, Hildeshein, Germany, 1962.

[78] U. Mosco and J. l. Joly, A propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles, *J. Funct. Anal.* **34**, 107-137 (1979).

[79] S. M. Momani, S. B. Hadid and Z. M. Alawenh, Some analytical properties of solutions of differential equations of noninteger order, *Int. J. Math. Math. Sci.* **2004**(2004), 697–701.

[80] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations,John Wiley, New York, 1993.

[81] D. O'Regan and R. Precup, Fixed point theorems for set-valued maps and existence principles for integral inclusions, *J. Math. AnaL. Appl.* 245 (2000), 594-612.

[82] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[83] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, *Fract. Cal. Appl. Anal.* **5** (2002), 367-386.

[84] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary and L. Dorčak, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, *Nonl. Dynam.* **29** (2002), 281-296.

[85] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives: Theory and applications, Gordon and Breach (1993).

[86] W. Shammakh, A study of Caputo-Hadamard-Type fractional differential equations with nonlocal boundary conditions, *J. Func. Spac.*, (2016), ID 7057910 .

[87] France Vaillancourt, Méthode variationnelles appliquées á l'équation du pendule forcé sans conservation, *mémoire présenté en maitre és sciences (M. Sc.)*, Faculté des sciences, Université de Sherbrooke, Québec, Canada, Juillet 2000.

[88] M. Sowmya and A.S. Vatsala, Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence, *Nonl. Dyn. and Syst. They*, **15** (2) (2015) 198208.

[89] M.S,Souid , Thèses de doctorat (Es sciences), Une Contribution Dans L'étude Des Équation Différentielles Fractionnaires implicite,Université Djillali Liabes de Sidi Bel Abbrès (2015).

[90] S. Szufla, On the application of measure of noncompactness to existence theorems, *Rend.del.Sem. Math. Della Universita di Padova* **75** (1986), 1-14.

[91] P. Thiramanus, S. K. Ntouyas and J. Tariboon, Existence and uniqueness results for Hadamard- type fractional differential equations with nonlocal fractional integral boundary conditions, *Abstr. Appl. Anal.* (2014), Art. ID 902054, 9 pp.

[92] K. Wainwright and A. Chiang ,Fundamental Methods of Mathematical Economis,(4 th Edition), Mc Graw Hill 2005.

[93] A.Zahed and S. Hamani ,Boundary Value Problems for Hadamard-Caputo Fractional Differential Inclusions with Nonlocal Conditions, *Pan. Math, J* **29** ,(2019) No 4,15-28.

[94] A.Zahed , S. Hamani and J. Henderson, Boundary Value Problems for Caputo-Hadamard Fractional Differential Inclusions with Integral Conditions *Moroccan J. of Pure and Appl. Anal.***6** (2020), No. ,62-75.

[95] A.Zahed , S. Hamani and J. Henderson, Integrable Solutions for Caputo-Hadamard Implicit Fractional Order Differential Equations, *Adv. in Dyn. Syst. and Appl.(ADSA).* **15** (2020), No. **1**,17-28.

[96] A.Zahed , S. Hamani and J.R. Graef , Caputo-Hadamard implicit Fractional Differential inclusion with nonlocal condition, *Arc. Math.(Brno)***57** ,(2021) No 5, 285297.

[97] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations,'*Electron. J. Diff. Equ.* (2006), No. 36, pp, 1-12.