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Abstract

In this thesis, we study a new class of natural metrics denoted by ${}^{f}g$ and called the vertical rescaled metric on the cotangent bundle. We calculate its Levi-Civita connection and Riemannian curvature tensor. Also we present some new properties for bi-f-harmonic map between generalized warped product manifolds, for example we calcul the bi-f-harmonicity of the inclusion maps, the bi-f-harmonicity of the projection maps, and bi-f-harmonicity of the prduct maps with harmonic factor.

Keywords:Horizontal lift, Vertical lift, Harmonic maps, Cotangent bundle, Harmonic map, Bi-f-harmonic map, Bi-harmonic map, Generalized warped product manifolds .

Résumé

Dans cette thése, nous étudions une nouvelle classe de métriques naturelles notées par ${}^{f}g$, et appelée la métrique déformé verticale sur le fibré cotangent. Nous calculons les connexions de Levi Civita et les courbures Riemanniennes. De plus, Nous présentons également des nouvelles propriétés pour les applications bi- f-harmonique dans les variétés produit généralisé, par exemple nous calculons la bi-f-harmonicité de l' application inclusion, la bi-f-harmonicité des applications de projection, et la bi-f-harmonicité des applications avec facteur harmonique.

ملخـــــــص الأطـــــــروحة

في هذه الأطروحة نهتم بدراسة أصناف جديدة للقياسات الطبيعية المعرفة بالرمز g^f والتي تسمى بالمقياس العمودي المشوه للحزم التماسية و نقوم بحساب المشتقات المتغايرة والانحاءات الريمانية اضافة على ذالك نقوم بدراسة الخصائص الهندسية وحساب التطبيقات الثنائية-f-التوافقية للمنواعات الريمانية المرفقة بالجداء الريماني المتري العام على سبيل المثال حساب الثنائية-f-التوافقية لتطبيق الاحتواء, حساب الثنائية-f-التوافقية لتطبيقات الاسقاط و حساب الثنائية-f-التوافقية للتطبيقات المزودة بوسيط.

Introduction

The theory of harmonic maps between Riemannian manifolds was first established by Eells and Sampson (Chiang's Ph.D. adviser) [22] in 1964. In the last two decades, there were many new developments in harmonic maps achieved by a number of mathematicians. They were first explored by a number of physicists in the 1950s, and since then there were many new developments in this subject. Biharmonic maps, which generalize harmonic maps, they were first studied by Jiang [30, 31, 32] in 1986. In recent years, there has been progress in biharmonic maps, accomplished by quite a few mathematicians.

Harmonic maps were first introduced by Sampson in the hope of obtaining a homotopy version of the highly successful Hodge theory of cohomology in 1952. Not long after that his then colleague, John Nash (one of the three Nobel laureates in Economics in 1994) proposed a quite different but equivalent definition both of them were Moore Instructors at MIT at the time.Fuller [25] also came upon harmonic maps in 1954. The definition, whether in terms of the energy functional or the Euler-Lagrange equations, seems very natural to us today, but it was not so obvious half a century ago.

Eells and Sampson [22] collaborated on the first paper on harmonic maps of Riemannian manifolds at the Institute for Advanced Study at Princeton in 1964. This paper is usually considered as the pioneering work in harmonic maps. They also published a second and third joint paper [23, 24] afterwards.

With an eye toward the physical concept of kinetic energy $(\frac{mv^2}{2})$, a harmonic map $f : (M^m, g_{ij}) \to (N^n, h_{\alpha\beta})$ from an $m-$dimensional Riemannian manifold into an $n-$dimensional Riemannian manifold is defined as a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M |df|^2 dv = \frac{1}{2}\int_M h_{\alpha\beta} f_i^\alpha f_j^\beta g^{ij} dv, \quad i,j = 1..m, \quad \alpha,\beta = 1..n \tag{1}$$

where dv is the volume form of M determined by the metric g. In order to derive the associated Euler-Lagrange equations, we consider a one-parameter family of maps $f_t \in C^\infty(M \times [0,1], N)$ from a compact manifold M (without boundary) into a Riemannian manifold N such that f_t is the endpoint of a segment starting at $f(x) = f_0(x)$ determined in length and direction by the vector field $f(x)$. If M is a non-closed manifold, we assume that $f(x)$ has compact support, which is contained in the interior of M. We now compute the first variation of the energy functional:

$$\begin{aligned}\dot{E}(f) &= \frac{d}{dt}E(f_t)|_{t=0} = \int_M (df_t, D_t df_t)|_{t=0} dv = \int_M (df, D\dot{f})dv, \\ &= \int_M div(w)dv - \int_M (\tau f, \dot{f})dv = -\int_M (\tau f, \dot{f})dv, \forall \dot{f}, \end{aligned} \tag{2}$$

by the divergence theorem, where $\tau^\alpha(f) = trace_g(Ddf)$, D is the connection on $T^* \otimes f^{-1}TN$ induced by the Levi-Civita connections on M and N, $div(w) = w^j_{|j}$, and $w^j = h_{\alpha\beta} f^\alpha_i \dot{f}^\beta g^{ij}$ is a vector field on M. The map $f : M \to N$ is harmonic if the tension field

$$\begin{aligned}\tau^\alpha(f) &= trace_g(Ddf) = g^{ij} f^\alpha_{i_{|j}} = g^{ij}(f^\alpha_{i,j} + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j), \\ &= g^{ij}(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j), \end{aligned} \tag{3}$$

vanishes identically, where $f^\alpha_i = \frac{\partial f^\alpha}{\partial x_i}$, $f^\alpha_{ij} = \frac{\partial^2 f^\alpha}{\partial x_i \partial x_j}$, $f^\alpha_{i,j} = f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k$, and Γ^k_{ij} and $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connections on M and N, respectively.

Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n, respectively and let $\lambda : M \longrightarrow \mathbb{R}_+$ and $\mu : N \longrightarrow \mathbb{R}_+$ be smooth functions. A doubly warped product manifold $G = M \times_{(\lambda,\mu)} N$ is the product manifold $M \times N$ metric endowed with the doubly warped product metric $\bar{g} = \mu^2 g + \lambda^2 h$. W.J. Lu [56] study the basic properties of f-bi-harmonic and bi$f$-harmonique maps of this product ($i.e.$ for $f \in \mathcal{C}^\infty(\mathcal{M})$). In this work we study the basic properties of f-bi-harmonic and bi$f$-harmonique maps of generalized warped product manifolds $M \times_f N$ where f is a smooth positive function defined by $f : M \times N \longrightarrow \mathbb{R}_+$.

The thesis is organized in four chapters. In the first chapter, we give definitions of manifolds, differentiable manifolds, tangent spaces, Riemannian metrics and we introduce basic concepts of curvature, harmonic maps and biharmonic maps.

In the second and the third chapters we give definitions of tangent and cotangent bundles of Riemannian manifolds, examples of tangent bundles and cotangent bundles of Riemannian manifolds, curvature and connection in tangent and cotangent bundles in Riemannian manifolds.

In the last chapter, we introduce a very large generalization of harmonic maps called f-bi-harmonic maps as the critical points of f-bi-energy functional, and then derive the Euler-Lagrange equation of f-bi-energy functional given by the vanishing of f-bi-tension field. Subsequently, we study some properties of f-bi-harmonic maps between the same dimensional manifolds and give a non-trivial example.

Chapter 1

Preliminaries

In this first chapter, we give definitions of manifolds, differentiable manifolds, tangent spaces, Riemannian metrics and we introduce basic concepts of curvature, harmonic maps and biharmonic maps. For more details, we invite the reader to consult the references ([17],[33], [36], [37], [40], [45]). This chapter doesn't contain any proof, we refer to the cited references above..

1.1 Riemannian manifolds

1.1.1 Manifolds and Differentiable Manifolds

A topological space is a set $\mathcal{M}$ together with a family $\mathcal{O}$ of subsets of $\mathcal{M}$ satisfying the following properties:

(i) $\Omega_1,\, \Omega_2 \in \mathcal{O} \Rightarrow \Omega_1 \cap \Omega_2 \in \mathcal{O}$,

(ii) for any index set $I\,:(\Omega)_{\alpha\in I} \subset \mathcal{O} \Rightarrow \bigcup_{\alpha\in I} \Omega_\alpha \in \mathcal{O}$,

(iii) $\emptyset,\, \mathcal{M} \in \mathcal{O}$.

The sets from $\mathcal{O}$ are called open. A topological space is called Hausdorff if for any two distinct points $p_1,\, p_2 \in \mathcal{M}$ there exists open sets $\Omega_1,\, \Omega_2 \in \mathcal{O}$ with $p_1 \in \Omega_1,\, p_2 \in \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset$. A covering $(\Omega_\alpha)_{\alpha\in I}$ (I an arbitrary index set) is called locally finite if each $p \in \mathcal{M}$ has a neighborhood that intersects only finitely many Ω_α. $\mathcal{M}$ is called paracompact if any open covering possesses a locally finite refinement. This means that for any open covering $(\Omega_\alpha)_{\alpha\in I}$ there exists a locally finite open covering $(\Omega'_\beta)_{\beta\in J}$ where

$$\forall\beta \in J,\ \exists\alpha \in I,\ \ \Omega'_\beta \subset \Omega_\alpha.$$

A map between topological spaces is called continuous if the preimage of any open set is open. A bijective map which is continuous in both directions is called a homeomorphism.

Definition 1. *A manifold $\mathcal{M}$ of dimension d is a connected paracompact Hausdorff which every point has a neighborhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^d$. Such a homeomorphism*

$$x\,: U \to \Omega.$$

is called a (coordinate) chart .

An atlas is a family $\{(U_\alpha,\, x_\alpha)\}$ of charts for which the U_α constitute an open covering of $\mathcal{M}$.

Definition 2. *An atlas* $\{(U_\alpha,\, x_\alpha)\}$ *on a manifold is called differentiable if all chart transitions*

$$x_\beta \circ x_\alpha^{-1} \,:\, x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ *(in case* $U_\alpha \cap U_\beta \neq \emptyset$*).*

A maximal differentiable atlas is called a differentiable structure, and a differentiable manifold of dimension d is a manifold of dimension d with a differentiable stricture. From now on all atlases are supposed to be differentiable. Two atlases are called compatible if their union is again an atlas. In general, a chart is called compatible with an atlas id adding the chart to the atlas yields again an atlas. An atlas is called maximal if any compatible with it is already contained in it.

Definition 3. *An atlas for a differentiable manifold is called oriented if all chart transitions have positive functional determinant.*
A differentiable manifold is called orientable if it possesses an oriented atlas.

It is customary to write the Euclidean coordinates of $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ open, as

$$x = (x^1, ..., x^d) \tag{1.1}$$

and these then are considered as local coordinates our manifold
$\mathcal{M}$ when $x\,:U \to \Omega$ is a chart.

Example 1. *The* $n-$*sphere* $S^n = \{(x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} \,/\, \sum_{i=1}^{n+1}(x^i)^2 = 1\}$ *is an* $n-$*dimentional differentiable manifold. Charts can be given as follows : On*
$U_1 := S^n \setminus \{(0, ..., 0, 1)\}$*, we put*

$$\begin{aligned} f_1(x^1, ..., x^{n+1}) &:= (f_1^1(x^1, ..., x^{n+1}), ..., f_1^n(x^1, ..., x^{n+1})) \\ &:= \left(\tfrac{x^1}{1-x^{n+1}}, ..., \tfrac{x^n}{1-x^{n+1}}\right) \end{aligned}$$

and on $U_2 := S^n \setminus \{(0, ..., 0, -1)\}$*, we have*

$$\begin{aligned} f_2(x^1, ..., x^{n+1}) &:= (f_2^1(x^1, ..., x^{n+1}), ..., f_2^n(x^1, ..., x^{n+1})) \\ &:= \left(\tfrac{x^1}{1+x^{n+1}}, ..., \tfrac{x^n}{1+x^{n+1}}\right). \end{aligned}$$

Definition 4. *A map* $h\,:\mathcal{M} \to \mathcal{M}'$ *between differentiable manifolds* $\mathcal{M}$ *and* $\mathcal{M}'$ *with charts* $\{(U_\alpha,\, x_\alpha)\}$ *and* $\{(U'_\alpha,\, x'_\alpha)\}$ *is called differentiable if all maps* $x'_\beta \circ h \circ x_\alpha^{-1}$ *are differentiable of class* C^∞*, as always where defined. Such a map is called a diffeomorphism if bijective and differentiable in both direction.*

For purposes of differentiation a differentiable manifold locally has the structure of Euclidean space. Thus, the differentiability of a map can be tested in local coordinates. The diffeomorphism requirement for the chart transitions then guarantees that differentiability defined in this manner is a consistent notion, *i.e.* independent of the choice of a chart.

Lemma 1. *Let $\mathcal{M}$ be a differentiable manifold, $(U_i)_{i\in I}$ an open covering. Then there exists a partition of unity,subordinate to (U_i). This means that there exists a locally finite refinement $(V_j)_{j\in J}$ of (U_i) and C_0^∞ (i.e. C^∞ functions φ_j with $\{x \in \mathcal{M}/\varphi_j(x) \neq 0\}$ having compact closure) functions $\varphi_j : \mathcal{M} \to \mathbb{R}$ with*

(1) $supp\,\varphi_j \subset V_j \; for\ all\ j \in J,$

(2) $0 \leq \varphi_j(x) \leq 1 \; for\ all\ x \in \mathcal{M}, j \in J,$

(3) $\sum_{j\in J} \varphi_j(x) = 1 \; for\ all\ x \in \mathcal{M}.$

Note that in (3)*, there are only finitely many nonvanishing summands at each point since only finitely many φ_j are nonzero at any given point because the covering (V_j) is locally finite.*

1.1.2 Tangent Spaces

Let $x = (x^1, ..., x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is open, $x_0 \in \Omega$. The tangent space of Ω at the point x_0, $T_{x_0}\Omega$ is the space $\{x_0\} \times E$, where E is the $d-$dimensional vector space spanned by the basis $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^d}$. Here, $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^d}$ are the partial derivatives at the point x_0. If $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^c$ are open, and $f : \Omega \to \Omega'$ is differentiable, we define the derivative $df(x_0)$ for $x_0 \in \Omega$ as the linear map between tangent spaces

$$df(x_0) \; : T_{x_0}\Omega \to T_{f(x_0)}\Omega',$$

$$\upsilon = \upsilon^i \frac{\partial}{\partial x^i} \mapsto \upsilon^i \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial f^j}.$$

Here and in the sequel, we use the Einstein summation convention. An index occurring twice in a product is to be summed from 1 up to the space dimension. Thus $\upsilon^i \frac{\partial}{\partial x^i}$ is an abbreviation for

$$\sum_{i=1}^{d} \upsilon^i \frac{\partial}{\partial x^i},$$

$\upsilon^i \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial f^j}$ stands for

$$\sum_{i=1}^{d}\sum_{j=1}^{c} \upsilon^i \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial f^j}.$$

In the previous notations, we put

$$T\Omega := \Omega \times E \cong \Omega \times \mathbb{R}^d.$$

thus, $T\Omega$ is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$, hence in particular a differentiable manifold.

$$\pi \; : T\Omega \to \Omega, \;\; (\; projection\ into\ the\ first\ factor\,)$$

$$(x, \upsilon) \mapsto x,$$

is called a tangent bundle of Ω. $T\Omega$ is called the total space of the tangent bundle.
Likewise, we define

$$df \; : T\Omega \to T\Omega'$$

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}\right).$$

instead of

$$df(x,v)$$

we write

$$df(x)(v).$$

If in particular, $f : \Omega \to \mathbb{R}$ is a differentiable function, we have for $v = v^i \frac{\partial}{\partial x^i}$

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}.$$

In this case, we often write $v(f)(x)$ in place of $df(x)(v)$ when we want to express that the tangent vector v operates by differentiation on the function f.

Let now $\mathcal{M}$ be a differentiable manifold of dimension d, and $p \in \mathcal{M}$. We want to define the tangent space of $\mathcal{M}$ at the point p. Let $x : U \to \mathbb{R}^d$ be a chart with $p \in U$, U open in $\mathcal{M}$. We say that the tangent space $T_p\mathcal{M}$ is represented in the chart x by $T_{x(p)}x(U)$. Let $x' : U' \to \mathbb{R}^d$ be another chart with $p \in U'$, U' open in $\mathcal{M}$. $\Omega := x(U)$, $\Omega' := x'(U')$. The transition map

$$x' \circ x^{-1} : x(U \cap U') \to x'(U \cap U),$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)) : T_{x(p)}\Omega \to T_{x'(p)}\Omega'.$$

We say that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in $T_p\mathcal{M}$. Thus a tangent vector in $T_p\mathcal{M}$ is given by the family of its coordinate representations. This is motivated as follows: Let $f : \mathcal{M} \to \mathbb{R}$ be a differentiable function. Assume that the tangent vector $\omega \in T_p\mathcal{M}$ is represented by $v \in T_{x(p)}x(U)$. We then want to define $df(p)$ as a linear map from $T_p\mathcal{M}$ to $\mathbb{R}$. In the chart x, let $\omega \in T_p\mathcal{M}$ be represented by $v = v^i \frac{\partial}{\partial x^i} \in T_{x(p)}x(U)$. We then say that

$$df(p)(\omega),$$

in this chart is represented by

$$d(f \circ x^{-1})(x(p))(v).$$

Now

$$\begin{aligned} d(f \circ x^{-1})(x(p))(v) &= d(f \circ x'^{-1} \circ x' \circ x^{-1})(x(p))(v) \\ &= d(f \circ x'^{-1})(x'(p))(L(v)) \\ &= d(f \circ x'^{-1})(x'(p)) \circ d(x' \circ x^{-1})(x(p))(v). \end{aligned}$$

Thus in the chart (U', x'), ω is represented by $L(v)$. Here a fundamental idea emerges that will be essential for the understanding of the sequel. $T_p\mathcal{M}$ is a vector space of dimension d, hence isomorphic to $\mathbb{R}^d$. This isomorphism however is not canonical but depends on the choice of a chart.

A change of charts changes the isomorphism, namely at the point p by the linear transformation $L = d(x' \circ x^{-1})(x(p))$. Under a change of charts, also other objects then are correspondingly transformed, for example derivatives of functions or more generally of maps. In other words a chart yields local representations for tangent vectors derivatives etc., and under a change of charts these local representations need to be correctly transformed. Or in still other words: We know how to differentiate (differentiable) functions that are defined on open subsets of $\mathbb{R}^d$. If now a function is given a manifold, we pull it back by a chart, to an open subset of $\mathbb{R}^d$ and then differentiable the pulled back function. In order to obtain an object that does not depend the choice of chart, we have to know in addition the transformation behavior under chart changes. A tangent vector thus is determined by how it operates on functions by differentiation.

Likewise for a differentiable map $F : \mathcal{M} \to \mathcal{N}$ between differentiable manifolds, dF is represented in local charts $x : U \subset \mathcal{M} \to \mathbb{R}^d$, $y : V \subset \mathcal{N} \to \mathbb{R}^c$ by

$$d(y \circ F \circ x^{-1}).$$

In the sequel in our notation, we shall frequently drop reference to the charts and write instead of $d(y \circ F \circ x^{-1})$ simply dF, provided the choice of charts or at least the fact that charts have been chosen is obvious from the context. We can achieve this most simply as follows :
Let the local coordinates on U be

$$(x^1, ..., x^d),$$

and those on V be $(F^1, ..., F^c)$. We then consider $F(x)$ as abbreviation for

$$(F^1(x^1, ..., x^d), ..., F^c(x^1, ..., x^d)).$$

dF now induce a linear map

$$dF : T_x\mathcal{M} \to T_{F(x)}\mathcal{N},$$

which in our coordinates it represented by the matrix

$$\left(\frac{\partial F^\alpha}{\partial x^i}\right)_{\substack{\alpha \;= 1, ..., c \\ \beta \;= 1, ..., d}} .$$

A change of charts leads to a base change of the tangent spaces and the transformation behavior is determined by the chain rule. If

$$(x^1, ..., x^d) \mapsto (\xi^1, ..., \xi^d),$$

and

$$(F^1, ..., F^c) \mapsto (\Phi^1, ..., \Phi^c),$$

are coordinate changes, then dF is represented in the new coordinates by

$$\left(\frac{\partial \Phi^\beta}{\partial \xi^j}\right) = \left(\frac{\partial \Phi^\beta}{\partial F^\alpha}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial x^i}{\partial \xi^j}\right).$$

Note that the functional matrix of the coordinate change of the image $\mathcal{N}$, but the inverse of the functional matrix of the coordinate change of the domain $\mathcal{M}$ appears here. We also remark that for a function $\varphi : \mathcal{N} \to \mathbb{R}$ and a $v \in T_x\mathcal{M}$,

$$(dF(v)(\varphi))(F(x)) := d\varphi(dF(v))(F(x)),$$

by definition of the application of $dF(v) \in T_{F(x)}\mathcal{N}$ to $\varphi : \mathcal{N} \to \mathbb{R}$,

$$= d(\varphi \circ F)(v)(x) \textit{ by the chain rule}$$

$$= v(\varphi \circ F)(x),$$

by definition of the application of $v \in T_x M$ to $\varphi \circ F : \mathcal{M} \to \mathbb{R}$.

Instead of applying the tangent vector $dF(v)$ to the function one may also apply the tangent vector v to the "pulled back " function $\varphi \circ F$.
We want to collect the previous considerations in a formal definition:

Definition 5. *Let $p \in \mathcal{M}$. On $\{(x, v)/x : U \to \Omega \, chart \; with \; p \in U, \; v \in T_{x(p)}\Omega\}$ $(x, v) \sim (y, \omega) :\Leftrightarrow \omega = d(y \circ x^{-1})v$. The space of equivalence classes is called the tangent space to $\mathcal{M}$ at the point p and it is denoted by $T_p\mathcal{M}$.*

$T_p\mathcal{M}$ naturally carries the stricture of a vector space :
The equivalence class of $\lambda_1(x, v_1) + \lambda_2(x, v_2)$ $(\lambda_1, \lambda_2 \in \mathbb{R})$ is the one of $(x, \lambda_1 v_1 + \lambda_2 v_2)$. We now want to define the tangent bundle of a differentiable manifold of dimension d. $T\mathcal{M}$ is the disjoint union of the tangent space $T_p\mathcal{M}$, $p \in \mathcal{M}$, equipped with the following stricture of a differentiable manifold: First let $\pi : T\mathcal{M} \to \mathcal{M}$ with $\pi(\omega) = p$ for $\omega \in T\mathcal{M}$ be the projection onto the "base point". If $x : U \to \mathbb{R}^d$ is a chart for $\mathcal{M}$, we let TU be the disjoint union of the $T_p\mathcal{M}$ with $p \in U$ and define the chart

$$dx : TU \to Tx(U), \; (:= \bigcup_{p \in x(U)} T_p\mathcal{M}),$$

where $Tx(U)$ carries the differentiable stricture of $x(U) \times \mathbb{R}^d$

$$\omega \mapsto dx(\pi(\omega))(\omega) \in T_{x(\pi(\omega))}x(U).$$

The transition maps

$$dx' \circ (dx)^{-1} = d(x' \circ x^{-1}),$$

then are differentiable.π is locally represented by

$$x \circ \pi \circ dx^{-1},$$

and this maps $(x_0, v) \in Tx(U)$ to x_0.

Definition 6. *The triple $(T\mathcal{M}, \pi, \mathcal{M})$ is called the tangent bundle of $\mathcal{M}$, and $T\mathcal{M}$ is called the total space of the tangent bundle.*

1.2 Submanifolds

A differentiable map $f : \mathcal{M} \to \mathcal{N}$ is called an immersion, if for any $x \in \mathcal{M}$

$$df : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$$

is injective. In particular, in this case $m := dim\mathcal{M} \leq n := dim\mathcal{N}$. If an immersion $f : \mathcal{M} \to \mathcal{N}$ maps $\mathcal{M}$ homeomorphically onto its image in $\mathcal{N}$, f is called differentiable embedding. The following lemma shows that locally, any immersion is a differentiable embedding :

Lemma 2. *Let $f : \mathcal{M} \to \mathcal{N}$ be an immersion, $dim\mathcal{M} = m$, $dim\mathcal{N} = n$, $x \in \mathcal{M}$, Then there exists a neighborhood U of x and a chart (V, y) on $\mathcal{N}$ with $f(x) \in V$, such that*

(i) $f_{/U}$ is a differentiable embedding, and

(ii) $y^{m+1}(p) = ... = y^n(p) = 0$ for all $p \in f(U) \cap V$.

Proof. This follows from the implicit function theorem. In local coordinates $(z^1, ..., z^n)$ on $\mathcal{N}$, $(x^1, ..., x^m)$ local coordinates on $\mathcal{M}$ (since $df(x)$ is injective)

$$\left(\frac{\partial z^\alpha(f(x))}{\partial x^i}\right)_{i,\alpha=1,...,m}$$

be nonsingular .
We consider

$$F(z, x) := (z^1 - f^1(x), ..., z^n - f^n(x)),$$

which has maximal rank in $x^1, ..., x^m, z^{m+1}, ..., z^n$. By the implicit function theorem, there locally exists a map

$$(z^1, ..., z^m) \mapsto (\varphi^1(z^1, ..., z^m), ..., \varphi^n(z^1, ..., z^m)),$$

with

$$\begin{aligned} F(z, x) = 0 \quad \Leftrightarrow \quad & x^1 = \varphi^1(z^1, ..., z^m), ..., x^m = \varphi^m(z^1, ..., z^m), \\ & z^{m+1} = \varphi^{m+1}(z^1, ..., z^m), ..., z^n = \varphi^n(z^1, ..., z^m), \end{aligned}$$

for which $\left(\frac{\partial \varphi^i}{\partial z^\alpha}\right)_{\alpha,i=1,..,m}$ has maximal rank.
As new coordinates, we now choose

$$\begin{aligned} (y^1, ..., y^n) \quad = \quad & \varphi^1(z^1, ..., z^m), ..., \varphi^m(z^1, ..., z^m), \\ & z^{m+1} - \varphi^{m+1}(z^1, ..., z^m), ..., z^n - \varphi^n(z^1, ..., z^m), \end{aligned}$$

Then

$$\begin{aligned} z = f(x) \quad \Leftrightarrow & \qquad F(x, z) = 0 \\ \Leftrightarrow & \quad (y^1, ..., y^n) = (x^1, ..., x^n, 0, ..., 0), \end{aligned}$$

and the claim follows. □

If $f : \mathcal{M} \to \mathcal{N}$ is a differentiable embedding, $f(\mathcal{M})$ is called a differentiable submanifold of $\mathcal{N}$. A subset $\mathcal{N}'$ of $\mathcal{N}$, equipped with the relative topology, thus is a differentiable submanifold of $\mathcal{N}$, if $\mathcal{N}'$ is a manifold and the inclusion is a differentiable embedding.
Chrts on $\mathcal{N}'$ then are simply given by restriction of charts of $\mathcal{N}$ to $\mathcal{N}'$, and Lemma 2 shows that one may here always find a particularly convenient stricture of the charts.
Similarly the implicit function theorem implies.

Lemma 3. *Let $f : \mathcal{M} \to \mathcal{N}$ be a differentiable map, $dim\mathcal{M} = m$, $dim\mathcal{N} = n$, $n, m \geq n$, $p \in \mathcal{N}$. Let $df(x)$ have rank n for all $x \in \mathcal{M}$ with $f(x) = p$. Then $f^{-1}(p)$ is a union of differentiable submanifolds of $\mathcal{M}$ of dimension $m - n$.*

Proof. We again represent the situation in local coordinates around $x \in \mathcal{M}$ and $p = f(x) \in \mathcal{N}$. Of course in these coordinates $df(x)$ still has rank n. By the implicit function theorem, there exists a neighborhood U of x and a differentiable map

$$g(x^{n+1}, ..., x^m) : U_2 \subset \mathbb{R}^{m-n} \to U_1 \subset \mathbb{R}^n,$$

with

$$U = U_1 \times U_2,$$

and

$$f(x) = p \Leftrightarrow (x^1, ..., x^n) = g(x^{n+1}, ..., x^m),$$

with

$$y^\alpha = x^\alpha - g(x^{n+1}, ..., x^m),\ for\ \alpha = 1, ..., n,$$

$$y^s = x^s,\ for\ s = n+1, ..., m,$$

we then get coordinates for which

$$f(x) = 0 \Leftrightarrow y^\alpha = 0,\ for\ \alpha = 1, ..., n.$$

$(y^{n+1}, ..., y^m)$ thus yield local coordinates for $\{f(x) = p\}$ and this implies in some neighborhood of x $\{f(x) = p\}$ is a submanifold of $\mathcal{M}$ of dimension $m - n$. □

Let $\mathcal{M}$ be a differentiable submanifold of $\mathcal{N}$ and let $i : \mathcal{M} \to \mathcal{N}$ be the inclusion. For $p \in \mathcal{M}$, $T_p\mathcal{M}$ can be considered as subspace of $T_p\mathcal{N}$, namely as the image $di(T_p\mathcal{M})$. The standard example is the sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}.$$

By the Lemme 3, S^n is a submanifold of $\mathbb{R}^{n+1}$.

Example 2. *In the situation of Lemma3, we have for the submanifold* $X = f^{-1}(p)$ *and* $q \in X$

$$T_qX = Kerdf(q) \subset T_q\mathcal{M}.$$

Proof. Let $v \in T_qM$, (φ, U) a chart on X with $q \in U$. Let γ be any smooth curve in $\varphi(U)$ with $\gamma(0) = \varphi(q)$, $\dot\gamma(0) := \frac{d}{dt}\gamma(t)|_{t=0} = d\varphi(v)$, for example, $\gamma(t) = \varphi(q) + td\varphi(v)$.
$c := \varphi^{-1}(\gamma)$ then is a curve in X with $\dot c(0) = v$. Because of $X = f^{-1}(p)$,

$$f \circ c(t) = p\ \forall t.$$

Hence $df(q) \circ \dot c(0) = 0$ and consequently $v = \dot c(0) \in Kerdf(q)$.
Since also $dimT_qX = dimKerdf(q) = m - n$, the claim follows. □

For our example S^n, we may choose

$$f : \mathbb{R}^{n+1} \to \mathbb{R},\ f(x) = |x|^2.$$

Then

$$T_xS^n = Kerdf(x) = \{v \in \mathbb{R}^{n+1}, : x.v = x^iv^i = 0\}.$$

1.3 Riemannian Metric

We now want to introduce metric structures on differentiable manifolds. Again, we shall start from infinitesimal considerations. We would like to able to measure the lengths of curve and the angles between tangent vectors. Then one may for example obtain the length of a differentiable curve by integration. In a vector space such a notion of measurement is usually given by a scalar product. We thus define

Definition 7. *A Riemannian metric on a differentiable manifold $\mathcal{M}$ is given by a scalar product on each tangent space $T_p\mathcal{M}$ which depends smoothly on the base point p.*
A Riemannian manifold is a differentiable manifold equipped with a Riemannian metric.

In order to understand the concept of a Riemannian metric, we again need to study local coordinate representations and the transformation behavior of these expressions.
Thus, let $x = (x^1, ..., x^d)$ be local coordinates. In these coordinates a metric is represented by a positive definite symmetric matrix

$$(g_{ij}(x))_{i,j=1,..,d},$$

($i.e.$ $g_{ij} = g_{ji}$ for all i, j, ,$g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, ..., \xi^d) \neq 0$), where the coefficients depend smoothly on x. The transformation (1.4) below will imply that this smoothness does not depend on the choice of coordinates. There fore smooth dependence on the base point as required in Definition 7 can be expressed in local coordinates.
The product of two tangent vectors υ, $\omega \in T_p\mathcal{M}$ with coordinate representations $(\upsilon^1, ..., \upsilon^d)$ and $(\omega^1, ..., \omega^d)$ (i.e $\upsilon = \upsilon^i\frac{\partial}{\partial x^i}$, $\omega = \omega^j\frac{\partial}{\partial x^j}$) then is

$$< \upsilon, \omega >:= g_{ij}(x(p))\upsilon^i\omega^j. \tag{1.2}$$

in particular, $< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} >= g_{ij}$.
Similarly the length of υ is given by

$$\|\upsilon\| :=< \upsilon, \upsilon >^{\frac{1}{2}} .$$

We now want to study the transformation behavior. Let $y = f(x)$ define different local coordinates. In these coordinates, υ and ω have representations $(\widetilde{\upsilon}^1, .., \widetilde{\upsilon}^d)$ and $(\widetilde{\omega}^1, .., \widetilde{\omega}^d)$ with $\widetilde{\upsilon}^j = \upsilon^i\frac{\partial f^j}{\partial x^i}$, $\widetilde{\omega}^j = \omega^i\frac{\partial f^j}{\partial x^i}$. Let the metric in the new coordinates be given by $h_{kl}(y)$.
It follows that

$$h_{kl}(f(x))\widetilde{\upsilon}^k\widetilde{\omega}^l =< \upsilon, \omega >= g_{ij}(x)\upsilon^i\omega^j, \tag{1.3}$$

hence

$$h_{kl}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}\upsilon^i\omega^j = g_{ij}(x)\upsilon^i\omega^j,$$

and since this holds for all tangent vectors υ, ω,

$$h_{kl}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j} = g_{ij}(x). \tag{1.4}$$

Formula (1.4) gives the transformation behavior of a metric under coordinate changes.
The simplest example of a Riemannian metric of course is the Euclidean one.
For $\upsilon = (\upsilon^1, ..., \upsilon^d)$, $\omega = (\omega^1, ..., \omega^d) \in T_x\mathbb{R}^d$, the Euclidean scalar product is simply

$$\delta_{ij}\upsilon^i\omega^j = \upsilon^i\omega^i,$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

is the standard Kronecker symbol.

Theorem 1. *Each differentiable manifold may be equipped with a Riemannian metric.*

Proof. Let $\{(x_\alpha, U_\alpha) : \alpha \in I\}$ be an atlas , $(\varphi_\alpha)_{\alpha \in A}$ a partition of unity subordinate to $(U_\alpha)_{\alpha \in I}$ (see Lemma 1(for simplicity of notation, we use the same index set for (φ_α) and (U_α), this may be justified by replacing the original covering (U_α) by a locally finite refinement).
For v, $\omega \in T_p\mathcal{M}$ and $\alpha \in I$ with $p \in U_\alpha$ let the coordinate representations be $(v_\alpha^1, ..., v_\alpha^d)$ and $(\omega_\alpha^1, ..., \omega_\alpha^d)$. Then we put

$$< v, \omega > := \sum_{\substack{\alpha \in I \\ with \quad p \in U_\alpha}} \varphi_\alpha(p) v_\alpha^i \omega_\alpha^i.$$

This defines a Riemannian metric. (The metric is simply obtained by piecing the Euclidean metrics of the coordinate images together with the help of a partition of unity). □

Let now $[a, b]$ be a closed interval in $\mathbb{R}$, $\gamma : [a, b] \to \mathcal{M}$ a smooth curve, where "smooth ", as always, mean "of class C^∞ ."
The length of γ then is defined as

$$L(\gamma) := \int_a^b \| \frac{d\gamma}{dt}(t) \| dt$$

and the energy of γ as

$$E(\gamma) := \frac{1}{2} \int_a^b \| \frac{d\gamma}{dt}(t) \|^2 dt$$

In physics, $E(\gamma)$ is usually called "action of γ" where γ is considered as the orbit of a mass point. Of course, these expressions can be computed in local coordinates.
Working with the coordinates $(x^1(\gamma(t)), ..., x^d(\gamma(t)))$ we use the abbreviation

$$\dot{x}^i(t) := \frac{d}{dt}(x^i(\gamma(t))).$$

Then

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t)} dt$$

and

$$E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) dt.$$

We also remark for later technical purposes that the length of a (continuous and) piecewise smooth curve may be defined as the sum of the lengths of the smooth pieces, and the same

hold for the energy.
On a Riemannian manifold $\mathcal{M}$, the distance between two points p, q can be defined

$$d(p,q) := inf\{L(\gamma)\, : [a,b] \to \mathcal{M}\, piecewise\, smooth\, curve\, with\, \gamma(a)=p,\, \gamma(b)=q\}.$$

We first remark, that any two points p, $q \in \mathcal{M}$ can be connected by a piecewise smooth curve, and $d(p,q)$ therefore is always defined. Namely, let

$$E_p := \{q \in \mathcal{M}\, : p \text{ and } q \text{ can be connected by a piecewise smooth curve}\}.$$

With the help of local coordinates one sees that E_p is open. But then also $\mathcal{M}\backslash E_p = \bigcup_{q \in E_P} E_p$ is open. Since $\mathcal{M}$ is connected and $E_P \neq \emptyset\,(p \in E_p)$, we conclude $\mathcal{M} = E_p$.
The distance function satisfies the usual axioms:

Lemma 4. *(i) $d(p,q) \geq 0$ for all p, q, and $d(p,q) > 0$ for all $p \neq q$*

(ii) $d(p,q) = d(q,p)$,

(iii) $d(p,q) \leq d(p,r) + d(r,q)$ (triangle inequality) for all p, q and $r \in \mathcal{M}$.

Proof. (i) and (ii) are obvious. For (i), we only have to show $d(p,q) > 0$ for $p \neq q$. For this purpose,let $x\, : U \to \mathbb{R}^d$ be a chart with $p \in U$. Then there exists $\epsilon > 0$ with

$$D_\epsilon(x(p)) := \{y \in \mathbb{R}^d\, : |y - x(p)| \leq \epsilon\} \subset x(U)$$

(the bars denote the Euclidean absolute value) and

$$q\; no\, in\; x^{-1}(D_\epsilon(x(p))). \tag{1.5}$$

Let the metric be represented by $(g_{ij}(x))$ in our chart. Since $(g_{ij}(x))$ is positive definite and smooth, hence continuous in x and $D_\epsilon(x(p))$ is compact, there exists $\lambda > 0$ with

$$g_{ij}(y)\xi^i\xi^j \geq \lambda|\xi|^2, \tag{1.6}$$

for all $y \in D_\epsilon(x(p))$, $\xi = (\xi^1, ..., \xi^d) \in \mathbb{R}^d$. Therefore, for any curve $\gamma\, : [a,b] \to \mathcal{M}$ with $\gamma(a) = p$, $\gamma(b) = q$

$$\begin{aligned} L(\gamma) &\geq L(\gamma \cap x^{-1}(D_\epsilon(x(p)))) \\ &\geq \lambda\epsilon > 0, \end{aligned} \tag{1.7}$$

because $x(\gamma)$ by (1.5) has to contain a point $z \in \partial(D_\epsilon(x(p))))$, *i.e.* a point whose Euclidean distance from $x(p)$ is ϵ. By (1.6), z then has distance from $x(p)$ at least $\lambda\epsilon$ w.r.t the metric (g_{ij}). □

Corollary 1. *The topology on $\mathcal{M}$ induced by the distance function d coincides with the original manifold topology of $\mathcal{M}$.*

Proof. It suffices to show that in each chart the topology induced by d coincides with the one of $\mathbb{R}^d$, *i.e.* the one induced by the Euclidean distance function. Now for every x in some chart, there exists $\epsilon > 0$ for which $D_\epsilon(x)$ is contained in the same chart, and positive constant λ, μ with

$$\lambda^2|\xi|^2 \leq g_{ij}(y)\xi^i\xi^j \leq \mu^2|\xi|^2 \; for\, all\, y \in D_\epsilon(x),\, \xi \in \mathbb{R}^d.$$

Thus

$$\lambda|y-x| \leq d(y,x) \leq \mu|y-x| \ for\, all\, y \in D_\epsilon(x),$$

and thus each Euclidean distance ball contains a distance ball for d, and vice versa, (with

$$B(z,\delta) := \{y \in \mathcal{M} \,/ d(z,y) \leq \delta\}$$

we have

$$\dot{D}_{\lambda\delta}(x) \subset \dot{B}(x,\delta) \subset \dot{D}_{\mu\delta}(x),$$

if $\mu\delta \leq \epsilon$). □

We now return to the length and energy functionals.

Lemma 5. *For each smooth curve* $\gamma : [a,b] \to \mathcal{M}$

$$L(\gamma)^2 \leq 2(b-a)E(\gamma), \tag{1.8}$$

and equality holds if and only if $\|\frac{d\gamma}{dt}\| \equiv const.$

Proof. By Hölder 's inequality

$$\int_a^b \|\frac{d\gamma}{dt}\|dt \leq (b-a)^{\frac{1}{2}} \left(\|\frac{d\gamma}{dt}\|^2 dt\right)^{\frac{1}{2}}$$

with equality precisely if $\|\frac{d\gamma}{dt}\| \equiv const.$ □

Lemma 6. *If* $\gamma : [a,b] \to \mathcal{M}$, *is a smooth curve,and* $\psi : [\alpha,\beta] \to [a,b]$ *is a change of parameter, then*

$$L(\gamma \circ \psi) = L(\gamma).$$

Proof. By the chain rule,

$$L(\gamma \circ \psi) = \int_a^b \left(g_{ij}(x(\gamma(\psi(\tau))))\dot{x}^i(\psi(\tau))\dot{x}^j(\psi(\tau))\left(\frac{d\psi}{d\tau}\right)^2\right)^{\frac{1}{2}} d\tau$$

and by a change of variables,

$$= L(\gamma).$$

□

1.4 Curvature

In this section,we introduce some basic notions of curvature in a Riemannian manifold and state basic properties.The proof can be found in many books, e.g. in [11] ,[17],[26],[33],[36],[37],[40] and [45]. Throughout, we assume all manifolds to be of class $\mathbb{C}^\infty$ and the manifold topology should be Hausdroff and satisfy the second countable axiom.

Definition 8. *We define* $\mathfrak{X}(M)$ *to be the set of all smooth vector fields on* M.

Definition 9. *Let $\mathcal{M}$ be a Riemannian manifold. The Riemannian curvature tensor is the map $R : \mathfrak{X}(\mathcal{M}) \times \in \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \longrightarrow \mathfrak{X}(\mathcal{M})$ defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ denotes the Levi-Civita connection in $\mathcal{M}$.

Remark 1. *Various definitions of the Riemannian curvature tensor may differ by a sign. The definition used here is also used in [11] , [26],[33],[36],[37],[40] and [45].*

Proposition 1. *The map R is a $(1, 3)-$ tensor field and satisfies*

1. $R(X, Y)Z = -R(Y, X)Z$,
2. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

The second equation is called the First Bianchi Identity.

We want to give a local expression of R.
Let $(\varphi = (x^1, x^2, ..., x^n), U)$ be a local coordinate system of $\mathcal{M}$. We write $\partial_i = \frac{\partial}{\partial x^i}$. Let the functions g_{ij} and Γ_{ij}^k on U be defined by

$$g_{ij} = g(\partial_i, \partial_j), \quad \nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k.$$

The coefficients g_{ij} are called the local expression of the Riemannian metric. The Γ_{ij}^k are called the Christoffel symbols of the Levi-Civita connection of $\mathcal{M}$.
Using the properties of the Levi-Civita connection, it is not hard to see that

$$\sum_l \Gamma_{ij}^l g_{lk} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{1.9}$$

Let

$$R(\partial_i, \partial_j)\partial_k = \sum_l R_{ijk}^l \partial_l. \tag{1.10}$$

The R_{ijk}^l are the components of R in the coordinate system (φ, U).

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k = \nabla_{\partial_i} \left(\sum_l \Gamma_{jk}^l \partial_l \right) - \nabla_{\partial_j} \left(\sum_l \Gamma_{ik}^l \partial_l \right),$$

which by a direct calculation yields

$$R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \frac{\partial}{\partial x^j} \Gamma_{ik}^s - \frac{\partial}{\partial x^i} \Gamma_{jk}^s \tag{1.11}$$

Remark 2. *If $\mathcal{M}$ is $1-$ dimensional, then $R = 0$: Let X, Y, $Z \in \mathfrak{X}(\mathcal{M})$ and let $\{E\}$ be a local frame field. Then we can write $X = f.E$, $Y = g.E$ for smooth functions f and g and therefore,*

$$R(X, Y)Z = R(f.E, g.E)Z = fg.R(E, E)Z = 0$$

Consider the expression

$$(X,\, Y,\, Z\, W) \mapsto \langle R(X,\, Y)Z, W\rangle\,,\ X, Y, Z \in \mathfrak{X}(M).$$

By proposition (1), it is a $(0,4)-$ tensor field. It satisfies the following symmetries.

Proposition 2. *For all* $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, *we have*

$$\langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle = 0,$$

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= -\langle R(Y,X)Z, W\rangle\,,\\ \langle R(X,Y)Z, W\rangle &= -\langle R(X,Y)Z, W\rangle\,,\\ \langle R(X,Y)Z, W\rangle &= -\langle R(Z,W)X, Y\rangle\,.\end{aligned}$$

Definition 10. *Let* $\mathcal{M}^n$, $n \geq 2$ *be a Riemannian manifold,* $p \in \mathcal{M}$ *and* $\sigma \subset T_p\mathcal{M}$ *be a two-dimensional subspace of* $T_p\mathcal{M}$. *The sectional curvature of* σ *is defined as*

$$K_p(\sigma) := \frac{\langle R_p(x,y), x, y\rangle_p}{< x, x >_p < y, y >_p - < x, y >_p^2},$$

where $\{x, y\}$ *is a basic of* σ.

By straightforward calculation, it is shown,that $K_p(\sigma)$ is well defined, *i.e.* it is independent of the particular choice of the basic $\{x, y\}$. We write $K_p(\sigma)$ or $K_p(x, y)$ for a basic $\{x, y\}$ of σ.

Lemma 7. *Let* $\mathcal{M}$ *be a Riemannian manifold. If the sectional curvature* K *of* $\mathcal{M}$ *is constant for all* $p \in \mathcal{M}$ *and all two dimensional subspaces* $\sigma \subset T_p\mathcal{M}$, *then*

$$\langle R(X,Y)Z, W\rangle = K[< X, Z >< Y, W > - < Y, Z >< X, W >],\ \forall X, Y, Z, W \in \mathfrak{X}(\mathcal{M})$$

Lemma 8. *Let* $(\mathcal{M}, g)$ *be a Riemannian manifold,* $\lambda > 0$ *and* $\overline{g} = \lambda.g$. *Let* K, $\overline{K}$ *be the sectional curvature with respect to the metric* g, $\overline{g}$, *respectively. Then*

$$\overline{K} = \frac{1}{\lambda}K.$$

Proof. Let $p \in \mathcal{M}$ and $(\varphi = (x^1, ..., x^n), U)$ local coordinates around p. We will distinguish the various objects corresponding to g, $\overline{g}$ by a bar. By (1.9), $\Gamma_{ij}^k = \overline{\Gamma}_{ij}^k$, and therefore,by (1.11), $R_{ijk}^l = \overline{R}_{ijk}^l$. It follows that the curvature tensors R and $\overline{R}$ are equal. Thus, for x, $y \in T_p\mathcal{M}$ linearly independent,

$$\begin{aligned}\overline{K}_p(x,y) &= \tfrac{\overline{g}_p(\overline{R}_p(x,y)x,y)}{\overline{g}_p(x,x).\overline{g}_p(y,y)-\overline{g}_p(x,y)^2}\\ &= \tfrac{\lambda.g(R_p(x,y)x,y)}{\lambda^2[g_p(x,x).g_p(y,y)-g_p(x,y)^2]}\\ &= \tfrac{1}{\lambda}K_p(x,y).\end{aligned}$$

□

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold equipped with the Riemannian metric g.

Definition 11. *For any smooth function f on $\mathcal{M}$, the gradient ∇f is a vector field on $\mathcal{M}$, which is locally coordinates $x_1, x_2 ....., x_n$ has the form*

$$(\nabla f)_i = g^{ij} \frac{\partial f}{\partial x_j},$$

where summation is assumed over repeated indices. For any smooth vector field F on $\mathcal{M}$, the divergence $divF$ is a scalar function on $\mathcal{M}$, which is given in local coordinates by

$$divF = \frac{1}{\sqrt{detg_{ij}}} \frac{\partial}{\partial x_i} (\sqrt{detg_{ij}} F_i)$$

Let ν be the Riemannian volume on $\mathcal{M}$, that is

$$\nu = \sqrt{detg_{ij}} dx_1 ..... dx_n.$$

By the divergence theorem, for any smooth function f and a smooth vector field F, such that either f or F has compact support,

$$\int_{\mathcal{M}} f divF d\nu = - \int_{\mathcal{M}} < \nabla f, F > d\nu. \tag{1.12}$$

where $< .,. >= g(.,.)$. In particular, if $F = \nabla \psi$ for a function ψ then we obtain

$$\int_{\mathbb{M}} f div \nabla \psi d\nu = - \int_{\mathcal{M}} < \nabla f, \nabla \psi > d\nu. \tag{1.13}$$

provided one of the functions f, ψ has compact support. The operator

$$\Delta := div \circ \nabla$$

is called the Laplace (or Laplace-Beltrami) operator of the Riemannian manifold $\mathcal{M}$. From (1.13) we obtain the Green formulas

$$\int_{\mathcal{M}} f \Delta \psi d\nu = - \int_{\mathcal{M}} < \nabla f, \nabla \psi > d\nu = \int_{\mathcal{M}} \psi \Delta f d\nu. \tag{1.14}$$

1.5 Covering spaces

Next we recall some basic of the theory of covering spaces. For more details, see [29].

Definition 12. *Let X be a topological space. A covering of X is a continuous map $\pi : \overline{X} \to X$ such that for each $x \in X$ there exists an open neighborhood U with the following properties: These exists an index set A and disjoint open sets $\overline{U}_\lambda \subset \overline{X}$, $\lambda \in A$ such that $\pi^{-1}(U) = \bigcup_{\lambda \in A} \overline{U}_\lambda$ and for each $\lambda \in A$, $\pi|_{\overline{U}_\lambda} : \overline{U}_\lambda \to U$ is a homeomorphism. We say that U is relatively covered.*
In this regard,we call X the basis, $\overline{X}$ the covering space and π the covering map. For $x \in X$, the discrete set $\pi^{-1}(x) \subset \overline{X}$ is called the fiber over x.

Definition 13. *Two coverings* $\pi_1\,; \overline{X}_1 \to X$ *and* $\pi_2\,; \overline{X}_2 \to X$ *are called isomorphic if there exists a homeomorphism* $\varphi\,; \overline{X}_1 \to \overline{X}_2$ *such that* $\pi_2 \circ \varphi = \pi_1$. *If* $\overline{X}_1 = \overline{X}_2$ *and* $\pi_1 = \pi_2$, φ *is called a covering transformation. The covering transformation of* $\overline{X}$ *form a subgroup of the homeomorphism group of* $\overline{X}$, *which we denote by* $C(\overline{X})$.

Example 3. *The map* $\pi\ : \mathbb{R} \to \mathbb{S}^1$, *defined by* $\pi(t) = e^{2\pi i t}$ *is a covering map. For each* $k \in \mathbb{Z}$, $\varphi_k\ : t \mapsto t + k$ *is a covering transformation.*

Proposition 3. *(Lifting property) Let* $\varphi,\ : \overline{X} \to X$ *be a covering and* Y *be locally path-connected and simply-connected. Fix a point* $y \in Y$ *and let* $f\ : Y \to X$ *be continuous. Then for each* $\overline{x} \in \pi^{-1}(f(y))$, *there exists a unique continuous map* $\overline{f}\ : Y \to \overline{X}$ *such that* $\pi \circ \overline{f} = f$ *and* $f(y) = \overline{x}$.

Proof. See [29], Proposition 1.33 and Proposition 1.34. □

Such a map $\overline{f}$ is called a lift of f over π. In particular, if $\alpha\ : [0, a] \to X$ is a path in X and a point $\overline{p}x \in \pi^{-1}(\alpha(0))$ is fixed, there exists a unique path $\overline{\alpha}\ : [0, a] \to \overline{X}$ such that $\pi \circ \overline{\alpha} = \alpha$ and $\overline{\alpha}(0) = \overline{p}x$.

Definition 14. *A covering* $\pi\ : \overline{X} \to X$ *is called a universal covering if* $\overline{X}$ *is simply connected.*

Remark 3. *A universal covering* $\pi\ : \overline{X} \to X$ *is the largest covering of* X *in the following sense. If* X *is connected and locally path-connected and* $\pi_1\ : \overline{X} \to X$ *is another covering with* $\overline{X}_1$ *connected, there exists* $\pi_0\ : \overline{X} \to \overline{X}_1$ *such that* $\pi_1 \circ \pi_0 = \pi$.

A universal covering is uniquely determined up to isomorphy: Let $\pi_1\ : \overline{X}_1 \to X$ *and* $\pi_2\ : \overline{X}_2 \to X$ *be two universal coverings. Fix* $x \in X$ *and* $\overline{x}_i \in \pi^{-1}(x)$, $i = 1, 2$. *By proposition 3, there exists a map* $\varphi\ : \overline{X}_1 \to \overline{X}_2$ *such that* $\pi_2 \circ \varphi = \pi_1$ *and* $\varphi(\overline{x}_1) = \overline{x}_2$ *and a map* $\psi\ : \overline{X}_2 \to \overline{X}_1$ *satisfying* $\pi_1 \circ \varphi = \pi_2$ *and* $\psi(\overline{x}_2) = \overline{x}_1$. *By proposition 3,* $\psi \circ \varphi = id_{\overline{X}_1}$ *and* $\varphi \circ \psi = id_{\overline{X}_2}$.
In particular, if X *is simply connected, each universal covering of* X *is isomorphic to the trivial covering* $id\ : X \to X$.
A universal covering does not always exist.Some connectedness conditions are required.

Theorem 2. *If* X *is connected,locally path-connected and semilocally simply-connected, there exists a universal covering of* X.

Proof. See ([29]). □

Proposition 4. *Let* X *satisfy the conditions of Theorem 2 and* $\pi\ : \overline{X} \to X$ *be the universal covering of* X. *Then*

(i) The group $C(\overline{X})$ *is isomorphic to the fundamental group* $\pi_1(X)$.

(ii) For each $x \in X$, $C(\overline{X})$ *is acting sharply transitive on* $\pi^{-1}(x)$, *i.e. for* $\overline{x}_1, \overline{x}_2 \in \pi^{-1}(x)$, *there exists a unique element* $\varphi \in C(\overline{X})$ *such that* $\varphi(\overline{x}_1) = \overline{x}_2$.

Proof. For a proof of (i), see [29]. By Proposition 3 there exist unique maps $\varphi,\, \phi\ : \overline{X} \to X$ satisfying $\phi(\overline{x}_1) = \overline{x}_2, \varphi(\overline{x}_2) = \overline{x}_1$ and $\pi \circ \phi = \pi \circ \varphi = \pi$. By uniqueness in Proposition 3,$\phi \circ \phi = \varphi \circ \phi = id_{\overline{M}}$, so $\phi,\, \varphi \in C(X)$ which proves (ii). □

Example 4. *Let* $\mathbb{R}P^n$, $n \geq 2$ *be the real projective space, obtained from identifying the antipodal points of* S^n. *Then the canonical projection* $\pi : S^n \to \mathbb{R}P^n$ *is the universal covering of* $\mathbb{R}P^n$ *and the covering transformation group* $\mathcal{C}(S^n)$ *consists precisely of the maps* $\pm id : p \mapsto \pm p$. *By Proposition 4(i), the fundamental group* $\pi_1(\mathbb{R}P^n)$ *is isomorphic to* $\mathbb{Z}_2$.

If $\mathcal{M}$ is a manifold and $\pi : \overline{\mathcal{M}} \to \mathcal{M}$ is a covering,then there exists a unique smooth stricture on $\overline{\mathcal{M}}$ such that π is a local diffeomorphism. Then all covering transformations are diffeomorphisms on $\overline{\mathcal{M}}$. Since $\mathcal{M}$ is locally diffeomorphic to $\mathbb{R}^n$, there exists an universal covering of $\mathcal{M}$ if $\mathcal{M}$ is connected. If two smooth maps $\pi_i : \overline{\mathcal{M}}_i \to \mathcal{M}\, i = 1, 2$ are universal coverings,there exists a diffeomorphism $\phi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$, such that $\pi_2 \circ \phi = \pi_1$.
Let $(\mathcal{M}, g)$ be a connected Riemannian manifold and $\pi : \overline{\mathcal{M}} \to \mathcal{M}$ be the universal covering of $\mathcal{M}$. Then we define a Riemannian metric $\overline{g}$ on $\overline{\mathcal{M}}$ by $\overline{g} = \pi^* g$,i.e.$\overline{g}_{\overline{p}}(v, w) := g_{\pi(\overline{p})}(T_p\pi(v), T_p\pi(w))$. This metric on $\overline{\mathcal{M}}$ is called the covering metric. It is the unique metric on $\overline{\mathcal{M}}$ such that $\pi : \overline{\mathcal{M}} \to \mathcal{M}$ is local isometry.

Proposition 5. *Let* $\mathcal{M}$ *be a connected Riemannian manifold and* $\pi : \overline{\mathcal{M}} \to \mathcal{M}$ *the universal covering of* $\mathcal{M}$ *with the covering metric. To each vector field* $X \in \mathfrak{X}(\mathcal{M})$ *we associate a vector field* $\pi^*(X) \in \mathfrak{X}(\overline{\mathcal{M}})$, *defined by* $\pi^*(X)_{\overline{p}} = (T_{\overline{p}}\pi)^{-1}(X_{\pi(\overline{p})})$. *Then we have*

(i) $\nabla^{\overline{g}}_{\pi^*(X)}\pi^*(Y) = \nabla^g_X Y$ *for all* X, $Y \in \mathfrak{X}(\mathcal{M})$.

(ii) If γ *is a geodesic in* $\mathcal{M}$, *then each lift of* γ *over* π *is a geodesic in* $\overline{\mathcal{M}}$.

(i i i) $R^{\overline{g}}(\pi^*(X), \pi^*(Y))\pi^*(Z) = R^g(X, Y)Z$ *for all* X, Y, $Z \in \mathfrak{X}(\mathcal{M})$.

(iv) $K^{\overline{g}}_{\overline{p}}(\sigma) = K^{\overline{g}}_{\pi(\overline{p})}(T_{\overline{p}}\pi(\sigma))$ *for any* $\overline{p} \in \overline{\mathcal{M}}$ *and any two-dimensional subspace* $\sigma \subset T_p\overline{\mathcal{M}}$.

(v) If $\mathcal{M}$ *is complete,* $\overline{\mathcal{M}}$ *is also complete.*

Proof. Properties (i)-(iv) are general properties of local isometries. To show that completeness of $\overline{M}$, let $\overline{\gamma}_0$ be a geodesic in $\overline{\mathcal{M}}$, starting at $\overline{p}$. By completeness of $\mathcal{M}$, the geodesic $\gamma_0 = \pi \circ \overline{\gamma}_0$ can be extend to a geodesic γ, which is defined for all time. Its lift starting at $\overline{p}$ is an extension of $\overline{\gamma}_0$ which is defined for all time. This proves (v). □

1.6 Space forms

We now want to describe manifolds of constant sectional curvature K. We may assume, by similarity, that the sectional curvature $K = -1, 0, 1$, cf.Lemma8. Let $\mathbb{R}^n$ be equipped with the usual metric, that is $g_{ij} = \delta_{i,j}$ in the natural coordinate system. By (1.9) and (1.11), it is easy to see that $R \equiv 0$ and therefore $K \equiv 0$. In Chapter 6 of [17], example 2.8 it is shown that the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the usual metric has constant sectional curvature $K = 1$. Both spaces are complete and simply connected.
We also want to give an example of a complete and simply connected manifold with constant sectional curvature $K = -1$. Consider the half-space of $\mathbb{R}^n$ given by

$$\mathbb{H}^n = \{(x_1, ..., x_n) | x_n > 0\}$$

and introduce on $\mathbb{H}^n$ the metric

$$g_x(v,w) = \frac{1}{x_n^2}(v_1w_1 + ... + v_nw_n).$$

The space $\mathbb{H}^n$ together with this metric is called the Poincaré half-plane model. Clearly, $\mathbb{H}^n$ is simply connected.

Proposition 6. *The Riemannian manifold $\mathbb{H}^n$ is completes and its sectional curvature K satisfies $K \equiv -1$.*

Theorem 3. *Let $\mathcal{M}^n$ be a complete Riemannian manifold with constant sectional curvature K. Then the universal covering $\overline{M}$ of $\mathcal{M}$ is isometric to :*

(i) $\mathbb{H}^n$ *if* $K = -1$,

(ii) $\mathbb{R}^n$ *if* $K = 0$,

(i i i) $\mathbb{S}^n$ *if* $K = 1$.

Remark 4. *Let $\mathcal{M}^n$ as above and $[K] \neq 1$. From Lemma 8, we obtain the following*

(i) If $K > 0$ then $\overline{\mathcal{M}}$ is isometric to $\mathbb{S}^n(K)$, the sphere with radius $\frac{1}{\sqrt{K}}$.

(ii) If $K < 0\,\overline{\mathcal{M}}$ is isometric to $H^n(K)$, the half-sphere of $\mathbb{R}^n$ with the metric

$$g_x(v,w) = \frac{1}{K.x_n^2}(v_1w_1 + ... + v_nw_n).$$

Definition 15. *The complete simply-connected Riemannian manifolds with constant sectional curvature are called spaces forms.*

1.7 Harmonic maps and biharmonic maps

Definition 16. *Consider a smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds, for any compact domain D of M the energy functional of φ is defined by*

$$E(\varphi; D) = \frac{1}{2}\int_D |d\varphi|^2\, v_g, \tag{1.15}$$

where $|d\varphi|$ is the Hilbert Schmidt norm of differential of the map φ given by

$$|d\varphi|^2 = \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i))$$

and $\{e_1, \ldots, e_m\}$ be an orthonormal frame on M

Definition 17. *A variation of φ to support in a compact domain $D \subset M$, is a smooth family maps $(\varphi_t)_{t\in(-\epsilon,\epsilon)} : M \to N$, such that $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ on $M \setminus \operatorname{int}(D)$.*

Definition 18. *A map is called harmonic if it is a critical point of the energy functional over any compact subset D of M. i.e*

$$\frac{d}{dt}E(\varphi_t; D)\bigg|_{t=0} = 0.$$

1.7.1 First variation of energy

Theorem 4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E(\varphi_t; D)\bigg|_{t=0} = -\int_D h(v, \tau(\varphi))\, v_g,$$

where $v = \dfrac{d\varphi_t}{dt}\bigg|_{t=0}$ *denotes the variation vector field of* $\{\varphi_t\}$*,*

$$\tau(\varphi) = \mathrm{trace}_g \nabla d\varphi = \sum_{i=1}^{m} \left\{\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^{M}_{e_i} e_i)\right\} \tag{1.16}$$

is called tension field of φ *where* $\{e_1, ..., e_m\}$ *is an orthonormal frame on* (M^m, g)*.*

Proof. Defined $\phi : M\times(-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$, let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have $[\partial_t, X] = 0$. Using (1.15) we obtain

$$\begin{aligned}
\frac{d}{dt}E(\varphi_t; D)\bigg|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^{m} h(d\varphi_t(e_i), d\varphi_t(e_i))\, v_g\bigg|_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^{m} h(d\phi(e_i, 0), d\phi(e_i, 0))\, v_g\bigg|_{t=0} \\
&= \frac{1}{2}\int_D \frac{\partial}{\partial t}\sum_{i=1}^{m} h(d\phi(e_i, 0), d\phi(e_i, 0))\, v_g\bigg|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(e_i, 0), d\phi(e_i, 0))\, v_g\bigg|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\phi}_{(e_i,0)} d\phi(0, \frac{d}{dt}), d\phi(e_i, 0))\, v_g\bigg|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{N}_{d\varphi(e_i)} v, d\varphi(e_i))\, v_g \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} v, d\varphi(e_i))\, v_g.
\end{aligned} \tag{1.17}$$

Define an 1-form on M by

$$\omega(X) = h(v, d\varphi(X)), \quad X \in \Gamma(TM).$$

We have

$$\begin{aligned}
\operatorname{div}^M\omega &= (\nabla_{e_i}\omega)(e_i)\\
&= \sum_{i=1}^{m}\left\{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i}e_i)\right\}\\
&= \sum_{i=1}^{m}\left\{h(\nabla^{\varphi}_{e_i}v, d\varphi(e_i)) + h(v, \nabla^{\varphi}_{e_i}d\varphi(e_i)) - h(v, d\varphi(\nabla^M_{e_i}e_i))\right\}\\
&= \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i}v, d\varphi(e_i)) + h(v, \tau(\varphi)), \qquad (1.18)
\end{aligned}$$

according to formulas (1.17), (1.18), and if

$$\int_D \operatorname{div}(\omega)v_g = 0 \qquad (1.19)$$

we obtain

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi))v_g.$$

□

Theorem 5. *A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is harmonic if and only if*

$$\tau(\varphi) = \mathrm{trace}\nabla d\varphi = 0.$$

Example 5. *The second fundamental form of the identity mapping $\mathrm{Id_M} : (M, g) \longrightarrow (M, g)$ is zero, i.e. $\mathrm{Id_M}$ is totally geodesic, therefore $\mathrm{Id_M}$ is harmonic.*

Example 6. *Let (M, g) be a Riemannian manifold and let $f : M \longrightarrow \mathbb{R}$ be a smooth function, then*

$$\begin{aligned}
\tau(f) &= \mathrm{trace}\nabla df\\
&= \nabla df(e_i, e_i)\\
&= \nabla^f_{e_i}df(e_i) - df(\nabla^M_{e_i}e_i)\\
&= e_i(e_i(f)) - (\nabla^M_{e_i}e_i)(f)\\
&= g(\nabla_{e_i}\mathrm{grad}\, f, e_i)\\
&= \mathrm{div}\,\mathrm{grad} f\\
&= \triangle(f),
\end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on M.

Example 7. *If $M =]a, b[$ be an interval of $\mathbb{R}$, then a curve $\gamma : (a, b) \longrightarrow (N^n, h)$ is harmonic if*

$$\frac{d^2\gamma^\alpha}{dt^2} +^N \Gamma^\alpha_{\beta\delta}\frac{d\gamma^\beta}{dt}\frac{d\gamma^\delta}{dt} = 0,$$

therefore, γ is harmonic if and only if it is a geodesic.

1.8 Biharmonic maps

The bi-energy functional of a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is defined by

$$E_2(\varphi, D) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v_g. \tag{1.20}$$

Definition 19. *A map is called biharmonic if it is a critical point of the bi-energy functional over any compact subset D of M.*

1.8.1 First variation of bi-energy

Theorem 6. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then*

$$\frac{d}{dt}E_2(\varphi_t; D)|_{t=0} = -\int_D h(v, \tau_2(\varphi))v_g,$$

where $v = \frac{d\varphi_t}{dt}|_{t=0}$ denotes the variation vector field of φ and in locale frame at $x \in M$, we have

$$\begin{aligned}
\tau_2(\varphi) &= -\mathrm{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \mathrm{trace}_g(\nabla^{\varphi})^2\tau(\varphi)\\
&= -\sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m}\{\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)\\
&\quad -\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)\}
\end{aligned} \tag{1.21}$$

$\tau_2(\varphi)$ is called the bi-tension field of φ.

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ by $\phi(x, t) = \varphi_t(x)$.
First note that

$$\frac{d}{dt}E_2(\varphi_t; D)|_{t=0} = \int_D \sum_{i=1}^{m} h\left(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i, 0), (e_i, 0)), \nabla d\phi((e_i, 0), (e_i, 0))\right) v_g|_{t=0}. \tag{1.22}$$

Calculating in a normal frame at $x \in M$ we have

$$\begin{aligned}
\nabla^{\phi}_{(0,\frac{d}{dt})}d\phi(e_i, 0) &= \nabla^{\phi}_{(e_i,0)}d\phi(0, \frac{d}{dt}) + d\phi([(0, \frac{d}{dt}), (e_i, 0)])\\
&= \nabla^{\phi}_{(e_i,0)}d\phi(0, \frac{d}{dt}).
\end{aligned} \tag{1.23}$$

$$\nabla^{\phi}_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i, 0) = \nabla^{\phi}_{(\nabla^M_{e_i}e_i,0)}d\phi(0, \frac{d}{dt}). \tag{1.24}$$

$$\begin{aligned}
\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i, 0), (e_i, 0)) &= \nabla^{\phi}_{(0,\frac{d}{dt})}\nabla^{\phi}_{(e_i,0)}d\phi(e_i, 0) - \nabla^{\phi}_{(0,\frac{d}{dt})}d\phi\left(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i, 0)\right)\\
&= R^N(d\phi(0, \frac{d}{dt}), d\phi(e_i, 0))d\phi(e_i, 0) + \nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{(0,\frac{d}{dt})}d\phi(e_i, 0)\\
&\quad +\nabla^{\phi}_{[(0,\frac{d}{dt}),(e_i,0)]}d\phi(e_i, 0) - \nabla^{\phi}_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i, 0).\\
&= R^N(d\phi(0, \frac{d}{dt}), d\phi(e_i, 0))d\phi(e_i, 0) + \nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{(e_i,0)}d\phi(0, \frac{d}{dt})\\
&\quad -\nabla^{\phi}_{(\nabla^M_{e_i}e_i,0)}d\phi(0, \frac{d}{dt}).
\end{aligned} \tag{1.25}$$

From where

$$\begin{aligned}&h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\ &h(R^N(v,d\varphi(e_i))d\varphi(e_i),\tau(\varphi)) + h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,\tau(\varphi)) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i}v,\tau(\varphi)).\end{aligned} \tag{1.26}$$

Let $\omega \in \Gamma(T^*M)$, be a 1-form to support in D, defined by:

$$\omega(X) = h(\nabla^{\varphi}_X v,\tau(\varphi)), \qquad X \in \Gamma(TM).$$

We calculate the divergence of ω

$$\begin{aligned}\operatorname{div}^M\omega &= \sum_{i=1}^{m}\{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i}e_i)\}\\ &= \sum_{i=1}^{m}\{e_i(h(\nabla^{\varphi}_{e_i}v,\tau(\varphi))) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i}v,\tau(\varphi))\}\\ &= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,\tau(\varphi)) + h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i}v,\tau(\varphi))\}.\end{aligned} \tag{1.27}$$

From the formulas (1.26) and (1.27), we obtain:

$$\begin{aligned}&\sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\ &\sum_{i=1}^{m} h(R^N(v,d\varphi(e_i))d\varphi(e_i),\tau(\varphi)) + \operatorname{div}^M\omega - \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)).\end{aligned} \tag{1.28}$$

Let $\eta \in \Gamma(T^*M)$, be an 1-form to support in D, given by

$$\eta(X) = h(v,\nabla^{\varphi}_X\tau(\varphi)), \qquad X \in \Gamma(TM).$$

We calculate the divergence of η

$$\begin{aligned}\operatorname{div}^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i}e_i)\}\\ &= \sum_{i=1}^{m}\{e_i(h(v,\nabla^{\varphi}_{e_i}\tau(\varphi))) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi))\}\\ &= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) + h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi))\}.\end{aligned} \tag{1.29}$$

Substituting (1.29) in (1.28), we obtain

$$\begin{aligned}&\sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\ &\sum_{i=1}^{m} h(R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i),v) + \mathrm{div}^M\omega - \mathrm{div}^M\eta\\ &+\sum_{i=1}^{M} h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) - \sum_{i=1}^{M} h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)).\end{aligned} \tag{1.30}$$

From the formulas (1.22), (1.30) and according and if

$$\int_D \mathrm{div}(\omega)v_g = 0, \tag{1.31}$$

we obtain

$$\frac{d}{dt}E_2(\varphi_t;D)|_{t=0} = -\int_D \sum_{i=1}^{m} h\left(-R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi),v\right)v_g.$$

□

Theorem 7. *Let $\varphi : (M^m,g) \longrightarrow (N^n,h)$ be a smooth map between two Riemannian manifolds, then φ is said biharmonic if and only if*

$$\tau_2(\varphi) = -\mathrm{trace}_g R^N(\tau(\varphi),d\varphi)d\varphi - \mathrm{trace}_g(\nabla^{\varphi})^2\tau(\varphi) = 0. \tag{1.32}$$

1. The equation (1.32) is called the Euler-Lagrange equation.

2. Let M and N be two Riemanian manifolds with the coordinates (x^i) and (y^α) respectively, then, in the neighborhood of the points $x \in M$ and $\varphi(x) \in N$ we have

$$\begin{aligned}\tau_2(\varphi) \ =\ & g^{ij}\{\frac{\partial^2\tau^\sigma}{\partial x^i\partial x^j} + 2\frac{\partial\tau^\sigma\partial\tau^\beta}{\partial x^j\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} + \tau^\alpha\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta}\\ &+\tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial^N\Gamma^\sigma_{\alpha\beta}}{\partial x^j} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\rho}{\partial x^j}{}^N\Gamma^\upsilon_{\alpha\beta}{}^N\Gamma^\sigma_{\upsilon\rho}\\ &-{}^M\Gamma^k_{ij}(\frac{\partial\tau^\sigma}{\partial x^k} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^k}{}^N\Gamma^\sigma_{\alpha\beta}) - \tau^\upsilon\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N R^\sigma_{\beta\alpha\upsilon}\}\frac{\partial}{\partial y^\sigma}\circ\varphi,\end{aligned}$$

   where $\tau^\gamma = g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} + \frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \frac{\partial\varphi^\gamma}{\partial x^k}{}^M\Gamma^k_{ij}\right)$ and ${}^N R^\sigma_{\beta\alpha\upsilon}$ designate the components of the curvature tensor of (N^n,h).

3. Any harmonic map is a biharmonic.

4. Biharomnic maps are not generally harmonic maps.

Example 8. 1. *The polynomials of degrees 3 on $\mathbb{R}$ are biharmonic non-harmonic maps.*

2. *The identity map $Id : (M^m,g) \longrightarrow (M^m,g)$ is biharmonic.*

3. *A smooth map $\varphi : (M^m,g) \longrightarrow (\mathbb{R}^n,<,>_{\mathbb{R}^n})$, is biharmonic if and only if $\Delta^M(\Delta^M\varphi^\sigma) = 0$, for all $\sigma = 1,\dots,n$.*

Chapter 2

Geometry of tangent and cotangent bundle with Sasaki metric

Sasakian metrics on tangent bundle were introduce in 1958 by the Japanese geometer Sasaki [47]. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in ([53], [34]) . In this chapter is devoted to the geometry of T^*M endowed with the Sasaki metric g^s and obtain several important geometry consequences on curvature properties, sectional curvature and scalar Curvature.

2.1 Basic Notions and Definition on TM.

Let (M,g) be an n-dimensional Riemannian manifold and (TM,π,M) be its tangent bundle. A local chart $(U,x^i)_{i=1\ldots n}$ on M induces a local chart $(\pi^{-1}(U),x^i,y^i)_{i=1\ldots n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\begin{aligned}\mathcal{V}_{(x,u)} &= \ker(d\pi_{(x,u)})\\ &= \{a^i\frac{\partial}{\partial y^i}_{|(x,u)};a^i\in\mathbb{R}\},\\ \mathcal{H}_{(x,u)} &= \{a^i\frac{\partial}{\partial x^i}_{|(x,u)}-a^iu^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}_{|(x,u)};a^i\in\mathbb{R}\},\end{aligned}$$

where $(x,u)\in TM$, such that $T_{(x,u)}TM=\mathcal{H}_{(x,u)}\oplus\mathcal{V}_{(x,u)}$.
Let $X=X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V=X^i\frac{\partial}{\partial y^i}, \tag{2.1}$$

and

$$X^H=X^i\frac{\delta}{\delta x^i}=X^i\Big\{\frac{\partial}{\partial x^i}-y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}\Big\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1..n}$ is a local adapted frame in TTM.

Note that the maps $X \mapsto X^H$ and $X \mapsto X^V$ are vector space isomorphisms between T_pM and the subspaces $\mathcal{H}_{(x,u)}$ and $\mathcal{V}_{(x,u)}$, respectively, and each tangent vector $Z \in T_{(p,u)}M$ can be decomposed uniquely as

$$Z = X^H + Y^V,$$

where $X, Y \in T_pM$.

Definition 20. *([27]).The Sasaki metric g^s on the tangent bundle TM of M is given by*

1. $g^s(X^H, Y^H) = g(X,Y) \circ \pi$,
2. $g^s(X^H, Y^V) = 0$,
3. $g^s(X^V, Y^V) = g(X,Y) \circ \pi$,

for all vector fields $X, Y \in \Gamma(TM)$.

Proposition 7. *([27]). Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then*

$$\begin{aligned}(\widehat{\nabla}_{X^H}Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V,\\ (\widehat{\nabla}_{X^H}Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H,\\ (\widehat{\nabla}_{X^V}Y^H)_{(x,u)} &= \frac{1}{2}(R_x(u,X)Y)^H,\\ (\widehat{\nabla}_{X^V}Y^V)_{(x,u)} &= 0,\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x,u) \in TM$.

Definition 21. *Let (M,g) be a Riemannian manifold and $F \in \mathfrak{T}^1_1(M)$ be a tensor of type (1,1). Then we define a vertical and horizontal vector fields VF, HF on TM by*

$$\begin{aligned} VF : TM &\longrightarrow TTM\\ (x,u) &\longmapsto (F(u))^V,\end{aligned}$$

and

$$\begin{aligned} HF : TM &\longrightarrow TTM\\ (x,u) &\longmapsto (F(u))^H.\end{aligned}$$

Proposition 8. *([12]). Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. If $F \in \mathfrak{T}^1_1(M)$ is a tensor of type (1,1), then*

$$\begin{aligned}(\widehat{\nabla}_{X^H}HF)_{(x,u)} &= H(\nabla_X F)_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V,\\ (\widehat{\nabla}_{X^H}VF)_{(x,u)} &= V(\nabla_X F)_{(x,u)} + \frac{1}{2}(R_x(u, F_x(u))X_x)^H,\\ (\widehat{\nabla}_{X^V}HF)_{(x,u)} &= (F(X))^H_{(x,u)} + \frac{1}{2}(R_x(u, X_x)F(u))^H,\\ (\widehat{\nabla}_{X^V}VF)_{(x,u)} &= (F(X))^V_{(x,u)},\end{aligned}$$

where $(x,u) \in TM$ *and* $X \in \Gamma(TM)$.

Proposition 9. *([27]). Let* (M,g) *be a Riemannian manifold and* $\widehat{R}$ *be the Riemann curvature tensor of the tangent bundle* (TM, g^s) *equipped with the Sasaki metric. The the following formulae hold.*

$$\begin{aligned}
1.\widehat{R}(X^V,Y^V)Z^V &= 0,\\
2.\widehat{R}(X^V,Y^V)Z^H &= \Big[R(X,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}5u,Y)(R(u,X)Z)\Big]_x^H,\\
3.\widehat{R}(X^H,Y^V)Z^V &= -\Big[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X)\Big]_x^H,\\
4.\widehat{R}(X^H,Y^V)Z^H &= \Big[\frac{1}{4}R(R(u,Y)Z,X)u + \frac{1}{2}R(X,Z)Y\Big]_x^V + \frac{1}{2}\Big[(\nabla_X R)(u,Y)Z\Big]_x^H,\\
5.\widehat{R}(X^H,Y^H)Z^V &= \Big[R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u - \frac{1}{4}R(R(u,Z)X,Y)u\Big]_x^V\\
&+ \frac{1}{2}\Big[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\Big]_x^H,\\
6.\widehat{R}(X^H,Y^H)Z^H &= \frac{1}{2}\Big[(\nabla_Z R)(X,Y)u\Big]_x^V\\
&+ \Big[R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X + \frac{1}{4}R(u,R(X,Z)u)Y\\
&+ \frac{1}{2}R(u,R(X,Y)u)Z\Big]_x^H,
\end{aligned}$$

for all vectors $u, X, Y, Z \in T_xM$.

2.2 Cotangent bundles T^*M

Let (M,g) be a m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T_x^*M, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im_s^r(M)$ (resp. $\Im_s^r(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r,s). Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

Let $X = X^i \dfrac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $U \subset M$ of a vector and covector (1-form) field $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, respectively. Then the complete and horizontal lifts X^C, $X^H \in \Im_0^1(T^*M)$ of $X \in \Im_0^1(M)$ and the vertical lift $\omega^V \in \Im_0^1(T^*M)$ of $\omega \in \Im_1^0(M)$ are defined, respectively by

$$X^C = X^i \frac{\partial}{\partial x^i} - p_h \frac{\partial X^h}{\partial x^i}\frac{\partial}{\partial x^{\bar{i}}}, \tag{2.3}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.4}$$

$$\omega^V = \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.5}$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on (M,g) (see [53] for more details).

From (2.4) and (2.5) we see that $(\frac{\partial}{\partial x^i})^H$ and $(dx^i)^V$ have respectively local expressions of the form

$$\tilde{e}_i = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} + p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{2.6}$$

$$\tilde{e}_{\bar{i}} = (dx^i)^V = \frac{\partial}{\partial x^{\bar{i}}}. \tag{2.7}$$

The set $\{\tilde{e}_\alpha\} = \{\tilde{e}_i, \tilde{e}_{\bar{i}}\}$ is called the frame adapted to Levi-Civita connection ∇ on (M,g). The indices $\alpha, \beta, \ldots = \overline{1,2m}$ indicate the indices with respect to the adapted frame.

Using (2.4), (2.5) we have.

$$X^H = X^i \tilde{e}_i \ , \ X^H = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{2.8}$$

$$\omega^V = \omega_i \tilde{e}_{\bar{i}} \ , \ \omega^V = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \tag{2.9}$$

with respect to the adapted frame $\{\tilde{e}_\alpha\}_{\alpha=\overline{1,2m}}$, (see [53] for more details).

Lemma 9. *([53]) Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

(1) $[\omega^V, \theta^V] = 0$,

(2) $[X^H, \theta^V] = (\nabla_X \theta)^V$,

(3) $[X^H, Y^H] = [X,Y]^H + (pR(X,Y))^V$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Let (M,g) be a Riemannian manifold, we define the map

$$\begin{aligned} \sharp : \Im^0_1(M) &\to \Im^1_0(M) \\ \omega &\mapsto \sharp\omega \end{aligned}$$

by for all $X \in \Im^1_0(M)$, $g(\sharp\omega, X) = \omega(X)$, the map $\sharp$ is $C^\infty(M)$-isomorphism.
Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\sharp\omega = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).
For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by

$$g^{-1}(\omega, \theta) = g(\sharp\omega, \sharp\theta) = g^{ij}\omega_i \theta_j.$$

If ∇ be the Levi-Civita connection of (M,g) we have

$$\nabla_X(\sharp\omega) = \sharp(\nabla_X\omega), \tag{2.10}$$

$$Xg^{-1}(\omega,\theta) = g^{-1}(\nabla_X\omega,\theta) + g^{-1}(\omega,\nabla_X\theta), \tag{2.11}$$

for all $X \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$.

In the following, we noted $\sharp\omega$ by $\tilde{\omega}$ for all $\omega \in \Im^0_1(M)$.

Definition 22. *([48]) Let (M,g) be a Riemannian manifold. We define the Sasaki metric g^s on the cotangent bundle by*

$$\begin{aligned} g^s(X^H,Y^H) &= g(X,Y)\circ\pi, & (2.12)\\ g^s(X^H,\theta^V) &= 0, & (2.13)\\ g^s(\omega^V,\theta^V) &= g^{-1}(\omega,\theta), & (2.14) \end{aligned}$$

where $X,Y \in \Im^1_0(M)$, $\omega,\theta \in \Im^0_1(M)$.

2.2.1 Levi-Civita connection of g^s

We shall calculate the Levi-Civita connection ∇^s of T^*M with the Sasaki metric g^s. This connection is characterized by the Koszul formula:

Lemma 10. *([48]) Let (M,g) be a Riemannian manifold and (T^*M,g^s) its cotangent bundle equipped with the Sasaki metric. If ∇ (resp. ∇^s) denote the Levi-Civita connection of (M,g) (resp. (T^*M,g^s)), then we have*

$$\begin{aligned} 1)\ g^s(\nabla^s_{X^H}Y^H,Z^H) &= g^s((\nabla_XY)^H,Z^H),\\ 2)\ g^s(\nabla^s_{X^H}Y^H,\eta^V) &= \frac{1}{2}g^s((pR(X,Y))^V,\eta^V),\\ 3)\ g^s(\nabla^s_{X^H}\theta^V,Z^H) &= \frac{1}{2}g^s((R(\tilde{p},\tilde{\theta})X)^H,Z^H),\\ 4)\ g^s(\nabla^s_{X^H}\theta^V,\eta^V) &= g^s((\nabla_X\theta)^V,\eta^V),\\ 5)\ g^s(\nabla_{\omega^V}Y^H,Z^H) &= \frac{1}{2}g^s((R(\tilde{p},\tilde{\omega})Y)^H,Z^H),\\ 6)\ g^s(\nabla^s_{\omega^V}Y^H,\eta^V) &= 0,\\ 7)\ g^s(\nabla^s_{\omega^V}\theta^V,Z^H) &= 0,\\ 8)\ g^s(\nabla^s_{\omega^V}\theta^V,\eta^V) &= 0, \end{aligned}$$

for all $X,Y,Z \in \Im^1_0(M)$ and $\omega,\theta,\eta \in \Im^0_1(M)$.

Theorem 8. *([48]) Let (M,g) be a Riemannian manifold and (T^*M,g^s) its cotangent bundle equipped with the Sasaki metric. If ∇ (resp ∇^s) denote the Levi-Civita connection of (M,g)*

(resp (T^*M, g^s)), then we have

$$\begin{array}{rcl} (1)\ \nabla^s_{X^H} Y^H & = & (\nabla_X Y)^H + \dfrac{1}{2}(pR(X,Y))^V, \\ (2)\ \nabla^s_{X^H} \theta^V & = & (\nabla_X \theta)^V, \\ (3)\ \nabla_{\omega^V} Y^H & = & \dfrac{1}{2}(R(\tilde{p},\tilde{\omega})Y)^H, \\ (4)\ \nabla^s_{\omega^V} \theta^V & = & 0, \end{array}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where R denote the curvature tensor of (M, g).

Definition 23. *Let (M, g) be a Riemannian manifold and K is a tensor field of type $(1,1)$ on M. Then the vertical and horizontal vector fields VK and HK respectively are defined on T^*M by*

$$\begin{array}{rrl} VK: & T^*M \to & TT^*M \\ & (x,p) \mapsto & (pK)^V \end{array}, \qquad \begin{array}{rrl} HK: & T^*M \to & TT^*M \\ & (x,p) \mapsto & (K(\tilde{p}))^H, \end{array}$$

locally we have

$$VK = p_i(dx^i K)^V, \tag{2.15}$$

$$HK = \tilde{p}^i (K(\frac{\partial}{\partial x^i}))^H, \tag{2.16}$$

where $\tilde{p} = \tilde{p}^i \dfrac{\partial}{\partial x^i} = p_j g^{ij} \dfrac{\partial}{\partial x^i}$.

Proposition 10. *Let (M, g) be a Riemannian manifold and (T^*M, g^s) its tangent bundle equipped with the Sasaki metric. If ∇ (resp ∇^s) denote the Levi-Civita connection of (M, g) (resp (T^*M, g^s)) and K is a tensor field of type $(1,1)$ on M, then we have*

$$\begin{array}{rcl} 1.\ \nabla^s_{\omega^V} VK & = & (\omega K)^V, \\ 2.\ \nabla^s_{X^H} VK & = & V(\nabla_X K) + \dfrac{1}{2}(R(\tilde{p}, \widetilde{pK})X)^H, \\ 3.\ \nabla^s_{\omega^V} HK & = & (K(\widetilde{\omega}))^H + \dfrac{1}{2}(R(\tilde{p},\tilde{\omega})K(\tilde{p}))^H, \\ 4.\ \nabla^s_{X^H} HK & = & H(\nabla_X K) + \dfrac{1}{2}(pR(X, K(\tilde{p})))^V, \end{array}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, where R denote the curvature tensor of (M, g).

2.2.2 Riemannian curvature tensor of g^s

We shall calculate the Riemannian curvature tensor R^s of T^*M with the Sasaki metric g^s. This curvature tensor is characterized by the formula:

$$R^s(\widetilde{U}, \widetilde{V})\widetilde{W} = {}^f\nabla_{\widetilde{U}} \nabla^s_{\widetilde{V}} \widetilde{W} - \nabla^s_{\widetilde{V}} \nabla^s_{\widetilde{U}} \widetilde{W} - \nabla^s_{[\widetilde{U}, \widetilde{V}]} \widetilde{W}, \tag{2.17}$$

for all $\widetilde{U}, \widetilde{V}, \widetilde{W} \in \Im^1_0(T^*M)$.

Theorem 9. *([48]) Let (M,g) be a Riemannian manifold and (T^*M, g^s) its tangent bundle equipped with the Sasaki metric. If R (resp R^s) denote the Riemannian curvature tensor of (M,g) (resp (T^*M,g^s)), then we have the following formulas*

$$\begin{aligned} 1)\ R^s(X^H,Y^H)Z^H &= (R(X,Y)Z)^H + \frac{1}{4}(R(\tilde{p}, \widetilde{pR(X,Z)})Y)^H \\ &-\frac{1}{4}(R(\tilde{p}, \widetilde{pR(Y,Z)})X)^H + \frac{1}{2}(R(\tilde{p}, \widetilde{pR(X,Y)})Z)^H \\ &+\frac{1}{2}((\nabla_Z(pR))(X,Y))^V, \end{aligned} \tag{2.18}$$

$$2)\ R^s(X^H,\theta^V)\eta^V = \frac{-1}{2}(R(\tilde{\theta},\tilde{\eta})X)^H - \frac{1}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\eta})X)^H, \tag{2.19}$$

$$\begin{aligned} 3)\ R^s(\omega^V,\theta^V)Z^H &= (R(\tilde{\omega},\tilde{\theta})Z)^H + \frac{1}{4}(R(\tilde{p},\tilde{\omega})R(\tilde{p},\tilde{\theta})Z)^H \\ &-\frac{1}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\omega})Z)^H, \end{aligned} \tag{2.20}$$

$$\begin{aligned} 4)\ R^s(X^H,\theta^V)Z^H &= \frac{1}{2}((\nabla_X R)(\tilde{p},\tilde{\theta})Z)^H \\ &+\frac{1}{2}(\theta R(X,Z))^V + \frac{1}{4}(pR(X,R(\tilde{p},\tilde{\theta})Z))^V, \end{aligned} \tag{2.21}$$

$$\begin{aligned} 5)\ R^s(X^H,Y^H)\eta^V &= \frac{1}{2}((\nabla_X R)(\tilde{p},\tilde{\eta})Y)^H - \frac{1}{2}((\nabla_Y R)(\tilde{p},\tilde{\eta})X)^H \\ &+(\eta R(X,Y))^V + \frac{1}{4}(pR(X,R(\tilde{p},\tilde{\eta})Y))^V \\ &-\frac{1}{4}(pR(Y,R(\tilde{p},\tilde{\eta})X))^V, \end{aligned} \tag{2.22}$$

$$6)\ R^s(\omega^V,\theta^V)\eta^V = 0, \tag{2.23}$$

for all $X,Y,Z \in \Im^1_0(M)$ and $\omega,\theta,\eta \in \Im^0_1(M)$, where $\widetilde{pR(Y,Z)} = g^{-1}\circ pR(Y,Z)$ and $(\nabla_X(pR))(Y,Z) = \nabla_X(pR(Y,Z)) - (\nabla_X p)R(Y,Z) - pR(\nabla_X Y, Z) - pR(Y,\nabla_X Z)$.

2.2.3 Sectional curvature

It is known that the sectional curvature on (T^*M,g^s) for $P(V,W)$ is given by

$$K^s(V,W) = \frac{g^s(R^s(V,W)W,V)}{g^s(V,V)g^s(W,W) - g^s(V,W)^2}, \tag{2.24}$$

where $P(V,W)$ denotes the plane spanned by $\{V,W\}$.
If $\{X_i\}_{i=\overline{1,m}}$ and $\{\omega^i\}_{i=\overline{1,m}}$ be a local orthonormal frame and coframe on M, respectively, we see that

$$\{X_1^H,\cdots,X_m^H,(\omega^1)^V,\cdots,(\omega^m)^V\}, \tag{2.25}$$

is a local orthonormal frame on T^*M. Let $K^s(X^H, Y^H)$, $K^s(X^H, \theta^V)$ and $K^s(\omega^V, \theta^V)$ denote the sectional curvature of the plane spanned by $\{X^H, Y^H\}$, $\{X^H, \theta^V\}$ and $\{\omega^V, \theta^V\}$ on (T^*M, g^s) respectively, where $X, Y \in \{X_i\}_{i=\overline{1,m}}$ and $\omega, \theta \in \{\omega^i\}_{i=\overline{1,m}}$.

Proposition 11. *([48]) Let (M, g) be a Riemannian manifold and (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric. Then we have the following*

$$\begin{array}{rcl} i)\ g^s(R^s(X^H, Y^H)Y^H, X^H) &=& g(R(X,Y)Y,X) - \dfrac{3}{4}\|pR(X,Y)\|^2, \\ ii)\ g^s(R^s(X^H, \theta^V)\theta^V, X^H) &=& \dfrac{1}{4}\|R(\tilde{p}, \tilde{\theta})X\|^2, \\ iii)\ g^s(R^s(\omega^V, \theta^V)\theta^V, \omega^V) &=& 0. \end{array}$$

Proof. $i)$ From the formula (2.18), we have

$$\begin{array}{rcl} g^s(R^s(X^H, Y^H)Y^H, X^H) &=& g^s[(R(X,Y)Y)^H + \dfrac{1}{4}(R(\tilde{p}, \widetilde{pR(X,Y)})Y)^H \\ && -\dfrac{1}{4}(R(\tilde{p}, \widetilde{pR(Y,Y)})X)^H + \dfrac{1}{2}(R(\tilde{p}, \widetilde{pR(X,Y)})Y)^H \\ && +\dfrac{1}{2}((\nabla_Y(pR))(X,Y))^V, X^H] \\ &=& g(R(X,Y)Y,X) - \dfrac{3}{4}g(R(R(X,Y)\tilde{p}, \tilde{p})Y, X)) \\ &=& g(R(X,Y)Y,X) - \dfrac{3}{4}g(R(X,Y)\tilde{p}, R(X,Y)\tilde{p}) \\ &=& g(R(X,Y)Y,X) - \dfrac{3}{4}\|pR(X,Y)\|^2, \end{array}$$

because

$$\widetilde{pR(Y,X)} = R(Y,X)\tilde{p}. \tag{2.26}$$

$ii)$ From the formula (2.19), we have

$$\begin{array}{rcl} g^s(R^s(X^H, \theta^V)\theta^V, X^H) &=& \dfrac{-1}{4}g(R(\tilde{p}, \tilde{\theta})R(\tilde{p}, \tilde{\theta})X, X) \\ &=& \dfrac{1}{4}\|R(\tilde{p}, \tilde{\theta})X\|^2. \end{array}$$

$iii)$ The result follows immediately from the formula (2.23)

$$g^s(R^s(\omega^V, \theta^V)\theta^V, \omega^V) \ = \ 0.$$

□

Theorem 10. *([48]) Let (M, g) be a Riemannian manifold and (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric. If K (resp., K^s) denote the sectional curvature tensor of (M, g) (resp., (T^*M, g^s)). Then we have the following*

$$\begin{array}{rcl} (1)\ K^s(X^H, Y^H) &=& K(X,Y) - \dfrac{3}{4}\|pR(X,Y)\|^2, \\ (2)\ K^s(X^H, \theta^V) &=& \dfrac{1}{4}\|R(\tilde{p}, \tilde{\theta})X\|^2, \\ (3)\ K^s(\omega^V, \theta^V) &=& 0. \end{array}$$

Proof. Using Proposition 11 we have

$$\begin{aligned}
(1)\, K^s(X^H, Y^H) &= \frac{g^s(R^s(X^H, Y^H)Y^H, X^H)}{g^s(X^H, X^H)g^s(Y^H, Y^H) - g^s(X^H, Y^H)^2} \\
&= g^s(R^s(X^H, Y^H)Y^H, X^H) \\
&= g(R(X,Y)Y, X) - \frac{3}{4}\|pR(X,Y)\|^2 \\
&= K(X,Y) - \frac{3}{4}\|pR(X,Y)\|^2.
\end{aligned}$$

$$\begin{aligned}
(2)\, K^s(X^H, \theta^V) &= \frac{g^s(R^s(X^H, \theta^V)\theta^V, X^H)}{g^s(X^H, X^H)g^s(\theta^V, \theta^V) - g^s(X^H, \theta^V)^2} \\
&= g^s(R^s(X^H, \theta^V)\theta^V, X^H) \\
&= \frac{1}{4}\|R(\tilde{p}, \tilde{\theta})X\|^2.
\end{aligned}$$

$$\begin{aligned}
(3)\, K^s(\omega^V, \theta^V) &= \frac{g^s(R^s(\omega^V, \theta^V)\theta^V, \omega^V)}{g^s(\omega^V, \omega^V)g^s(\theta^V, \theta^V) - g^s(\omega^V, \theta^V)^2} \\
&= g^s(R^s(\omega^V, \theta^V)\theta^V, \omega^V).
\end{aligned}$$

□

Proposition 12. *([48]) Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric. If K^s denote the sectional curvature tensor of (T^*M, g^s). Then we have the following*

$$\begin{aligned}
(1)\, K^s(X^H, Y^H) &= \lambda - \frac{3\lambda^2}{4}\big[g(X, \tilde{p})^2 + g(Y, \tilde{p})^2\big], \\
(2)\, K^s(X^H, \theta^V) &= \frac{\lambda^2}{4}\big[g(X, \tilde{\theta})^2\|p\|^2 - 2g(X, \tilde{\theta})g(X, \tilde{p})g(\tilde{\theta}, \tilde{p}) + g(X, \tilde{p})^2\big], \\
(3)\, K^s(\omega^V, \theta^V) &= 0.
\end{aligned}$$

Proof. M has constant curvature λ then

$$R(X,Y)Z = \lambda\big[g(Y,Z)X - g(X,Z)Y\big]$$

then a direct calculations we get

$$\begin{aligned}
\|pR(X,Y)\|^2 &= \|R(Y,X)\tilde{p}\|^2 = \lambda^2\big[g(X, \tilde{p})^2 + g(Y, \tilde{p})^2\big], \\
\|R(\tilde{p}, \tilde{\theta})X\|^2 &= \lambda^2\big[g(X, \tilde{\theta})^2\|p\|^2 - 2g(X, \tilde{\theta})g(X, \tilde{p})g(\tilde{\theta}, \tilde{p}) + g(X, \tilde{p})^2\big],
\end{aligned}$$

this completes the proof. □

Corollary 2. *Let (M, g) be a m-dimensional Riemannian manifold of constant sectional curvature and (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric, then (T^*M, g^s) has zero constant sectional curvature if and only if (M, g) is flat.*

2.2.4 Scalar curvature

Theorem 11. *([48]) Let (M,g) be a m-dimensional Riemannian manifold and (T^*M,g^s) its cotangent bundle equipped with the Sasaki metric. If σ (resp., σ^s) denote the scalar curvature of (M,g) (resp., (T^*M,g^s)), then we have*

$$\sigma^s \;=\; \sigma - \frac{1}{4}\sum_{i,j=1}^{m} \|pR((E_i,(E_j)\|^2. \tag{2.27}$$

where $(E_i)_{i=\overline{1,m}}$ be a local orthonormal frame on M

Proof. Let $(E_i)_{i=\overline{1,m}}$ and $(\omega^i)_{i=\overline{1,m}}$ be a local orthonormal frame and coframe on M, $\left\{E_1^H,\cdots,E_m^H,(\omega^1)^V,\cdots,(\omega^m)^V\right\}$ is a local orthonormal frame on T^*M.
Using Theorem (10) and definition of scalar curvature, we have.

$$\begin{aligned}
\sigma^s &= \sum_{i,j=1}^{m} K^s(E_i^H,E_j^H) + 2\sum_{i,j=1}^{m} K^s(E_i^H,(\omega^j)^V) + \sum_{i,j=1}^{m} K^s((\omega^i)^V,(\omega^j)^V) \\
&= \sum_{i,j=1}^{m} \left[K(E_i,E_j) - \frac{3}{4}\|pR(E_i,E_j)\|^2\right] \\
&\quad +2\sum_{i,j=1}^{m} \big[\frac{1}{4}\|R(\tilde{p},\widetilde{\omega^j})E_i\|^2 + \frac{1}{4}E_i \\
&= \sigma - \frac{3}{4}\sum_{i,j=1}^{m} \|pR(E_i,E_j)\|^2 + \frac{1}{2}\sum_{i,j=1}^{m} \|R(\tilde{p},\widetilde{\omega^j})E_i\|^2.
\end{aligned}$$

In order to simplify this last expression.
From the formula (2.26) we have

$$\begin{aligned}
\widetilde{\omega^j} &= \sum_{i=1}^{m} g(\widetilde{\omega^j},E_i)E_i = \sum_{i,a,b=1}^{m} g_{ab}\widetilde{\omega^j}^{a} E_i^b E_i = \sum_{i,a,b,k=1}^{m} g_{ab}g^{ka}\omega_k^j E_i^b E_i \\
&= \sum_{i,b,k=1}^{m} \delta_b^k \omega_k^j E_i^b E_i = \sum_{i,k=1}^{m} \omega_k^j E_i^k E_i = \sum_{i=1}^{m} \omega^j(E_i)E_i = \sum_{i=1}^{m} \delta_i^j E_i = E_j,
\end{aligned}$$

and

$$\sum_{i,j=1}^{m} \|R(E_j,E_i)\tilde{p}\|^2 \;=\; \sum_{i,j=1}^{m} \|R(\tilde{p},E_j)E_i\|^2,$$

this completes the proof. □

Corollary 3. *Let (M,g) be a flat m-dimensional Riemannian manifold and (T^*M,g^s) its cotangent bundle equipped with the Sasaki metric. If σ (resp., σ^s) denote the scalar curvature of (M,g) (resp., (T^*M,g^s)), then we have*

$$\sigma^s \;=\; 0. \tag{2.28}$$

Corollary 4. *Let (M, g) be a m-dimensional Riemannian manifold, (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric . If σ (resp., σ^s) denote the scalar curvature of (M, g) (resp., (T^*M, g^s)), then we have*

$$\sigma^s = \sigma - \frac{1}{4}\sum_{i,j=1}^{m} \|pR(E_i, E_j)\|^2, \tag{2.29}$$

where (E_i) is a local orthonormal frame of M.

Corollary 5. *Let (M, g) be a m-dimensional Riemannian manifold of constant sectional curvature λ, (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric . Then (T^*M, g) has zero constant scalar curvature if and only if M is flat.*

Proposition 13. *Let (M, g) be a m-dimensional Riemannian manifold $(m \geq 2)$ of constant sectional curvature λ and (T^*M, g^s) its cotangent bundle equipped with the Sasaki metric. If σ^s denote the scalar curvature of (T^*M, g^s), then we have*

$$\sigma^s = (m-1)\lambda\Big(m - \frac{1}{2}\lambda\|p\|^2\Big). \tag{2.30}$$

Proof. M has constant curvature λ then $R(X, Y)Z = \lambda\big[g(Y, Z)X - g(X, Z)Y\big]$
and

$$\sigma = m(m-1)\lambda,$$

for $i \neq j$ we get

$$\sum_{i,j=1}^{m} \|pR(E_i, E_j)\|^2 = 2(m-1)\lambda^2\|p\|^2.$$

This completes the proof. □

Chapter 3

Geometry of the cotangent bundle with vertical rescaled metric

In this chapter, we introduce a new class of natural metrics denoted by ${}^{f}g$ and called the vertical rescaled metric on the cotangent bundle $T^{*}M$. We calculate its Levi-Civita connection and Riemannian curvature tensor. We study the geometry of $(T^{*}M, {}^{f}g)$ and several important results are obtained on curvature, scalar and sectional curvatures. Thus, we generalized the results in the article [48].

3.1 Vertical rescaled metric

Definition 24. *Let* (M, g) *be a Riemannian manifold and* $f : M \rightarrow]0, +\infty[$ *be a strictly positive smooth function on* M*. We define the vertical rescaled metric* ${}^{f}g$ *on the cotangent bundle by*

$$\begin{aligned} {}^{f}g(X^{H}, Y^{H}) &= g(X, Y) \circ \pi, & (3.1)\\ {}^{f}g(X^{H}, \theta^{V}) &= 0, & (3.2)\\ {}^{f}g(\omega^{V}, \theta^{V}) &= f g^{-1}(\omega, \theta), & (3.3) \end{aligned}$$

where $X, Y \in \Im^{1}_{0}(M)$*,* $\omega, \theta \in \Im^{0}_{1}(M)$*.*

Note that, if $f = 1$, then ${}^{f}g$ is the Sasaki metric [48].

Since any tensor field of type $(0, s)$ on $T^{*}M$ where $s \geq 1$ is completely determined with the vector fields of type X^{H} and ω^{V} where $X \in \Im^{1}_{0}(M)$ and $\omega \in \Im^{0}_{1}(M)$ (see [53]). In the particular case the metric ${}^{f}g$ is tensor field of type $(0, 2)$ on $T^{*}M$. it follows that ${}^{f}g$ is completely determined by its formulas (3.1), (3.2) and (3.3).

By means of (2.3) and (2.4), the complete lift X^{C} of $X \in \Im^{1}_{0}(M)$ is given by

$$X^{C} = X^{H} - (p(\nabla X))^{V}, \qquad (3.4)$$

where $p(\nabla X) = p_{h}(\nabla_{i} X^{h}) dx^{i} = p_{h}(\dfrac{\partial X^{h}}{\partial x^{i}} + \Gamma^{h}_{ij} X^{j}) dx^{i}$.

Taking account of (3.1), (3.2), (3.3) and (3.4), we obtain

$$
{}^{f}g(X^{C},Y^{C}) \;=\; g(X,Y)^{V}+fg^{-1}(p(\nabla X),p(\nabla Y)). \tag{3.5}
$$

Since the tensor field ${}^{f}g\in\Im^{0}_{2}(T^{*}M)$ is completely determined also by its action on vector fields of type X^{C} and Y^{C} (see [53]), we say that the formula (3.5) is an alternative characterization of ${}^{f}g$.

Remark 5. *From the formulas (3.1), (3.2) and (3.3) we see that*

$$
\begin{aligned}
{}^{f}g_{ij} &= {}^{f}g(\tilde{e}_{i},\tilde{e}_{j})=g(\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial x^{j}})^{V}=g_{ij},\\
{}^{f}g_{i\bar{j}} &= {}^{f}g(\tilde{e}_{i},\tilde{e}_{\bar{j}})=0,\\
{}^{f}g_{\bar{i}\bar{j}} &= {}^{f}g(\tilde{e}_{\bar{i}},\tilde{e}_{\bar{j}})=fg^{ij}.
\end{aligned}
$$

Then the Mus-Sasaki metric ${}^{f}g$ has components with respect to the adapted frame $\{\tilde{e}_{\alpha}\}_{\alpha=\overline{1,2m}}$

$$
{}^{f}g=\begin{pmatrix} g_{ij} & 0\\ 0 & fg^{ij}\end{pmatrix}. \tag{3.6}
$$

Lemma 11. *([48]) Let (M,g) be a Riemannian manifold and $(T^{*}M,{}^{f}g)$ its cotangent bundle equipped with the vertical rescaled metric, for all $X\in\Im^{1}_{0}(M)$ and $\omega,\theta\in\Im^{0}_{1}(M)$, we have*

1. $X^{H}\,{}^{f}g(\theta^{V},\eta^{V})=\frac{1}{f}X(f)\,{}^{f}g(\theta^{V},\eta^{V})+{}^{f}g((\nabla_{X}\theta)^{V},\eta^{V})+{}^{f}g(\theta^{V},(\nabla_{X}\eta)^{V})$,
2. $\omega^{V}\,{}^{f}g(\theta^{V},\eta^{V})=0$.

Proof.

$$
\begin{aligned}
1.\ X^{H}\,{}^{f}g(\theta^{V},\eta^{V}) &= X^{H}\big[fg^{-1}(\theta,\eta)\big]\\
&= X(f)g^{-1}(\theta,\eta)+fXg^{-1}(\theta,\eta)\\
&= X(f)g^{-1}(\theta,\eta)+fg^{-1}(\nabla_{X}\theta,\eta)+fg^{-1}(\theta,\nabla_{X}\eta)\\
&= \frac{1}{f}X(f)\,{}^{f}g(\theta^{V},\eta^{V})+{}^{f}g((\nabla_{X}\theta)^{V},\eta^{V})+{}^{f}g(\theta^{V},(\nabla_{X}\eta)^{V}),
\end{aligned}
$$

$$
\begin{aligned}
2.\ \omega^{V}\,{}^{f}g(\theta^{V},\eta^{V}) &= \omega^{V}\big[fg^{-1}(\theta,\eta)\big]\\
&= 0.
\end{aligned}
$$

□

3.1.1 Levi-Civita connection of ${}^{f}g$

We shall calculate the Levi-Civita connection ${}^{f}\nabla$ of $T^{*}M$ with the vertical rescaled metric ${}^{f}g$. This connection is characterized by the Koszul formula:

$$
\begin{aligned}
2\,{}^{f}g({}^{f}\nabla_{\widetilde{U}}\widetilde{V},\widetilde{W}) &= \widetilde{U}\,{}^{f}g(\widetilde{V},\widetilde{W})+\widetilde{V}\,{}^{f}g(\widetilde{W},\widetilde{U})-\widetilde{W}\,{}^{f}g(\widetilde{U},\widetilde{V})\\
&\quad +{}^{f}g(\widetilde{W},[\widetilde{U},\widetilde{V}])+{}^{f}g(\widetilde{V},[\widetilde{W},\widetilde{U}])-{}^{f}g(\widetilde{U},[\widetilde{V},\widetilde{W}]),
\end{aligned} \tag{3.7}
$$

for all $\widetilde{U},\widetilde{V},\widetilde{W}\in\Im^{1}_{0}(T^{*}M)$.

Lemma 12. *([48]) Let (M,g) be a Riemannian manifold and $(T^*M,{}^fg)$ its cotangent bundle equipped with the vertical rescaled metric. If ∇ (resp. ${}^f\nabla$) denote the Levi-Civita connection of (M,g) (resp. $(T^*M,{}^fg)$), then we have*

$$\begin{array}{rcl}
1)\ {}^fg({}^f\nabla_{X^H}Y^H,Z^H) &=& {}^fg((\nabla_XY)^H,Z^H),\\
2)\ {}^fg({}^f\nabla_{X^H}Y^H,\eta^V) &=& \dfrac{1}{2}{}^fg((pR(X,Y))^V,\eta^V),\\
3)\ {}^fg({}^f\nabla_{X^H}\theta^V,Z^H) &=& \dfrac{f}{2}{}^fg((R(\tilde{p},\tilde{\theta})X)^H,Z^H),\\
4)\ {}^fg({}^f\nabla_{X^H}\theta^V,\eta^V) &=& {}^fg((\nabla_X\theta)^V,\eta^V)+\dfrac{1}{2f}X(f){}^fg(\theta^V,\eta^V),\\
5)\ {}^fg({}^f\nabla_{\omega^V}Y^H,Z^H) &=& \dfrac{f}{2}{}^fg((R(\tilde{p},\tilde{\omega})Y)^H,Z^H),\\
6)\ {}^fg({}^f\nabla_{\omega^V}Y^H,\eta^V) &=& \dfrac{1}{2f}Y(f){}^fg(\omega^V,\eta^V),\\
7)\ {}^fg({}^f\nabla_{\omega^V}\theta^V,Z^H) &=& \dfrac{-1}{2}g^{-1}(\omega,\theta){}^fg((grad\, f)^H,Z^H),\\
8)\ {}^fg({}^f\nabla_{\omega^V}\theta^V,\eta^V) &=& 0,
\end{array}$$

for all $X,Y,Z\in\Im^1_0(M)$ and $\omega,\theta,\eta\in\Im^0_1(M)$.

Proof.
The proof of Lemma 12 follows directly from Koszul formula (3.7), Lemma 9, Definition 24 and Lemma 11.
1)

$$\begin{array}{rcl}
2{}^fg({}^f\nabla_{X^H}Y^H,Z^H) &=& X^H{}^fg(Y^H,Z^H)+Y^H{}^fg(Z^H,X^H)-Z^H{}^fg(X^H,Y^H)\\
&& +{}^fg(Z^H,[X^H,Y^H])+{}^fg(Y^H,[Z^H,X^H])-{}^fg(X^H,[Y^H,Z^H])\\
&=& Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+{}^fg(Z^H,[X,Y]^H)\\
&& +{}^fg(Y^H,[Z,X]^H)-{}^fg(X^H,[Y,Z]^H)\\
&=& Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y])\\
&& +g(Y,[Z,X])-g(X,[Y,Z])\\
&=& 2g(\nabla_XY,Z)\\
&=& 2{}^fg((\nabla_XY)^H,Z^H).
\end{array}$$

2)

$$\begin{array}{rcl}
2{}^fg({}^f\nabla_{X^H}Y^H,\eta^V) &=& X^H{}^fg(Y^H,\eta^V)+Y^H{}^fg(\eta^V,X^H)-\eta^V{}^fg(X^H,Y^H)\\
&& +{}^fg(\eta^V,[X^H,Y^H])+{}^fg(Y^H,[\eta^V,X^H])-{}^fg(X^H,[Y^H,\eta^V])\\
&=& {}^fg(\eta^V,[X^H,Y^H])\\
&=& {}^fg((pR(X,Y))^V,\eta^V).
\end{array}$$

3)

$$
\begin{aligned}
2\,{}^fg({}^f\nabla_{X^H}\theta^V, Z^H) &= X^H\,{}^fg(\theta^V, Z^H) + \theta^V\,{}^fg(Z^H, X^H) - Z^H\,{}^fg(X^H, \theta^V) \\
&\quad + {}^fg(Z^H, [X^H, \theta^V]) + {}^fg(\theta^V, [Z^H, X^H]) - {}^fg(X^H, [\theta^V, Z^H]) \\
&= {}^fg(\theta^V, [Z^H, X^H]) \\
&= {}^fg((pR(Z,X))^V, \theta^V) \\
&= f g^{-1}(pR(Z,X), \theta) \\
&= f g^{kl}(pR(Z,X))_k \theta_l \\
&= f p_s R^s_{ijk} Z^i X^j \tilde{\theta}^k \\
&= f g_{st} \tilde{p}^t R^s_{ijk} Z^i X^j \tilde{\theta}^k \\
&= f R_{ijkt} Z^i X^j \tilde{\theta}^k \tilde{p}^t \\
&= f g(R(Z,X)\tilde{\theta}, \tilde{p}) \\
&= f g(R(\tilde{p}, \tilde{\theta})X, Z) \\
&= f\,{}^fg((R(\tilde{p}, \tilde{\theta})X)^H, Z^H).
\end{aligned}
$$

4)

$$
\begin{aligned}
2\,{}^fg({}^f\nabla_{X^H}\theta^V, \eta^V) &= X^H\,{}^fg(\theta^V, \eta^V) + \theta^V\,{}^fg(\eta^V, X^H) - \eta^V\,{}^fg(X^H, \theta^V) \\
&\quad + {}^fg(\eta^V, [X^H, \theta^V]) + {}^fg(\theta^V, [\eta^V, X^H]) - {}^fg(X^H, [\theta^V, \eta^V]) \\
&= X^H\,{}^fg(\theta^V, \eta^V) + {}^fg(\eta^V, [X^H, \theta^V]) + {}^fg(\theta^V, [\eta^V, X^H])
\end{aligned}
$$

Using first formula of Lemma 11 we have

$$
\begin{aligned}
2\,{}^fg({}^f\nabla_{X^H}\theta^V, \eta^V) &= \frac{1}{f}X(f)\,{}^fg(\theta^V, \eta^V) + {}^fg((\nabla_X\theta)^V, \eta^V) + {}^fg(\theta^V, (\nabla_X\eta)^V) \\
&\quad + {}^fg(\eta^V, (\nabla_X\theta)^V) - {}^fg(\theta^V, (\nabla_X\eta)^V) \\
&= 2\,{}^fg((\nabla_X\theta)^V, \eta^V) + \frac{1}{f}X(f)\,{}^fg(\theta^V, \eta^V).
\end{aligned}
$$

7)

$$
\begin{aligned}
2\,{}^fg({}^f\nabla_{\omega^V}\theta^V, Z^H) &= \omega^V\,{}^fg(\theta^V, Z^H) + \theta^V\,{}^fg(Z^H, \omega^V) - Z^H\,{}^fg(\omega^V, \theta^V) \\
&\quad + {}^fg(Z^H, [\omega^V, \theta^V]) + {}^fg(\theta^V, [Z^H, \omega^V]) - {}^fg(\omega^V, [\theta^V, Z^H]) \\
&= -Z^H\,{}^fg(\omega^V, \theta^V) + {}^fg(\theta^V, [Z^H, \omega^V]) - {}^fg(\omega^V, [\theta^V, Z^H]).
\end{aligned}
$$

Using first formula of Lemma 11 we have

$$
\begin{aligned}
2\,{}^fg({}^f\nabla_{\omega^V}\theta^V, Z^H) &= \frac{-1}{f}Z(f)\,{}^fg(\omega^V, \theta^V) - {}^fg((\nabla_Z\omega)^V, \theta^V) - {}^fg(\omega^V, (\nabla_Z\theta)^V) \\
&\quad + {}^fg(\theta^V, (\nabla_Z\omega)^V) + {}^fg(\omega^V, (\nabla_Z\theta)^V) \\
&= \frac{-1}{f}Z(f)\,{}^fg(\omega^V, \theta^V) \\
&= -Z(f)g^{-1}(\omega, \theta) \\
&= -g^{-1}(\omega, \theta)\,{}^fg((grad\, f)^H, Z^H).
\end{aligned}
$$

Where ${}^f g((grad\, f)^H, Z^H) = g(grad\, f, Z) = Z(f)$.
8)

$$\begin{aligned} 2{}^f g({}^f\nabla_{\omega^V}\theta^V, \eta^V) &= \omega^V{}^f g(\theta^V, \eta^V) + \theta^V{}^f g(\eta^V, \omega^V) - \eta^V{}^f g(\omega^V, \theta^V) \\ &\quad + {}^f g(\eta^V, [\omega^V, \theta^V]) + {}^f g(\theta^V, [\eta^V, \omega^V]) - {}^f g(\omega^V, [\theta^V, \eta^V]) \\ &= \omega^V{}^f g(\theta^V, \eta^V) + \theta^V{}^f g(\eta^V, \omega^V) - \eta^V{}^f g(\omega^V, \theta^V). \end{aligned}$$

Using the second formula of Lemma 11 we have $2{}^f g({}^f\nabla_{\omega^V}\theta^V, \eta^V) = 0$. □

As a direct consequence of Lemma 12, we get the following theorem.

Theorem 12. *([48]) Let (M, g) be a Riemannian manifold and $(T^*M, {}^f g)$ its cotangent bundle equipped with the vertical rescaled metric. If ∇ (resp ${}^f\nabla$) denote the Levi-Civita connection of (M, g) (resp $(T^*M, {}^f g)$), then we have*

$$\begin{aligned} (1)\ {}^f\nabla_{X^H} Y^H &= (\nabla_X Y)^H + \frac{1}{2}(pR(X, Y))^V, \\ (2)\ {}^f\nabla_{X^H}\theta^V &= (\nabla_X \theta)^V + \frac{1}{2f}X(f)\theta^V + \frac{f}{2}(R(\tilde{p}, \tilde{\theta})X)^H, \\ (3)\ {}^f\nabla_{\omega^V} Y^H &= \frac{1}{2f}Y(f)\omega^V + \frac{f}{2}(R(\tilde{p}, \tilde{\omega})Y)^H, \\ (4)\ {}^f\nabla_{\omega^V}\theta^V &= \frac{-1}{2}g^{-1}(\omega, \theta)(grad\, f)^H, \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where R denote the curvature tensor of (M, g).

Definition 25. *Let (M, g) be a Riemannian manifold and K is a tensor field of type $(1, 1)$ on M. Then the vertical and horizontal vector fields VK and HK respectively are defined on T^*M by*

$$\begin{aligned} VK:\ & T^*M \to TT^*M \\ & (x,p) \mapsto (pK)^V \end{aligned}, \qquad \begin{aligned} HK:\ & T^*M \to TT^*M \\ & (x,p) \mapsto (K(\tilde{p}))^H, \end{aligned}$$

locally we have

$$VK = p_i(dx^i K)^V, \tag{3.8}$$

$$HK = \tilde{p}^i(K(\frac{\partial}{\partial x^i}))^H, \tag{3.9}$$

where $\tilde{p} = \tilde{p}^i\frac{\partial}{\partial x^i} = p_j g^{ij}\frac{\partial}{\partial x^i}$.

Proposition 14. *([39]) Let (M, g) be a Riemannian manifold and $(T^*M, {}^f g)$ its tangent bundle equipped with the vertical rescaled metric. If ∇ (resp ${}^f\nabla$) denote the Levi-Civita connection of (M, g) (resp $(T^*M, {}^f g)$) and K is a tensor field of type $(1, 1)$ on M, then we* *have*

$$\begin{aligned}
1.\, {}^f\nabla_{\omega^V} VK &= (\omega K)^V - \frac{1}{2} g(\omega, pK)(grad\, f)^H,\\
2.\, {}^f\nabla_{X^H} VK &= V(\nabla_X K) + \frac{1}{2f} X(f) VK + \frac{f}{2}(R(\tilde{p}, \widetilde{pK})X)^H,\\
3.\, {}^f\nabla_{\omega^V} HK &= (K(\widetilde{\omega}))^H + \frac{1}{2f} g(K(\tilde{p}), grad\, f)\omega^V + \frac{f}{2}(R(\tilde{p}, \tilde{\omega})K(\tilde{p}))^H,\\
4.{}^f\nabla_{X^H} HK &= H(\nabla_X K) + \frac{1}{2}(pR(X, K(\tilde{p})))^V,
\end{aligned}$$

for all $X \in \Im^1_0(M)$ *and* $\omega \in \Im^0_1(M)$*, where* R *denote the curvature tensor of* (M, g)*.*

Proof. By the Definition 25 and Theorem 12 we have:

$$\begin{aligned}
1.\, {}^f\nabla_{\omega^V} VK &= {}^f\nabla_{(\omega_i dx^i)^V}(p_k (dx^k K)^V)\\
&= \omega_i (dx^i)^V(p_k)(dx^k K)^V + \omega_i p_k {}^f\nabla_{(dx^i)^V}(dx^k K)^V\\
&= \omega_i \frac{\partial}{\partial p_i}(p_k)(dx^k K)^V - \frac{1}{2}\omega_i p_k g^{-1}(dx^i, dx^k K)(grad\, f)^H\\
&= \omega_k (dx^k K)^V - \frac{1}{2} g^{-1}(\omega, pK)(grad\, f)^H\\
&= (\omega K)^V - \frac{1}{2} g^{-1}(\omega, pK)(grad\, f)^H.
\end{aligned}$$

$$\begin{aligned}
2.\, {}^f\nabla_{X^H} VK &= {}^f\nabla_{X^H}(p_k (dx^k K)^V)\\
&= X^H(p_k)(dx^k K)^V + p_k {}^f\nabla_{X^H}(dx^k K)^V\\
&= p_a \Gamma^a_{ki} X^i (dx^k K)^V + p_k(\nabla_X dx^k K)^V + \frac{p_k}{2f} X(f)(dx^k K)^V\\
&+ \frac{p_k f}{2}(R(\tilde{p}, \widetilde{dx^k K})X)^H\\
&= -((\nabla_X p)K)^V + (\nabla_X pK)^V + \frac{1}{2f} X(f)(pK)^V + \frac{f}{2}(R(\tilde{p}, \widetilde{pK})X)^H\\
&= (p(\nabla_X K))^V + \frac{1}{2f} X(f) VK + \frac{f}{2}(R(\tilde{p}, \widetilde{pK})X)^H\\
&= V(\nabla_X K) + \frac{1}{2f} X(f) VK + \frac{f}{2}(R(\tilde{p}, \widetilde{pK})X)^H.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

3.1.2 Riemannian curvature tensor of ${}^f g$

We shall calculate the Riemannian curvature tensor ${}^f R$ of T^*M with the vertical rescaled metric ${}^f g$. This curvature tensor is characterized by the formula:

$$ {}^f R(\widetilde{U}, \widetilde{V})\widetilde{W} = {}^f\nabla_{\widetilde{U}} {}^f\nabla_{\widetilde{V}} \widetilde{W} - {}^f\nabla_{\widetilde{V}} {}^f\nabla_{\widetilde{U}} \widetilde{W} - {}^f\nabla_{[\widetilde{U}, \widetilde{V}]} \widetilde{W}, \tag{3.10}$$

for all $\widetilde{U}, \widetilde{V}, \widetilde{W} \in \Im^1_0(T^*M)$.

Theorem 13. *([39]) Let (M, g) be a Riemannian manifold and $(T^*M, {}^fg)$ its tangent bundle equipped with the vertical rescaled metric. If R (resp ${}^fR$) denote the Riemannian curvature tensor of (M, g) (resp $(T^*M, {}^fg)$), then we have the following formulas*

$$\begin{aligned} 1)\ {}^fR(X^H, Y^H)Z^H &= (R(X,Y)Z)^H + \frac{f}{4}(R(\tilde{p}, p\widetilde{R(Y,Z)})X)^H \\ &-\frac{f}{4}(R(\tilde{p}, p\widetilde{R(X,Z)})Y)^H - \frac{f}{2}(R(\tilde{p}, p\widetilde{R(X,Y)})Z)^H \\ &+\frac{1}{2}((\nabla_X(pR))(Y,Z))^V - \frac{1}{2}((\nabla_Y(pR))(X,Z))^V \\ &+\frac{1}{4f}X(f)(pR(Y,Z))^V - \frac{1}{4f}Y(f)(pR(X,Z))^V \\ &-\frac{1}{2f}Z(f)(pR(X,Y))^V, \end{aligned} \tag{3.11}$$

$$\begin{aligned} 2)\ {}^fR(X^H, \theta^V)\eta^V &= \frac{-f}{2}(R(\tilde{\theta}, \tilde{\eta})X)^H - \frac{f^2}{4}(R(\tilde{p}, \tilde{\theta})R(\tilde{p}, \tilde{\eta})X)^H \\ &+\frac{1}{4f}X(f)g^{-1}(\theta, \eta)(grad\, f)^H + \frac{1}{2}g^{-1}(\theta, \eta)(\nabla_X grad\, f)^H \\ &+\frac{1}{4}g^{-1}(\theta, \eta)(pR(X, grad\, f))^V - \frac{1}{4}g(grad\, f, R(\tilde{p}, \tilde{\eta})X)\theta^V, \end{aligned} \tag{3.12}$$

$$\begin{aligned} 3)\ {}^fR(\omega^V, \theta^V)Z^H &= f(R(\tilde{\omega}, \tilde{\theta})Z)^H + \frac{f^2}{4}(R(\tilde{p}, \tilde{\omega})R(\tilde{p}, \tilde{\theta})Z)^H \\ &-\frac{f^2}{4}(R(\tilde{p}, \tilde{\theta})R(\tilde{p}, \tilde{\omega})Z)^H + \frac{1}{4}g(grad\, f, R(\tilde{p}, \tilde{\theta})Z)\omega^V \\ &-\frac{1}{4}g(grad\, f, R(\tilde{p}, \tilde{\omega})Z)\theta^V, \end{aligned} \tag{3.13}$$

$$\begin{aligned} 4)\ {}^fR(X^H, \theta^V)Z^H &= \frac{f}{2}((\nabla_X R)(\tilde{p}, \tilde{\theta})Z)^H + \frac{1}{2}X(f)(R(\tilde{p}, \tilde{\theta})Z)^H \\ &+\frac{1}{4}Z(f)(R(\tilde{p}, \tilde{\theta})X)^H + \frac{1}{4}g^{-1}(\theta, pR(X,Z))(grad\, f)^H \\ &+\frac{1}{2}(\theta R(X,Z))^V + \frac{f}{4}(pR(X, R(\tilde{p}, \tilde{\theta})Z))^V \\ &+\big[\frac{1}{2f}g(Z, \nabla_X grad\, f) - \frac{1}{4f^2}X(f)Z(f)\big]\theta^V, \end{aligned} \tag{3.14}$$

$$\begin{aligned} 5)\ {}^fR(X^H, Y^H)\eta^V &= \frac{f}{2}((\nabla_X R)(\tilde{p}, \tilde{\eta})Y)^H - \frac{f}{2}((\nabla_Y R)(\tilde{p}, \tilde{\eta})X)^H \\ &+\frac{1}{4}X(f)(R(\tilde{p}, \tilde{\eta})Y)^H - \frac{1}{4}Y(f)(R(\tilde{p}, \tilde{\eta})X)^H \\ &+\frac{1}{2}g^{-1}(\eta, pR(X,Y))(grad\, f)^H + (\eta R(X,Y))^V \\ &+\frac{f}{4}(pR(X, R(\tilde{p}, \tilde{\eta})Y))^V - \frac{f}{4}(pR(Y, R(\tilde{p}, \tilde{\eta})X))^V, \end{aligned} \tag{3.15}$$

$$\begin{aligned} 6)\ {}^{f}R(\omega^V,\theta^V)\eta^V &= \frac{f}{4}\big[g^{-1}(\omega,\eta)(R(\tilde{p},\tilde{\theta})grad\, f)^H - g^{-1}(\theta,\eta)(R(\tilde{p},\tilde{\omega})grad\, f)^H\big] \\ &+\frac{1}{4f}\|grad\, f\|^2\big[g^{-1}(\omega,\eta)\theta^V - g^{-1}(\theta,\eta)\omega^V\big], \end{aligned} \tag{3.16}$$

for all $X,Y,Z\in\Im^1_0(M)$ *and* $\omega,\theta,\eta\in\Im^0_1(M)$*, where* $\widetilde{pR(Y,Z)} = g^{-1}\circ pR(Y,Z)$ *and* $(\nabla_X(pR))(Y,Z) = \nabla_X(pR(Y,Z)) - (\nabla_X p)R(Y,Z) - pR(\nabla_X Y,Z) - pR(Y,\nabla_X Z)$.

Proof. Let $X,Y,Z\in\Im^1_0(M)$ and $\omega,\theta,\eta\in\Im^0_1(M)$. By applying Definition 24, Lemma 9, Lemma 11, Theorem 12 and Proposition 14 we have:

1) ${}^{f}R(X^H,Y^H)Z^H = {}^{f}\nabla_{X^H}{}^{f}\nabla_{Y^H}Z^H - {}^{f}\nabla_{Y^H}{}^{f}\nabla_{X^H}Z^H - {}^{f}\nabla_{[X^H,Y^H]}Z^H$.

Let K be the bundle endomorphism given by,

$$\begin{aligned} K: T^*M &\to T^*M \\ p &\mapsto pR(Y,Z). \end{aligned}$$

Direct calculations give,

$$\begin{aligned} {}^{f}\nabla_{X^H}{}^{f}\nabla_{Y^H}Z^H &= {}^{f}\nabla_{X^H}\big[(\nabla_Y Z)^H + \frac{1}{2}VK\big] \\ &= (\nabla_X\nabla_Y Z)^H + \frac{1}{2}(pR(X,\nabla_Y Z))^V + \frac{1}{2}(\nabla_X(pR(Y,Z)))^V \\ &-\frac{1}{2}((\nabla_X p)R(Y,Z))^V + \frac{1}{4f}X(f)(pR(Y,Z))^V \\ &+\frac{f}{4}(R(\tilde{p},\widetilde{pR(Y,Z)})X)^H, \end{aligned}$$

and

$$\begin{aligned} {}^{f}\nabla_{Y^H}{}^{f}\nabla_{X^H}Z^H &= (\nabla_Y\nabla_X Z)^H + \frac{1}{2}(pR(Y,\nabla_X Z))^V + \frac{1}{2}(\nabla_Y(pR(X,Z)))^V \\ &-\frac{1}{2}((\nabla_Y p)R(X,Z))^V + \frac{1}{4f}Y(f)(pR(X,Z))^V \\ &+\frac{f}{4}(R(\tilde{p},\widetilde{pR(X,Z)})Y)^H, \end{aligned}$$

and

$$\begin{aligned} {}^{f}\nabla_{[X^H,Y^H]}Z^H &= {}^{f}\nabla_{[X,Y]^H}Z^H + {}^{f}\nabla_{(pR(X,Y))^V}Z^H \\ &= (\nabla_{[X,Y]}Z)^H + \frac{1}{2}(pR([X,Y],Z))^V + \frac{1}{2f}Z(f)(pR(X,Y))^V \\ &+\frac{f}{2}(R(\tilde{p},\widetilde{pR(X,Y)})Z)^H. \end{aligned}$$

Hence, we have:

$$\begin{aligned}
{}^fR(X^H,Y^H)Z^H &= (R(X,Y)Z)^H + \frac{f}{4}(R(\tilde{p},p\widetilde{R(Y,Z)})X)^H \\
&-\frac{f}{4}(R(\tilde{p},p\widetilde{R(X,Z)})Y)^H - \frac{f}{2}(R(\tilde{p},p\widetilde{R(X,Y)})Z)^H \\
&+\frac{1}{2}((\nabla_X(pR))(Y,Z))^V - \frac{1}{2}((\nabla_Y(pR))(X,Z))^V \\
&+\frac{1}{4f}X(f)(pR(Y,Z))^V - \frac{1}{4f}Y(f)(pR(X,Z))^V \\
&-\frac{1}{2f}Z(f)(pR(X,Y))^V.
\end{aligned}$$

2) ${}^fR(X^H,\theta^V)\eta^V = {}^f\nabla_{X^H}{}^f\nabla_{\theta^V}\eta^V - {}^f\nabla_{\theta^V}{}^f\nabla_{X^H}\eta^V - {}^f\nabla_{[X^H,\theta^V]}\eta^V.$

From direct calculation we get:

$$\begin{aligned}
{}^f\nabla_{X^H}{}^f\nabla_{\theta^V}\eta^V &= {}^f\nabla_{X^H}\big[\frac{-1}{2}g^{-1}(\theta,\eta)(grad\, f)^H\big] \\
&= -\frac{1}{2}\big[g^{-1}(\nabla_X\theta,\eta) + g^{-1}(\theta,\nabla_X\eta)\big](grad\, f)^H \\
&\quad -\frac{1}{2}g^{-1}(\theta,\eta)(\nabla_X grad\, f)^H - \frac{1}{4f}g^{-1}(\theta,\eta)(pR(X,grad\, f))^V.
\end{aligned}$$

Let K be the bundle map given by,

$$\begin{aligned}
K : T^*M &\rightarrow TM \\
p &\mapsto R(\tilde{p},\tilde{\eta})X.
\end{aligned}$$

Direct calculations give,

$$\begin{aligned}
{}^f\nabla_{\theta^V}{}^f\nabla_{X^H}\eta^V &= {}^f\nabla_{\theta^V}\big[\big(\nabla_X\eta\big)^V + \frac{1}{2f}X(f)\eta^V + \frac{f}{2}HK\big] \\
&= -\frac{1}{2}g^{-1}(\theta,\nabla_X\eta)(grad\, f)^H - \frac{1}{4f}X(f)g^{-1}(\theta,\eta)(grad\, f)^H \\
&\quad +\frac{f}{2}(R(\tilde{\theta},\tilde{\eta})X)^H + \frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\eta})X))\theta^V \\
&\quad +\frac{f^2}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\eta})X)^H,
\end{aligned}$$

and

$$ {}^f\nabla_{[X^H,\theta^V]}\eta^V = {}^f\nabla_{(\nabla_X\theta)^V}\eta^V = -\frac{1}{2}g^{-1}(\nabla_X\theta,\eta)(grad\, f)^H, $$

which gives,

$$\begin{aligned}
{}^fR(X^H,\theta^V)\eta^V &= \frac{-f}{2}(R(\tilde{\theta},\tilde{\eta})X)^H - \frac{f^2}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\eta})X)^H \\
&\quad +\frac{1}{4f}X(f)g^{-1}(\theta,\eta)(grad\, f)^H + \frac{1}{2}g^{-1}(\theta,\eta)(\nabla_X grad\, f)^H \\
&\quad +\frac{1}{4}g^{-1}(\theta,\eta)(pR(X,grad\, f))^V - \frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\eta})X)\theta^V.
\end{aligned}$$

3) Applying formula (3.12) and 1^{st} Bianchi identity.

$$^{f}R(\omega^V,\theta^V)Z^H = {}^{f}R(Z^H,\theta^V)\omega^V - {}^{f}R(Z^H,\omega^V)\theta^V,$$

we get

$$\begin{aligned} {}^{f}R(Z^H,\theta^V)\omega^V &= \frac{-f}{2}(R(\tilde{\theta},\tilde{\omega})Z)^H - \frac{f^2}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\omega})Z)^H \\ &+\frac{1}{4f}Z(f)g^{-1}(\theta,\omega)(grad\, f)^H + \frac{1}{2}g^{-1}(\theta,\omega)(\nabla_Z grad\, f)^H \\ &+\frac{1}{4}g^{-1}(\theta,\omega)(pR(Z,grad\, f))^V - \frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\omega})Z)\theta^V, \end{aligned}$$

and

$$\begin{aligned} {}^{f}R(Z^H,\omega^V)\theta^V &= \frac{-f}{2}(R(\tilde{\omega},\tilde{\theta})Z)^H - \frac{f^2}{4}(R(\tilde{p},\tilde{\omega})R(\tilde{p},\tilde{\theta})Z)^H \\ &+\frac{1}{4f}Z(f)g^{-1}(\omega,\theta)(grad\, f)^H + \frac{1}{2}g^{-1}(\omega,\theta)(\nabla_Z grad\, f)^H \\ &+\frac{1}{4}g^{-1}(\omega,\theta)(pR(Z,grad\, f))^V - \frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\theta})Z)\omega^V, \end{aligned}$$

which gives,

$$\begin{aligned} {}^{f}R(\omega^V,\theta^V)Z^H &= f(R(\tilde{\omega},\tilde{\theta})Z)^H + \frac{f^2}{4}(R(\tilde{p},\tilde{\omega})R(\tilde{p},\tilde{\theta})Z)^H \\ &-\frac{f^2}{4}(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\omega})Z)^H + \frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\theta})Z)\omega^V \\ &-\frac{1}{4}g(grad\, f, R(\tilde{p},\tilde{\omega})Z)\theta^V. \end{aligned}$$

The other formulas are obtained by a similar calculation. □

3.1.3 Sectional curvature

It is known that the sectional curvature on $(T^*M, {}^fg)$ for $P(V,W)$ is given by

$$^{f}K(V,W) = \frac{{}^fg({}^fR(V,W)W,V)}{{}^fg(V,V){}^fg(W,W) - {}^fg(V,W)^2}, \tag{3.17}$$

where $P(V,W)$ denotes the plane spanned by $\{V,W\}$.
If $\{X_i\}_{i=\overline{1,m}}$ and $\{\omega^i\}_{i=\overline{1,m}}$ be a local orthonormal frame and coframe on M, respectively, we see that

$$\{X_1^H,\cdots,X_m^H,\frac{1}{\sqrt{f}}(\omega^1)^V,\cdots,\frac{1}{\sqrt{f}}(\omega^m)^V\}, \tag{3.18}$$

is a local orthonormal frame on T^*M. Let ${}^fK(X^H,Y^H)$, ${}^fK(X^H,\theta^V)$ and ${}^fK(\omega^V,\theta^V)$ denote the sectional curvature of the plane spanned by $\{X^H,Y^H\}$, $\{X^H,\theta^V\}$ and $\{\omega^V,\theta^V\}$ on $(T^*M,{}^fg)$ respectively, where $X,Y\in\{X_i\}_{i=\overline{1,m}}$ and $\omega,\theta\in\{\omega^i\}_{i=\overline{1,m}}$.

Proposition 15. *([39]) Let (M,g) be a Riemannian manifold and $(T^*M,{}^fg)$ its cotangent bundle equipped with the vertical rescaled metric. Then we have the following*

$$\begin{aligned} i)\ {}^fg({}^fR(X^H,Y^H)Y^H,X^H) &= g(R(X,Y)Y,X)-\frac{3f}{4}\|pR(X,Y)\|^2,\\ ii)\ {}^fg({}^fR(X^H,\theta^V)\theta^V,X^H) &= \frac{f^2}{4}\|R(\tilde{p},\tilde{\theta})X\|^2+\frac{1}{4}X(f)^2+\frac{1}{2}g(\nabla_X grad\, f,X),\\ iii)\ {}^fg({}^fR(\omega^V,\theta^V)\theta^V,\omega^V) &= \frac{1}{4}\|grad\, f\|^2. \end{aligned}$$

Proof. *i)* From the formula (3.11), we have

$$\begin{aligned} {}^fg({}^fR(X^H,Y^H)Y^H,X^H) &= g(R(X,Y)Y,X)-\frac{f}{4}g(R(\tilde{p},\widetilde{pR(Y,Z)})Y,X)\\ &\quad+\frac{f}{2}g(R(\tilde{p},\widetilde{pR(Y,Z)})Y,X)\\ &= g(R(X,Y)Y,X)-\frac{3f}{4}g(R(Y,X)\tilde{p},\widetilde{pR(Y,Z)}), \end{aligned}$$

and

$$\widetilde{pR(Y,X)}=R(Y,X)\tilde{p}, \tag{3.19}$$

because for $Z\in\Im^1_0(M)$,

$$\begin{aligned} g(\widetilde{pR(Y,X)},Z) &= g_{st}(\widetilde{pR(Y,X)})^sZ^t=g_{st}g^{ks}(pR(Y,X))_kZ^t\\ &= \delta^k_t p_aR^a_{ijk}X^iY^jZ^t=g_{ab}\tilde{p}^bR^a_{ijk}X^iY^jZ^k\\ &= g(R(X,Y)Z,\tilde{p})=g(R(Y,X)\tilde{p},Z), \end{aligned}$$

then $\quad {}^fg({}^fR(X^H,Y^H)Y^H,X^H)=g(R(X,Y)Y,X)-\dfrac{3f}{4}\|pR(X,Y)\|^2.$

ii) From the formula (3.12), we have

$$\begin{aligned} {}^fg({}^fR(X^H,\theta^V)\theta^V,X^H) &= \frac{-f^2}{4}g(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\theta})X,X)+\frac{1}{4}X(f)^2g^{-1}(\theta,\theta)\\ &\quad+\frac{1}{2}g^{-1}(\theta,\theta)g(\nabla_X grad\, f,X)\\ &= \frac{f^2}{4}\|R(\tilde{p},\tilde{\theta})X\|^2+\frac{1}{4}X(f)^2+\frac{1}{2}g(\nabla_X grad\, f,X). \end{aligned}$$

iii) The result follows immediately from the formula (3.16)

$$\begin{aligned} {}^fg({}^fR(\omega^V,\theta^V)\theta^V,\omega^V) &= \frac{-1}{4f}\|grad\, f\|^2\left[fg^{-1}(\omega,\theta)^2-fg^{-1}(\theta,\theta)g^{-1}(\omega,\omega)\right]\\ &= \frac{1}{4}\|grad\, f\|^2. \end{aligned}$$

□

Theorem 14. *([39]) Let (M,g) be a Riemannian manifold and $(T^*M,{}^fg)$ its cotangent bundle equipped with the vertical rescaled metric. If K (resp., ${}^fK$) denote the sectional curvature tensor of (M,g) (resp., $(T^*M,{}^fg)$). Then we have the following*

$$\begin{aligned}
(1)\,{}^fK(X^H,Y^H) &= K(X,Y)-\frac{3f}{4}\|pR(X,Y)\|^2,\\
(2)\,{}^fK(X^H,\theta^V) &= \frac{f}{4}\|R(\tilde{p},\tilde{\theta})X\|^2+\frac{1}{4f}X(f)^2+\frac{1}{2f}g(\nabla_X grad\, f,X),\\
(3)\,{}^fK(\omega^V,\theta^V) &= \frac{1}{4f^2}\|grad\, f\|^2.
\end{aligned}$$

Proof. Using Proposition 15 we have

$$\begin{aligned}
(1)\,{}^fK(X^H,Y^H) &= \frac{{}^fg({}^fR(X^H,Y^H)Y^H,X^H)}{{}^fg(X^H,X^H){}^fg(Y^H,Y^H)-{}^fg(X^H,Y^H)^2}\\
&= {}^fg({}^fR(X^H,Y^H)Y^H,X^H)\\
&= g(R(X,Y)Y,X)-\frac{3f}{4}\|pR(X,Y)\|^2\\
&= K(X,Y)-\frac{3f}{4}\|pR(X,Y)\|^2.
\end{aligned}$$

$$\begin{aligned}
(2)\,{}^fK(X^H,\theta^V) &= \frac{{}^fg({}^fR(X^H,\theta^V)\theta^V,X^H)}{{}^fg(X^H,X^H){}^fg(\theta^V,\theta^V)-{}^fg(X^H,\theta^V)^2}\\
&= \frac{1}{f}{}^fg({}^fR(X^H,\theta^V)\theta^V,X^H)\\
&= \frac{f}{4}\|R(\tilde{p},\tilde{\theta})X\|^2+\frac{1}{4f}X(f)^2+\frac{1}{2f}g(\nabla_X grad\, f,X).
\end{aligned}$$

$$\begin{aligned}
(3)\,{}^fK(\omega^V,\theta^V) &= \frac{{}^fg({}^fR(\omega^V,\theta^V)\theta^V,\omega^V)}{{}^fg(\omega^V,\omega^V){}^fg(\theta^V,\theta^V)-{}^fg(\omega^V,\theta^V)^2}\\
&= \frac{1}{f^2}{}^fg({}^fR(\omega^V,\theta^V)\theta^V,\omega^V)\\
&= \frac{1}{4f^2}\|grad\, f\|^2.
\end{aligned}$$

□

Proposition 16. *([39]) Let (M,g) be a Riemannian manifold of constant sectional curvature λ and $(T^*M,{}^fg)$ its cotangent bundle equipped with the vertical rescaled metric. If ${}^fK$ denote the sectional curvature tensor of $(T^*M,{}^fg)$. Then we have the following*

$$\begin{aligned}
(1)\,{}^fK(X^H,Y^H) &= \lambda-\frac{3\lambda^2}{4}\big[g(X,\tilde{p})^2+g(Y,\tilde{p})^2\big],\\
(2)\,{}^fK(X^H,\theta^V) &= \frac{f\lambda^2}{4}\big[g(X,\tilde{\theta})^2\|p\|^2-2g(X,\tilde{\theta})g(X,\tilde{p})g(\tilde{\theta},\tilde{p})+g(X,\tilde{p})^2\big]\\
&\quad+\frac{1}{4f}X(f)^2+\frac{1}{2f}g(\nabla_X grad\, f,X),\\
(3)\,{}^fK(\omega^V,\theta^V) &= \frac{1}{4f^2}\|grad\, f\|^2.
\end{aligned}$$

Proof. M has constant curvature λ then

$$R(X,Y)Z = \lambda\big[g(Y,Z)X - g(X,Z)Y\big]$$

then a direct calculations we get

$$\begin{aligned} \|pR(X,Y)\|^2 &= \|R(Y,X)\tilde{p}\|^2 = \lambda^2\big[g(X,\tilde{p})^2 + g(Y,\tilde{p})^2\big], \\ \|R(\tilde{p},\tilde{\theta})X\|^2 &= \lambda^2\big[g(X,\tilde{\theta})^2\|p\|^2 - 2g(X,\tilde{\theta})g(X,\tilde{p})g(\tilde{\theta},\tilde{p}) + g(X,\tilde{p})^2\big], \end{aligned}$$

this completes the proof. □

Corollary 6. *([39]) Let (M,g) be a m-dimensional Riemannian manifold of constant sectional curvature and $(T^*M, {}^f g)$ its cotangent bundle equipped with the vertical rescaled metric. If f be a constant, then $(T^*M, {}^f g)$ has zero constant sectional curvature if and only if (M,g) is flat.*

3.1.4 Scalar curvature

Theorem 15. *([39]) Let (M,g) be a m-dimensional Riemannian manifold and $(T^*M, {}^f g)$ its cotangent bundle equipped with the vertical rescaled metric. If σ (resp., ${}^f\sigma$) denote the scalar curvature of (M,g) (resp., $(T^*M, {}^f g)$), then we have*

$$^f\sigma = \sigma - \frac{f}{4}\sum_{i,j=1}^{m}\|pR((E_i,(E_j)\|^2 + \frac{m}{f}\Big(\Delta(f) + \frac{m+2f}{4f}\|grad\, f\|^2\Big), \tag{3.20}$$

where $(E_i)_{i=\overline{1,m}}$ be a local orthonormal frame on M and $\Delta(f) = \sum_{i=1}^{m} g(\nabla_{E_i} grad\, f, E_i)$

Proof. Let $(E_i)_{i=\overline{1,m}}$ and $(\omega^i)_{i=\overline{1,m}}$ be a local orthonormal frame and coframe on M, $\big\{E_1^H,\cdots,E_m^H, \frac{1}{\sqrt{f}}(\omega^1)^V,\cdots,\frac{1}{\sqrt{f}}(\omega^m)^V\big\}$ is a local orthonormal frame on T^*M.

Using Theorem 14 and definition of scalar curvature, we have.

$$\begin{aligned}
{}^f\sigma &= \sum_{i,j=1}^{m} {}^fK(E_i^H, E_j^H) + 2\sum_{i,j=1}^{m} {}^fK(E_i^H, \frac{1}{\sqrt{f}}(\omega^j)^V) + \sum_{i,j=1}^{m} {}^fK(\frac{1}{\sqrt{f}}(\omega^i)^V, \frac{1}{\sqrt{f}}(\omega^j)^V) \\
&= \sum_{i,j=1}^{m} {}^fK(E_i^H, E_j^H) + 2\sum_{i,j=1}^{m} {}^fK(E_i^H, (\omega^j)^V) + \sum_{i,j=1}^{m} {}^fK((\omega^i)^V, (\omega^j)^V) \\
&= \sum_{i,j=1}^{m} \big[K(E_i, E_j) - \frac{3f}{4}\|pR(E_i, E_j)\|^2\big] \\
&\quad +2\sum_{i,j=1}^{m} \big[\frac{f}{4}\|R(\tilde{p}, \widetilde{\omega^j})E_i\|^2 + \frac{1}{4f}E_i(f)^2 + \frac{1}{2f}g(\nabla_{E_i} grad\, f, E_i)\big] \\
&\quad + \sum_{i,j=1}^{m} \frac{1}{4f^2}\|grad\, f\|^2 \\
&= \sigma - \frac{3f}{4}\sum_{i,j=1}^{m} \|pR(E_i, E_j)\|^2 + \frac{f}{2}\sum_{i,j=1}^{m} \|R(\tilde{p}, \widetilde{\omega^j})E_i\|^2 \\
&\quad + \frac{m}{2f}\|grad\, f\|^2 + \frac{m}{f}\Delta(f) + \frac{m^2}{4f^2}\|grad\, f\|^2 \\
&= \sigma - \frac{3f}{4}\sum_{i,j=1}^{m} \|pR(E_i, E_j)\|^2 + \frac{f}{2}\sum_{i,j=1}^{m} \|R(\tilde{p}, \widetilde{\omega^j})E_i\|^2 \\
&\quad + \frac{m}{f}\Delta(f) + \frac{m^2 + 2mf}{4f^2}\|grad\, f\|^2.
\end{aligned}$$

In order to simplify this last expression.
From the formula (3.19) we have

$$\begin{aligned}
\widetilde{\omega^j} &= \sum_{i=1}^{m} g(\widetilde{\omega^j}, E_i)E_i = \sum_{i,a,b=1}^{m} g_{ab}\widetilde{\omega^j}^a E_i^b E_i = \sum_{i,a,b,k=1}^{m} g_{ab}g^{ka}\omega_k^j E_i^b E_i \\
&= \sum_{i,b,k=1}^{m} \delta_b^k \omega_k^j E_i^b E_i = \sum_{i,k=1}^{m} \omega_k^j E_i^k E_i = \sum_{i=1}^{m} \omega^j(E_i)E_i = \sum_{i=1}^{m} \delta_i^j E_i = E_j,
\end{aligned}$$

and

$$\sum_{i,j=1}^{m} \|R(E_j, E_i)\tilde{p}\|^2 = \sum_{i,j=1}^{m} \|R(\tilde{p}, E_j)E_i\|^2,$$

then

$$\sum_{i,j=1}^{m} \|pR(E_i, E_j)\|^2 = \sum_{i,j=1}^{m} \|R(\tilde{p}, \widetilde{\omega^j})E_i\|^2,$$

this completes the proof. □

From Theorem 15, we deduce the result.

Corollary 7. *([39]) Let* (M, g) *be a flat m-dimensional Riemannian manifold and* $(T^*M, {}^f g)$ *its cotangent bundle equipped with the vertical rescaled metric. If* σ *(resp.,* ${}^f\sigma$*) denote the scalar curvature of* (M, g) *(resp.,* $(T^*M, {}^f g)$*), then we have*

$$ {}^f\sigma \;=\; \frac{m}{f}\Big(\Delta(f) + \frac{m+2f}{4f}\|grad\, f\|^2\Big). \tag{3.21}$$

Corollary 8. *([39]) Let* (M, g) *be a m-dimensional Riemannian manifold,* $(T^*M, {}^f g)$ *its cotangent bundle equipped with the vertical rescaled metric and* f *be a constant. If* σ *(resp.,* ${}^f\sigma$*) denote the scalar curvature of* (M, g) *(resp.,* $(T^*M, {}^f g)$*), then we have*

$$ {}^f\sigma \;=\; \sigma - \frac{f}{4}\sum_{i,j=1}^{m}\|pR(E_i, E_j)\|^2, \tag{3.22}$$

where (E_i) *is a local orthonormal frame of* M*.*

Corollary 9. *Let* (M, g) *be a m-dimensional Riemannian manifold of constant sectional curvature* λ*,* $(T^*M, {}^f g)$ *its cotangent bundle equipped with the vertical rescaled metric and* f *be a constant. Then* $(T^*M, {}^f g)$ *has zero constant scalar curvature if and only if* M *is flat.*

Proposition 17. *([39]) Let* (M, g) *be a m-dimensional Riemannian manifold* $(m \geq 2)$ *of constant sectional curvature* λ *and* $(T^*M, {}^f g)$ *its cotangent bundle equipped with the vertical rescaled metric. If* ${}^f\sigma$ *denote the scalar curvature of* $(T^*M, {}^f g)$*, then we have*

$$ {}^f\sigma \;=\; (m-1)\lambda\Big(m - \frac{f}{2}\lambda\|p\|^2\Big) + \frac{m}{f}\Big(\Delta(f) + \frac{m+2f}{4f}\|grad\, f\|^2\Big). \tag{3.23}$$

Proof. M has constant curvature λ then $R(X, Y)Z = \lambda\big[g(Y, Z)X - g(X, Z)Y\big]$
and

$$\sigma = m(m-1)\lambda,$$

for $i \neq j$ we get

$$\sum_{i,j=1}^{m}\|pR(E_i, E_j)\|^2 \;=\; 2(m-1)\lambda^2\|p\|^2,$$

this completes the proof. □

Chapter 4

Bi-f-harmonic Maps on Generalized Warped Product Manifolds

In this chapter, we introduce a very large generalization of harmonic maps called f-bi-harmonic maps as the critical points of f-bi-energy functional, and then derive the Euler-Lagrange equation of f-bi-energy functional given by the vanishing of f-bi-tension field. Subsequently, we study some properties of f-bi-harmonic maps between the same dimensional manifolds and give a non-trivial example. Furthermore, we also study the basic properties of f-bi-harmonic maps on a generalized warped product manifold. bi-f-harmonic maps generalize not only the f-harmonic maps but also bi-harmonic maps, whereas f-bi-harmonic maps only generalize the bi-harmonic maps. Thus, we generalized the results in the article [56].

4.1 Some results on generalized warped product manifolds

In this section, we give the definition and some geometric properties of generalized warped product manifolds

Definition 26. *Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds, and*

$$G_\lambda = \pi^* g + \lambda^2 \eta^* h, \tag{4.1}$$

where $\pi : (x, y) \in M \times N \longrightarrow x \in M$ *and* $\eta : (x, y) \in M \times N \longrightarrow y \in N$ *are the canonical projections. For all* $X, Y \in T(M \times N)$*, we have*

$$G_\lambda(X, Y) = g(d\pi(X), d\pi(Y)) + \lambda^2 h(d\eta(X), d\eta(Y)), \tag{4.2}$$

and we denote by $X \wedge_{G_{\lambda^2}} Y$*, the linear map:*

$$Z \in \Gamma(TM) \times \Gamma(TN) \longrightarrow (X \wedge_{G_{\lambda^2}} Y)Z = G_{\lambda^2}(Z, Y)X - G_{\lambda^2}(Z, X)Y \tag{4.3}$$

Proposition 18. *([46]) Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds. If* $\overline{\nabla}$ *denote the Levi-Civita connection on* $(M \times_\lambda N, G_\lambda)$*, then for all* $X_1, Y_1 \in \Gamma(TM)$ *and* $X_2, Y_2 \in \Gamma(TN)$

we have:

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + X(\ln\lambda)(0, Y_2) + Y(\ln\lambda)(0, X_2) \\ &- \frac{1}{2} h(X_2, Y_2)(grad_M \lambda^2, \frac{1}{\lambda^2} grad_N \lambda^2), \end{aligned} \tag{4.4}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ *and* $\nabla_X Y = (\nabla^M_{X_1} Y_1, \nabla^N_{X_2} Y_2)$.

Proposition 19. *([46]) Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds. and* $\lambda : M \times N \longrightarrow \mathbb{R}$ *be smooth positive function. If* R *and* $\overline{R}$ *denote the curvatures tensors of product manifold* $(M \times N, G)$ *and generalized warped product manifold* $(M \times_\lambda N, G_\lambda)$ *respectively, then*

$$\begin{aligned} \overline{R}(X,Y)Z - R(X,Y)Z &= (\nabla^M_{Y_1} grad_M \ln\lambda + Y_1(\ln\lambda) grad_M \ln\lambda, 0) \wedge G_\lambda(0, X_2)Z \\ &- (\nabla^M_{X_1} grad_M \ln\lambda X_1(\ln\lambda) grad_M \ln\lambda, 0) \wedge G_\lambda(0, Y_2)Z \\ &+ \frac{1}{\lambda^2}\Big[(0, \nabla^N_{Y_2} grad_N \ln\lambda - Y_2(\ln\lambda) grad_N \ln\lambda, 0) \wedge G_\lambda(0, X_2)Z \\ &- (0, \nabla^N_{X_2} grad_N \ln\lambda - X_2(\ln\lambda) grad_N \ln\lambda, 0) \wedge G_\lambda(0, Y_2)Z \\ &- (\lambda^2 |grad_M \ln\lambda|^2 + |grad_N \ln\lambda|^2)(0, X_2) \wedge G_\lambda(0, Y_2)\Big] \\ &+ [X_1(Z_2(\ln\lambda)) + X_2(Z_1(\ln\lambda))](0, Y^2) \\ &- [Y_1(Z_2(\ln\lambda)) + Y_2(Z_1(\ln\lambda))](0, X^2), \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM) \times \Gamma(TN)$, *where* $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ *and* $Z = (Z_1, Z_2)$.

4.2 Generalized bi-f-harmonic map

Definition 27. *The bi-f-energy functional of smooth map* $\phi : (M^m, g) \longrightarrow (N^n, h)$ *is defined by*

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M f(x, \varphi(x)) |\tau(\varphi)|^2 v_g, \tag{4.5}$$

where $f : (x, y) \in M \times N \longrightarrow f(x, y) \in (0, +\infty)$ *be a smooth positive function,* φ *is called bi-f-harmonic if it is a critical point of the bi-f-energy functional.*

4.2.1 First variation of the bi-f-energy

Theorem 16. *([38]) Let* $\phi : (M^m, g) \longrightarrow (N^n, h)$ *be a smooth map and let* $\{\varphi_t\}_t(-\epsilon < t < \epsilon)$, *be a smooth variation of* φ. *Then*

$$\frac{d}{dt} E_{f,2}(\varphi_t)|_{t=0} = -\int_M h(\tau_{2,f}(\varphi), v) v_g, \tag{4.6}$$

where $v = \frac{d\varphi_t}{dt}|_{t=0}$ *denotes the variation vector field of* $\{\varphi_t\}_t$,

$$\tau_{2,f}(\varphi) = -tr_g(\nabla^\varphi)^2 f_\varphi \tau(\varphi) - f_\varphi tr_g R^N(\tau(\varphi), d\varphi)d\varphi - \frac{1}{2}|\tau(\varphi)|^2 (grad^N f) \circ \varphi, \tag{4.7}$$

where f_φ *is a smooth function* $x \in M^m \longrightarrow f_\varphi(x) = f(x, \varphi(x)) \in (0, +\infty)$.

Proof.
Let $\phi : I \times M \longrightarrow N$ be a smooth map satisfying for all $t \in I$ and all $x \in M$

$$\phi(t,x) = \varphi_t(x),$$

and

$$\phi(0,x) = \varphi(x).$$

The variation vector field $v \in \Gamma(\varphi^{-1}TN)$ associated to the variation $\{\varphi_t\}_{t\in I}$ is given for all $x \in M$ by

$$v(x) = d_{(0,x)}\phi(\frac{d}{dt}),$$

we have

$$\begin{aligned}
\frac{d}{dt}E(\varphi_t)|_{t=0} &= \frac{1}{2}\int_M \frac{\partial}{\partial t}(f(x,\varphi_t(x)))|\tau(\varphi_t)|^2|)|_{(0,x)}v_g \\
&= \frac{1}{2}\int_M \Big\{\frac{\partial}{\partial t}(f(x,\varphi_t(x)))|_{(0,x)}|\tau(\varphi_t)|^2|) \qquad (4.8)\\
&+ f(x,\varphi(x))\frac{\partial}{\partial t}|\tau(\varphi_t)|^2\ |_{(0,x)}\ v_g\Big\}.
\end{aligned}$$

First, note that:

$$\begin{aligned}
\frac{\partial}{\partial t}(f(x,\varphi_t(x)))|_{(0,x)} &= \frac{\partial}{\partial t}(f(x,\phi(t,x)))|_{(0,x)} \\
&= d_{(x,\varphi(x))}f(0,v(x)) \\
&= d_{\varphi(x)}f_x(v(x)),
\end{aligned}$$

where f_x is a smooth function $y \in N \longrightarrow f_x(y) = f(x,y) \in (0,+\infty)$. Hence

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}(f(x,\varphi_t(x)))|_{(0,x)}|\tau_x(\varphi)|^2 &= \frac{1}{2}d_{\varphi(x)}f_x(v(x))|\tau_x(\varphi)|^2 \\
&= \frac{1}{2}h((grad^N f_x)_{\varphi(x)}, v(x))|\tau_x(\varphi)|^2 \\
&= h(\frac{1}{2}|\tau_x(\varphi)|^2(grad^N f_x)_{\varphi(x)}, v(x)). \qquad (4.9)
\end{aligned}$$

Other hand:
Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i,j = 1,...,m$. Then calculating at x gives

$$\begin{aligned}
\frac{1}{2}f_\varphi(x)\frac{\partial}{\partial t}|\tau(\varphi_t)|^2_{t=0} &= \frac{1}{2}f_\varphi(x)\frac{\partial}{\partial t}h(\tau(\varphi_t),\tau(\varphi_t))|_{t=0} \\
&= f_\varphi(x)h(\nabla^\phi_{\frac{\partial}{\partial t}}\tau(\varphi_t),\tau(\varphi_t))|_{t=0} \\
&= f_\varphi(x)h(\nabla^\phi_{\frac{\partial}{\partial t}}\tau(\varphi_t)|_{t=0},\tau(\varphi)). \qquad (4.10)
\end{aligned}$$

For a given $X \in \Gamma(TM)$, note that $[\frac{\partial}{\partial t}, X] = 0$, we get

$$\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(X) = \nabla^\phi_X d\phi(\frac{\partial}{\partial t}) + d\phi([\frac{\partial}{\partial t}, X]) = \nabla^\phi_X(\frac{\partial}{\partial t}).$$

Thus at x, becomes

$$\begin{aligned}
\nabla^{\phi}_{\frac{\partial}{\partial t}}\tau(\varphi_t) &= \sum_{i=1}^{m}\nabla^{\phi}_{\frac{\partial}{\partial t}}\nabla^{\phi}_{e_i}d\phi(e_i)\\
&= \sum_{i=1}^{m}\Big(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(e_i)+\nabla^{\phi}_{[\frac{\partial}{\partial t},e_i]}d\phi(e_i)+R^{\phi}(\frac{\partial}{\partial t},e_i)d\phi(e_i)\Big)\\
&= \sum_{i=1}^{m}\Big(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(e_i)+R^{\phi}(\frac{\partial}{\partial t},e_i)d\phi(e_i)\Big)\\
&= \sum_{i=1}^{m}\Big(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi(\frac{\partial}{\partial t})+R^{N}(d\phi(\frac{\partial}{\partial t}),d\phi(e_i))d\phi(e_i)\Big)\\
&= \sum_{i=1}^{m}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi(\frac{\partial}{\partial t})+\sum_{i=1}^{m}R^{N}(d\phi(\frac{\partial}{\partial t}),d\phi(e_i))d\phi(e_i). \qquad (4.11)
\end{aligned}$$

Substituting (4.11) into (4.9) and Taking into account the symmetric properties of Riemann-Christoffel tensor field, we have

$$\begin{aligned}
\frac{d}{dt}E_{2,f}(\varphi_t)|_{t=0} &= \int_M f_\varphi h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,\tau(\varphi))+h(R^N(v,d\varphi(e_i))d\varphi(e_i,\tau(\varphi))v_g\\
&= \int_M f_\varphi h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,\tau(\varphi))-h(R^N(d\varphi(e_i),\tau(\varphi))d\varphi(e_i,v)v_g\\
&= \int_M h(tr_g\nabla^{\varphi}_{*}\nabla^{\varphi}_{*}v,f_\varphi\tau(\varphi))-h(f_\varphi tr_g R^N(d\varphi(*),\tau(\varphi))d\varphi(*),v)v_g.
\end{aligned}$$

$$(4.12)$$

Denote by ν the outward unit normal vector of ∂M in M and by $i:\partial M\longrightarrow M$ the canonical inclusion. Note that

$$h(tr_g(\nabla^{\varphi}\nabla^{\varphi}v),f_\varphi\tau(\varphi))=h(v,tr_g(\nabla^{\varphi}\nabla^{\varphi}f_\varphi\tau(\varphi))+d^{\varphi}h(v,\nabla^{\varphi}f_\varphi\tau(\varphi)),$$

where

$$\begin{aligned}
d^{\varphi}h(v,\nabla^{\varphi}f_\varphi\tau(\varphi)) =&h(tr_g(\nabla^{\varphi}\nabla^{\varphi}v,f_\varphi\tau(\varphi))+h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}f_\varphi\tau(\varphi))\\
=&h(v,tr_g(\nabla^{\varphi}\nabla^{\varphi}f_\varphi\tau(\varphi))+h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}f_\varphi\tau(\varphi)).
\end{aligned}$$

By using the Divergence theorem, we can write (4.12) as

$$\begin{aligned}
\frac{d}{dt}E_{2,f}(\varphi_t)|_{t=0} &= \int_M h(tr_g(\nabla^{\varphi}f_\varphi\nabla^{\varphi}\tau(\varphi)),v)v_g+\int_M d^{\varphi}h(\nabla^{\varphi}v,f_\varphi\tau(\varphi))v_g\\
&- \int_M d^{\varphi}h(v,\nabla^{\varphi}f_\varphi\tau(\varphi)v_g+\int_M h(f_\varphi tr_g R^N(\tau(\varphi),d\varphi)d\varphi,v)v_g\\
&= \int_M h(tr_g h((\nabla^{\varphi}\nabla^{\varphi}f_\varphi\tau(\varphi)+f_\varphi tr_g R^N(\tau(\varphi),d\varphi)d\varphi,v)v_g\\
&+ \int_{\partial M} h(\nabla^{\varphi}_{\mu}\mu),f_\varphi\tau(\varphi))v_g-\int_{\partial M} h(v,\nabla^{\varphi}_{\mu}f_\varphi\tau(\varphi))v_g\\
&= \int_M h(tr_g\nabla^{\varphi}\nabla^{\varphi}f_\varphi\tau(\varphi)+f_\varphi tr_g R^N(\tau(\varphi),d\varphi)d\varphi,v)v_g. \qquad (4.13)
\end{aligned}$$

Thus, (4.13) gives

$$\frac{d}{dt}E_{2,f}(\varphi_t)|_{t=0} = -\int_M h(-tr_g(\nabla^\varphi)^2 f_\varphi \tau(\varphi) - f_\varphi tr_g R^N(\tau(\varphi), d\varphi)d\varphi, v)v_g.$$

□

Particular Cases

(1) If $f = 1$, then $\tau_{2,f}(\varphi) = \tau_2(\varphi)$ is the natural bi-tension field of φ.

(2) Let $f : M \longrightarrow (0, +\infty)$, be a smooth positif function. If $f(x, y) = f(x)$, for all $(x, y) \in M \times N$, then

$$\tau_{2,f}(\varphi) = -tr_g(\nabla^\varphi)^2 f\tau(\varphi) - f tr_g R^N(\tau(\varphi), d\varphi)d\varphi.$$

Next, we give the relation between bi-f-tension field $\tau_{2,f}(\varphi)$ and bi-tension field $\tau_2(\varphi)$.

Proposition 20. *([38]) Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. Then the relation between bi-f-tension field $\tau_{2,f}(\varphi)$ and bi-tension field $\tau_2(\varphi)$ is*

$$\tau_{2,f}(\varphi) = f_\varphi \tau_2(\varphi) - \Delta(f_\varphi)\tau(\varphi) - 2\nabla^\varphi_{grad(f_\varphi)}\tau(\varphi) - \frac{1}{2}|\tau(\varphi)|^2(grad^N f)\circ\varphi, \tag{4.14}$$

where f_φ is a smooth function $x \in M^m \longrightarrow f_\varphi(x) = f(x, \varphi(x)) \in (0, +\infty)$.

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, ..., m$, we have

$$\begin{aligned} tr_g(\nabla^\varphi)^2 f_\varphi \tau(\varphi) &= \sum_{i=1}^m \nabla^\varphi_{e_i}\nabla^\varphi_{e_i} f_\varphi \tau(\varphi) \\ &= f_\varphi \sum_{i=1}^m \Big(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) + e_i(e_i f)\tau(\varphi) + 2e_i(f)\nabla^\varphi_{e_i}\tau(\varphi)\Big) \\ &= f_\varphi tr_g(\nabla^\varphi)^2\tau(\varphi) + \Delta(f_\varphi)\tau(\varphi) + 2\nabla^\varphi_{grad(f_\varphi)}\tau(\varphi). \end{aligned}$$

Combining (1.21) and (4.7), we obtain (4.14). □

Example 9. *Let $M = (\mathbb{R}^*, dx^2)$, $N = (\mathbb{R}, dy^2)$, $\varphi : M \longrightarrow N$ be a smooth function, and let $f : M \times N \longrightarrow \mathbb{R}_+$ be a C^2 function. From Theorem 16, we have*

$$\tau_{2,f}(\varphi)|_x = \Big[\varphi'''(x)f(x, \varphi(x)) + \varphi''(x)\frac{\partial f}{\partial x}(x, \varphi(x)) + \frac{1}{2}\varphi''(x)^2\frac{\partial f}{\partial y}(x, \varphi(x))\Big]\frac{d}{dy}|_{\varphi(x)}.$$

If $f(x, y) = e^{xy}$, by (4.14) φ is bi-f-harmonic if and only if

$$\varphi'''(x) + \varphi(x)\varphi''(x) + \frac{1}{2}x\varphi''(x)^2 = 0. \tag{4.15}$$

A local solution of the equation (4.15) is $\varphi(x) = ax + b, \qquad a, b \in \mathbb{R}$.

4.3 Bi-f-harmonic map equations on generalized WPM

4.3.1 Bi-f-harmonicity of the inclusion maps

Proposition 21. *([38]) Let $f : N \longrightarrow (0, +\infty)$ be a smooth function and (M^m, g), (N^n, h) be two Riemannian manifolds and x_0 be an arbitrary point of M. Then the bi-f-tension fields of the inclusion*

$$
\begin{array}{rcl}
\bar{\phi} : (N, h) & \longrightarrow & (M \times_\lambda N, G_\lambda) \\
y & \longmapsto & (x_0, y),
\end{array}
$$

are given by:

$$
\begin{aligned}
\tau_{2,f}(\bar{\phi}) = \Big(&- \frac{n^2 e^{4\gamma}}{2} f(y) grad_M(|grad_M\gamma|^2 - e^{2\gamma} f(y)[2n^2 e^{2\gamma}|grad_M\gamma|^2 - 4\Delta_N\gamma)] grad_M\gamma \\
&- e^{2\gamma} f(y)[(n^2 - 4n - 4)|grad_N\gamma)|^2] grad_M\gamma + n\Delta_N f(y) e^{2\gamma} grad_M\gamma \\
&+ 2ne^{2\gamma}[2grad_N f(y)(\gamma) grad_M\gamma + 2(2-n)e^{2\gamma} 2grad_N f(y)(\gamma) grad_M\gamma], 0\Big) \\
&+ \Big(0, (n-2) f(y)(grad_N(\Delta_N\gamma) + 2Ricci_N(grad_N\gamma) \\
&+ f(y)[(2-n)^2|grad_N\gamma)|^2 - 2(2-n)\Delta_N\gamma)] grad_N\gamma + ne^{2\gamma} f(y) grad_N(|grad_M\gamma)|^2 \\
&+ f(y) trace_N(grad_M\gamma)(*(\gamma))(*) + \frac{(n-2)(6-n)}{2} f(y) grad_N(|grad_N\gamma)|^2 \\
&+ f(y)[2n(n-4)e^{2\gamma}|grad_M\gamma)|^2] grad_N\gamma + 2ne^{2\gamma}|grad_M\gamma|^2 grad_N f(y)) \\
&- (2-n) f(y)\Delta_N f(y) grad_N\gamma - (2-n)\nabla^N_{grad_N f(y)} grad_N\gamma \\
&- 2|grad_N\gamma|^2 grad_N f(y))\Big),
\end{aligned}
$$

where $\lambda(x, y) = e^{\gamma(x,y)}$.

Proof. Let $\{e_\alpha\}_{i=1}^n$ be an orthonormal frame on N. Similar to the proof of [[15], Theorem 1], we have

$$
\tau(\bar{\phi}) = \{-ne^{2\gamma}(grad_M\gamma, 0) + (2-n)(0, grad_N\gamma)\} \circ \bar{\phi}.
$$

$$
\begin{aligned}
\tau_2(\bar{\phi}) = \{&-\frac{n^2 e^{4\gamma}}{2}(grad_M(|grad_M\gamma|^2, 0) \\
&+ (n-2)(0, grad_N(\Delta_N\gamma) + 2Ricci_N(grad_N\gamma) \\
&- e^{2\gamma}[2n^2 e^{2\gamma}|grad_M\gamma|^2 - 4\Delta_N\gamma)](grad_M\gamma, 0) \\
&- e^{2\gamma}[(n^2 - 4n - 4)|grad_N\gamma)|^2](grad_M\gamma, 0) \\
&+ [(2-n)^2|grad_N\gamma)|^2 - 2(2-n)\Delta_N\gamma)](0, grad_N\gamma) \\
&+ [2n(n-4)e^{2\gamma}|grad_M\gamma)|^2](0, grad_N\gamma) \\
&+ ne^{2\gamma}[(0, grad_N(|grad_M\gamma)|^2) + trace_N(grad_M\gamma)(*(\gamma))(0, *))] \\
&+ \frac{(n-2)(6-n)}{2}(0, grad_N(|grad_N\gamma)|^2)\} \circ \bar{\phi}.
\end{aligned}
$$

Furthermore, we have

$$\begin{aligned}
\nabla^{\bar{\phi}}_{grad_N f(y)}\tau(\bar{\phi}) &= \overline{\nabla}_{(0,grad_N f(y))}\tau(\bar{\phi})\\
&= -ne^{2\gamma}[2grad_N f(y)(\gamma)(grad_M\gamma,0)+|grad_M\gamma|^2(0,grad_N f(y))]\\
&\quad+(2-n)[(0,\nabla^N_{grad_N f(y)}grad_N\gamma)+|grad_N\gamma|^2(0,grad_N f(y))]\\
&\quad-2(2-n)e^{2\gamma}grad_N f(y)(\gamma)(grad_M\gamma,0).
\end{aligned}$$

Combining these equations with (4.14), we obtain

$$\begin{aligned}
\tau_{2,f}(\bar{\phi}) = \Big(&-\frac{n^2e^{4\gamma}}{2}f(y)grad_M(|grad_M\gamma|^2-e^{2\gamma}f(y)[2n^2e^{2\gamma}|grad_M\gamma|^2-4\Delta_N\gamma)]grad_M\gamma\\
&-e^{2\gamma}f(y)[(n^2-4n-4)|grad_N\gamma)|^2]grad_M\gamma+n\Delta_N f(y)e^{2\gamma}grad_M\gamma\\
&+2ne^{2\gamma}[2grad_N f(y)(\gamma)grad_M\gamma+2(2-n)e^{2\gamma}2grad_N f(y)(\gamma)grad_M\gamma,0\Big)\\
&+\Big(0,(n-2)f(y)(grad_N(\Delta_N\gamma)+2Ricci_N(grad_N\gamma))\\
&+f(y)[2n(n-4)e^{2\gamma}|grad_M\gamma)|^2]grad_N\gamma)\\
&+f(y)[(2-n)^2|grad_N\gamma)|^2-2(2-n)\Delta_N\gamma)]grad_N\gamma)\\
&+f(y)trace_N(grad_M\gamma)(*(\gamma))(*)+\frac{(n-2)(6-n)}{2}f(y)grad_N(|grad_N\gamma)|^2\\
&-(2-n)f(y)\Delta_N f(y)grad_N\gamma+2ne^{2\gamma}|grad_M\gamma|^2grad_N f(y))\\
&+ne^{2\gamma}f(y)grad_N(|grad_M\gamma)|^2-(2-n)\nabla^N_{grad_N f(y)}grad_N\gamma\\
&-2|grad_N\gamma|^2grad_N f(y))\Big).
\end{aligned}$$

□

Corollary 10. *([38]) Let $f:(N,h)\longrightarrow(0,+\infty)$ be a smooth function. The inclusion map $\bar{\phi}:(N,h)\longrightarrow(M\times_\lambda N,G_\lambda)$ is an bi-f-harmonic map if and only if λ and f satisfy*

$$\begin{aligned}
&-\frac{n^2e^{4\gamma}}{2}f(y)grad_M(|grad_M\gamma|^2-e^{2\gamma}f(y)[2n^2e^{2\gamma}|grad_M\gamma|^2-4\Delta_N\gamma)]grad_M\gamma\\
&-e^{2\gamma}f(y)[(n^2-4n-4)|grad_N\gamma)|^2]grad_M\gamma+n\Delta_N f(y)e^{2\gamma}grad_M\gamma\\
&+2ne^{2\gamma}[2grad_N f(y)(\gamma)grad_M\gamma+2(2-n)e^{2\gamma}2grad_N f(y)(\gamma)grad_M\gamma=0.
\end{aligned}$$

$$\begin{aligned}
&(n-2)f(y)(grad_N(\Delta_N\gamma)+2Ricci_N(grad_N\gamma))\\
&+f(y)[2n(n-4)e^{2\gamma}|grad_M\gamma)|^2]grad_N\gamma)\\
&+f(y)[(2-n)^2|grad_N\gamma)|^2-2(2-n)\Delta_N\gamma)]grad_N\gamma)+ne^{2\gamma}f(y)grad_N(|grad_M\gamma)|^2\\
&+f(y)trace_N(grad_M\gamma)(*(\gamma))(*)+\frac{(n-2)(6-n)}{2}f(y)grad_N(|grad_N\gamma)|^2\\
&-(2-n)f(y)\Delta_N f(y)grad_N\gamma+2ne^{2\gamma}|grad_M\gamma|^2grad_N f(y))\\
&-(2-n)\nabla^N_{grad_N f(y)}grad_N\gamma-2|grad_N\gamma|^2grad_N f(y))=0.
\end{aligned}$$

4.4 Bi-f-harmonicity of the projection maps

We know that the second factor projection map $\pi_2 : M \times_\lambda N \longrightarrow N$ satisfies $\tau(\pi_2) = 0$, whereas the first projection map $\pi_1 : M \times_\lambda N \longrightarrow M$ satisfies $\tau(\pi_1) \neq 0$ with non-constant positive function λ. This shows that π_2 automatically becomes an bi-f-harmonic map. Thus, we only need to consider the projection map π_1.

Proposition 22. *([56]) Let $f : M \times_\lambda N \longrightarrow (0, +\infty)$ be a smooth function. The bi-f-tension field of $\pi_1 : M \times_\lambda N \longrightarrow M$ is given by*

$$\begin{aligned}\tau_{2,f}(\pi_1) = {}& nf(x,y)tr_g(\nabla^M)^2 grad_M \ln\lambda + \frac{n^2}{2} f(x,y) grad_M(|grad_M \ln\lambda|^2)\\ &+ nf(x,y)Ricci^M(grad_M \ln\lambda) - n(\Delta_{M\times_\lambda N} f(x,y)) grad_M \ln\lambda\\ &- 2n\nabla^M_{grad_M f} grad_M \ln\lambda|_{\pi_1}\end{aligned}$$

Corollary 11. *([56]) The projection map π_1 is a proper bi-f-harmonic map if and only if λ and f satisfy*

$$\begin{aligned}&f(x,y)tr_g(\nabla^M)^2 grad_M \ln\lambda + \frac{n}{2} f(x,y) grad_M(|grad_M \ln\lambda|^2)\\ &+ nf(x,y)Ricci^M(grad_M \ln\lambda) - (\Delta_{M\times_\lambda N} f(x,y)) grad_M \ln\lambda\\ &- 2\nabla^M_{grad_M f} grad_M \ln\lambda|_{\pi_1} = 0.\end{aligned}$$

Proposition 23. *([38]) Let $f : M \times_\lambda N \longrightarrow (0, +\infty)$ be a smooth function. The bi-f-tension field of $\pi_2 : M \times_\lambda N \longrightarrow N$ is given by*

$$\begin{aligned}\tau_{2,f}(\pi_2) = {}& -f(x,y)\frac{n-2}{\lambda^4}\Big[grad_N(\Delta_N \ln\lambda) + 2Ricci(grad_N)\ln\lambda\\ &+ \frac{n-6}{2} grad_N(|grad_N \ln\lambda|^2) + (4-n)|grad_N \ln\lambda|^2\Big]\\ &+ \frac{n-2}{\lambda^4} f(x,y)\Delta_N \ln\lambda grad_N \ln\lambda + \frac{2(n-2)^2}{\lambda^2} f(x,y)|grad_M \ln\lambda|^2\\ &+ \frac{2(n-2)}{\lambda^2} f(x,y)\Delta_M \ln\lambda grad_N\lambda + \frac{(2-n)}{\lambda^2}\Delta_{M\times_\lambda N}(x,y) grad_N \ln\lambda\\ &- 2\frac{(n-2)}{\lambda}\Big(2grad_N(\ln\lambda)f + \frac{1}{\lambda^3}\nabla^N_{grad_N f} grad_N \ln\lambda\Big).\end{aligned}$$

Proof. Similar to the [[15],Corollary 4], we have

$$\tau(\pi_2) = \frac{n-2}{\lambda^4} grad_N \ln\lambda,$$

and

$$\begin{aligned}\tau_2(\pi_2) = &-\frac{n-2}{\lambda^4}\Big[grad_N(\Delta_N \ln\lambda) + 2Ricci(grad_N)\ln\lambda \\ &+ \frac{n-6}{2}grad_N(|grad_N \ln\lambda|^2) + (4-n)|grad_N \ln\lambda|^2\Big] \\ &+ \frac{n-2}{\lambda^4}\Delta_N \ln\lambda grad_N \ln\lambda + \frac{2(n-2)^2}{\lambda^2}|grad_M \ln\lambda|^2 \\ &+ \frac{2(n-2)}{\lambda^2}\Delta_M \ln\lambda grad_N\lambda.\end{aligned}$$

$$\nabla^{\pi_2}_{(grad_M f, \frac{1}{\lambda^2}grad_M f)}\tau(\pi_2) = \frac{(n-2)}{\lambda}\Big(2grad_N(\ln\lambda)f + \frac{1}{\lambda^3}\nabla^N_{grad_N f}grad_N \ln\lambda\Big),$$

from which, we get

$$\begin{aligned}\tau_{2,f}(\pi_2) = &-f(x,y)\frac{n-2}{\lambda^4}\Big[grad_N(\Delta_N \ln\lambda) + 2Ricci(grad_N)\ln\lambda \\ &+ \frac{n-6}{2}grad_N(|grad_N \ln\lambda|^2) + (4-n)|grad_N \ln\lambda|^2\Big] \\ &+ \frac{n-2}{\lambda^4}f(x,y)\Delta_N \ln\lambda grad_N \ln\lambda + \frac{2(n-2)^2}{\lambda^2}f(x,y)|grad_M \ln\lambda|^2 \\ &+ \frac{2(n-2)}{\lambda^2}f(x,y)\Delta_M \ln\lambda grad_N\lambda + \frac{(2-n)}{\lambda^2}\Delta_{M\times_\lambda N}(x,y)grad_N \ln\lambda \\ &- 2\frac{(n-2)}{\lambda}\Big(2grad_N(\ln\lambda)f + \frac{1}{\lambda^3}\nabla^N_{grad_N f}grad_N \ln\lambda\Big).\end{aligned}$$

□

4.5 Bi-f-harmonicity of the product maps with harmonic factor

Now, we will consider two types of product maps given by

$$\phi_1 : M\times_\lambda N \longrightarrow (M\times N, G), \qquad \phi_1(x,y) = (x, \psi(y)), \tag{4.16}$$

and

$$\phi_2 : M\times N \longrightarrow (M\times_\lambda N, G_\lambda), \qquad \phi_2(x,y) = (x, \psi(y)). \tag{4.17}$$

Proposition 24. *([38]) Suppose that $\psi : M \longrightarrow N$ are harmonic maps and $f \in C^\infty(M\times_\lambda N)$ is a positive function. For the product map $\phi_1 : (M\times_\lambda N, G_\lambda) \longrightarrow (M\times N, G)$, its bi-$f$-tension* *field is given by*

$$\begin{aligned}\tau_{2,f}(\phi_1) &= (\tau_{2,f}(\pi_1),0)\\ &+ f(x,y)\frac{n-2}{\lambda^4}\Big(0, J_\psi(d\psi(grad_N \ln\lambda)) - (n-6)\nabla^{\psi}_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda)\Big)\\ &+ f(x,y)\frac{2(n-2)}{\lambda^4}[\Delta_N(\ln\lambda) + \lambda^2\Delta_M(\ln\lambda) + (n-4)|grad_N \ln\lambda|^2\\ &+ (n-2)|grad_M \ln\lambda|^2]\Big(0, d\psi(grad_N \ln\lambda\Big)\\ &+ (\Delta_{M\times_\lambda N} f(x,y))\frac{2-n}{\lambda^2}(0, d\psi(grad_N \ln\lambda))\\ &+ \frac{(4-2n)}{\lambda^2}(0, \nabla^N_{d\psi(grad_M f)} d\psi(\frac{grad_M \ln\lambda}{\lambda^2})).\end{aligned}$$

Proof. Similar to the proof of [[15],Theorem 6], we have

$$\tau(\phi_1) = n(grad_M \ln\lambda, 0) + \frac{n-2}{\lambda^2}(0, d\psi(grad_N \ln\lambda)),$$

$$\begin{aligned}\tau_2(\phi_1) &= -\Big(n grad_M(\Delta(\ln(\lambda))) + 2nRicci^M(grad_M \ln\lambda, 0\Big)\\ &- \frac{n^2}{2}\Big(grad_M(|grad_M(\ln\lambda)|^2), 0\Big)\\ &+ \frac{n-2}{\lambda^4}\Big(0, J_\psi(d\psi(grad_N \ln\lambda)) - (n-6)\nabla^{\psi}_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda)\Big)\\ &+ \frac{2(n-2)}{\lambda^4}[\Delta_N(\ln\lambda) + \lambda^2\Delta_M(\ln\lambda) + (n-4)|grad_N \ln\lambda|^2\\ &+ (n-2)|grad_M \ln\lambda|^2]\Big(0, d\psi(grad_N \ln\lambda\Big). \qquad (4.18)\end{aligned}$$

Furthermore,we get

$$\begin{aligned}\nabla^{\phi_1}_{grad_{M\times_\lambda N} f}\tau(\phi_1) &= n\nabla^{\phi_1}_{(grad_M f, \frac{1}{\lambda^2} grad_N f)}(grad_M \ln\lambda, 0)\\ &+ (n-2)\nabla^{\phi_1}_{(grad_M f, \frac{1}{\lambda^2} grad_N f)}(0, d\psi(\frac{1}{\lambda^2} grad_N \ln\lambda))\\ &= n(\nabla^M_{grad_M f} grad_M \ln\lambda, 0) + \frac{(n-2)}{\lambda^2}(0, \nabla^N_{d\psi(grad_M f)} d\psi(\frac{grad_M \ln\lambda}{\lambda^2}). \qquad (4.19)\end{aligned}$$

Putting (4.18) and (4.19) together, we have

$$\begin{aligned}
\tau_{2,f}(\phi_1) &= f\tau_2(\phi_1) - (\Delta_{M\times_\lambda N} f(x,y))\tau(\phi_1) - 2\nabla^{\phi_1}_{grad_{M\times_\lambda N} f(x,y)}\tau(\phi_1) \\
&= -\Big(nf(x,y)grad_M(\Delta(\ln(\lambda))) + 2nf(x,y)Ricci^M(grad_M \ln\lambda, 0\Big) \\
&- f(x,y)\frac{n^2}{2}\Big(grad_M(|grad_M(\ln\lambda)|^2), 0\Big) \\
&+ f(x,y)\frac{n-2}{\lambda^4}\Big(0, J_\psi(d\psi(grad_N \ln\lambda)) - (n-6)\nabla^\psi_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda)\Big) \\
&+ f(x,y)\frac{2(n-2)}{\lambda^4}[\Delta_N(\ln\lambda) + \lambda^2\Delta_M(\ln\lambda) + (n-4)|grad_N \ln\lambda|^2 \\
&+ (n-2)|grad_M \ln\lambda|^2]\Big(0, d\psi(grad_N \ln\lambda\Big) - n(\Delta_{M\times_\lambda N} f(x,y))(grad_M \ln\lambda, 0) \\
&+ (\Delta_{M\times_\lambda N} f(x,y))\frac{2-n}{\lambda^2}(0, d\psi(grad_N \ln\lambda)) - 2n(\nabla^M_{grad_M f} grad_M \ln\lambda, 0) \\
&+ \frac{(4-2n)}{\lambda^2}(0, \nabla^N_{d\psi(grad_M f)} d\psi(\frac{grad_M \ln\lambda}{\lambda^2})).
\end{aligned} \tag{4.20}$$

By using proposition 24, (4.20) can reduce to

$$\begin{aligned}
\tau_{2,f}(\phi_1) &= (\tau_{2,f}(\pi_1), 0) \\
&+ f(x,y)\frac{n-2}{\lambda^4}\Big(0, J_\psi(d\psi(grad_N \ln\lambda)) - (n-6)\nabla^\psi_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda)\Big) \\
&+ f(x,y)\frac{2(n-2)}{\lambda^4}[\Delta_N(\ln\lambda) + \lambda^2\Delta_M(\ln\lambda) + (n-4)|grad_N \ln\lambda|^2 \\
&+ (n-2)|grad_M \ln\lambda|^2]\Big(0, d\psi(grad_N \ln\lambda\Big) \\
&+ (\Delta_{M\times_\lambda N} f(x,y))\frac{2-n}{\lambda^2}(0, d\psi(grad_N \ln\lambda)) \\
&+ \frac{(4-2n)}{\lambda^2}(0, \nabla^N_{d\psi(grad_M f)} d\psi(\frac{grad_M \ln\lambda}{\lambda^2})).
\end{aligned}$$

□

Corollary 12. *([38]) If N is a surface of dimension 2 $\dim N = 2$ then the bi-f-tension field of is given by $\tau_{2,f}(\phi_1) = (\tau_{2,f}(\pi_1), 0)$.*

Proposition 25. *([38]) Suppose that $\psi : M \longrightarrow N$ is a harmonic map and $f \in C^\infty(M \times N)$ is a positive function. For the product map $\phi_2 : (M \times N, G) \longrightarrow (M \times_\lambda N, G_\lambda)$, its bi-$f$-tension* *field is given by*

$$
\begin{aligned}
\tau_{2,f}(\phi_2) = \Big(& - f(x,y)e(\psi)tr_g\nabla^2 grad_M\lambda^2 - f(x,y)d\psi(\Delta_N e(\psi))grad_M\lambda^2 \\
& - f(x,y)e(\psi)(Ricci_M(grad_M\lambda^2) + \frac{f(x,y)}{2}e(\psi)^2 grad_M(|grad_M\lambda^2|^2) \\
& + f(x,y)\Big[2d\psi(grad_N e(\psi))\ln\lambda + e(\psi), d\psi(\Delta^N \ln\lambda)\Big]grad_M\lambda^2 \\
& - e(\psi)\Delta_{M\times N}f(x,y)grad_M\lambda^2 \\
& + 2grad_N f(e(\psi))grad_M\lambda^2 + 2e(\psi)\nabla^M_{grad_M f}grad_M\lambda^2 + 2grad_N\ln\lambda)(f)grad_M\lambda^2 \\
& + 2\Big[2e(\psi)^2|grad_N\ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N\ln\lambda) - |d\psi(grad_N\ln\lambda)|^2 \\
& - 2d\psi(grad_N e(\psi))\ln\lambda\Big]grad_M\lambda^2, 0\Big) \\
& + f(x,y)\Big[- 2d\psi(\Delta_N e(\psi)) \\
& - (3e(\psi)+1)d\psi(grad_N\ln\lambda)\ln\lambda + e(\psi)(2e(\psi)-1)(\lambda^2|grad_M\ln\lambda|^2) \\
& + \Big(0, -\frac{f(x,y)}{\lambda^2}|grad_M\lambda^2|^2 d\psi(grad_N e(\psi)) + e(\psi)d\psi(\Delta_N\ln\lambda) \\
& - e(\psi)|grad_N\ln\lambda|^2\Big]grad_N\ln\lambda \\
& + f(x,y)\Big[10d\psi(grad_N\ln\lambda)\ln\lambda + 2(1-2e(\psi))(\lambda^2|grad_M\ln\lambda|^2 + |grad_N\ln\lambda|^2) \\
& + e(\psi)|grad_N\ln\lambda|^2(\frac{1}{2\lambda^2} - 8 - 4\lambda)\Big]d\psi(grad_N\ln\lambda) - 2e(\psi)f(x,y)J_\psi(grad_N\ln\lambda) \\
& + 2f(x,y)J_\psi(d\psi(grad_N\ln\lambda)) + 4e(\psi)f(x,y)(e(\psi)-1)grad_N(|grad_N\ln\lambda|^2) \\
& + 2(2+e(\psi))f(x,y)\nabla^\psi_{grad_N\ln\lambda}grad_N\ln\lambda - 4f(x,y)\nabla^\psi_{grad_N e(\psi)}grad_N\ln\lambda \\
& - 4|grad_N\ln\lambda|^2 f(x,y)d\psi(grad_N e(\psi)) \\
& + f(x,y)(1-2e(\psi))d\psi(grad_N(|grad_N\ln\lambda|^2)) \\
& + \frac{f(x,y)}{\lambda^2}\Big[2e(\psi)^2|grad_N\ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N\ln\lambda) - |d\psi(grad_N\ln\lambda)|^2 \\
& - 2d\psi(grad_N e(\psi))\ln\lambda\Big]grad_N\lambda^2 + 4f(x,y)\nabla^\psi_{grad_N\ln\lambda}grad_N\ln\lambda \\
& + f(x,y)\nabla_{grad_N\ln\lambda}d\psi(grad_N\ln\lambda) + f(x,y)d\psi(grad_N\ln\lambda)(d\psi(grad_N\ln\lambda)\ln\lambda) \\
& - 4(\Delta_{M\times N}f(x,y))d\psi(grad_N\ln\lambda) - \frac{2}{\lambda^2}(\Delta_{M\times N}f(x,y))grad_N\lambda^2 \\
& + 2\frac{e(\psi)}{2\lambda^2}|grad_M\lambda^2|^2 grad_N f - 8(grad_M f)(\ln\lambda)d\psi(grad_N(\ln\lambda\circ\psi)) \\
& - 4\nabla_{grad_N f}d\psi(grad_N(\ln\lambda\circ\psi)) - 2(d\psi(grad_N(\ln\lambda\circ\psi)))(\ln\lambda)grad_N f \\
& + \frac{1}{\lambda^2}(d\psi(grad_N(\ln\lambda\circ\psi)))(f)grad_N\lambda^2 + 4e(\psi)\nabla^N_{grad_N f}grad_N\ln\lambda \\
& - 2(grad_N\ln\lambda)(f)grad_N\ln\lambda - 2|grad_N\ln\lambda|^2 grad_N + \frac{2}{\lambda^2}grad_N\lambda^2\Big).
\end{aligned}
\tag{4.21}
$$

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis on (M,g) and $\{e_j\}_{j=1}^n$ on (N,h). Then $\{(e_i,0),(0,e_j)\}$ is a local orthonormal basis on the direct product manifold $M\times N$. Since $\tau(\psi)=0$ and

$e(\psi) = \sum_{j=1}^{n} h(d\psi(e_j), d\psi(e_j))$, we have

$$\begin{aligned}
\tau(\phi_2) &= tr_G \nabla d\phi_2 \\
&= \sum_{i=1}^{m} \Big(\overline{\nabla}_{(e_i,0)}(e_i, 0) - \nabla_{(e_i,0)}(e_i, 0) \Big) \\
&+ \sum_{j=1}^{n} \Big(\overline{\nabla}_{(0,d\psi(e_j)}(0, d\psi(e_j)) - (0, d\psi(\nabla^N_{e_j} e_j) \Big) \\
&= (0, \tau(\psi)) + 2(0, d\psi(grad_N(\ln\lambda \circ \psi)) - e(\psi)(grad_M \lambda^2, \frac{1}{\lambda^2} grad_N \lambda^2) \\
&= 2(0, d\psi(grad_N(\ln\lambda \circ \psi)) - e(\psi)(grad_M \lambda^2, \frac{1}{\lambda^2} grad_N \lambda^2).
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
\nabla^{\phi_2}_{grad_{M\times N} f(x,y)} \tau(\phi_2) &= -grad_N f(e(\psi))(grad_M \lambda^2, 0) - e(\psi)(\nabla^M_{grad_M f} grad_M \lambda^2, 0) \\
&- \frac{e(\psi)}{2\lambda^2} |grad_M \lambda^2|^2 (0, grad_N f) \\
&+ 4(grad_M f)(\ln\lambda)(0, d\psi(grad_N(\ln\lambda \circ \psi))) \\
&+ 2(0, \nabla_{grad_N f} d\psi(grad_N(\ln\lambda \circ \psi))) \\
&+ (d\psi(grad_N(\ln\lambda \circ \psi)))(\ln\lambda)(0, grad_N f) \\
&- \frac{1}{2}(d\psi(grad_N(\ln\lambda \circ \psi)))(f)(grad_M \lambda^2, \frac{1}{\lambda^2} grad_N \lambda^2) \\
&- 2e(\psi)(grad_M f)(\ln\lambda)(0, grad_N \ln\lambda) - 2grad_N(e(\psi))(0, grad_N \ln\lambda) \\
&- 2e(\psi)\Big((0, \nabla^N_{grad_N f} grad_N \ln\lambda) + (grad_N \ln\lambda)(f)(0, grad_N \ln\lambda) \\
&+ |grad_N \ln\lambda|^2 (0, grad_N f) - (grad_N \ln\lambda)(f)(grad_M \lambda^2, \frac{1}{\lambda^2} grad_N \lambda^2)\Big).
\end{aligned}$$

$$\begin{aligned}
tr_G (\nabla^{\phi_2})^2 \tau(\phi_2) &= \sum_{i=1}^{m} \Big(\overline{\nabla}_{(e_i,0)} \overline{\nabla}_{(e_i,0)} - \overline{\nabla}_{\nabla_{(e_i,0)}(e_i,0)} \Big) \tau(\phi_2) \\
&+ \sum_{j=1}^{n} \Big(\overline{\nabla}_{(0,d\psi(e_j))} \overline{\nabla}_{(0,d\psi(e_j))} - \overline{\nabla}_{\nabla_{(0,d\psi(e_j)}(0,d\psi(e_j)}\Big) \tau(\phi_2) \\
&= -e(\psi)(tr_g \nabla^2 grad_M \lambda^2, 0) - d\psi(\Delta_N e(\psi))(grad_M \lambda^2, 0) \\
&- \frac{e(\psi)}{2\lambda^2} |grad_M \lambda^2|^2 (0, \tau(\psi)) - \frac{1}{\lambda^2} |grad_M \lambda^2|^2 (0, d\psi(grad_N e(\psi))) \\
&+ \frac{1}{2\lambda^2} |grad_M \lambda^2|^2 e(\psi))^2 (grad_M \lambda^2, 0) \\
&+ \Big[- 2e(\psi)\Delta^M \ln\lambda - 2e(\psi)|grad_M \lambda^2|^2 - 2d\psi(\Delta e(\psi)) \\
&- 4e(\psi) d\psi(grad_N \ln\lambda) \ln\lambda \Big] (0, grad_N \ln\lambda) - 4(0, \nabla^{\psi}_{grad_N e(\psi)} grad_N \ln\lambda) \\
&- 6e(\psi)(0, \nabla^{\psi}_{grad_N \ln\lambda} grad_N \ln\lambda) - 4(0, d\psi(grad_N e(\psi))) \\
&+ (1 - 2e(\psi))(0, d\psi(grad_N(|grad_N \ln\lambda|^2)))
\end{aligned}$$

$$
\begin{aligned}
&+ \Big[(\frac{1}{2\lambda^2} - 4\lambda^2 - 8)e(\psi)|grad_N \ln\lambda|^2 + 2\Delta_M \ln\lambda + 2|grad_M \ln\lambda|^2 \\
&+ 8d\psi(grad_N \ln\lambda)\ln\lambda + 2d\psi(\Delta_N \ln\lambda)\Big](0, d\psi(grad_N \ln\lambda)) \\
&+ \Big[2d\psi(grad_N e(\psi))\ln\lambda + e(\psi), d\psi(\Delta^N \ln\lambda)\Big](grad_M\lambda^2, 0) \\
&- 2e(\psi)(0, tr_h\nabla^2 grad_N \ln\lambda) \\
&+ \Big[2d\psi(grad_N \ln\lambda)\ln\lambda - 2e(\psi)|grad_N \ln\lambda|^2\Big](0, \tau(\psi)) \\
&+ \Big[2e(\psi)^2|grad_N \ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N \ln\lambda) + \frac{1}{2}\tau(\psi) - |d\psi(grad_N \ln\lambda)|^2 \\
&- 2d\psi(grad_N e(\psi))\ln\lambda\Big](grad_M\lambda^2, \frac{1}{\lambda^2}grad_N\lambda^2) \\
&+ 2(0, tr_h\nabla^2 d\psi(grad_N \ln\lambda) \\
&+ 4(0, \nabla^{\psi}_{grad_N \ln\lambda} grad_N \ln\lambda) + (0, \nabla_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda)) \\
&+ (0, d\psi(grad_N \ln\lambda)(d\psi(grad_N \ln\lambda)\ln\lambda)). \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
tr_G\overline{R}(\tau(\phi_2), d\phi)d\phi &= \sum_{i=1}^{m} \overline{R}(\tau(\phi_2), (e_i, 0))(e_i, 0) \\
&+ \sum_{j=1}^{n} \overline{R}(\tau(\phi_2), (0, d\psi(e_j))(0, d\psi(e_j)) \\
&= -e(\psi)(Ricci_M(grad_M\lambda^2), 0) + \frac{1}{2}e(\psi)^2(grad_M(|grad_M\lambda^2|^2), 0) \\
&- \frac{1}{2\lambda^2}|grad_M\lambda^2|^2(grad_M\lambda^2), 0) \\
&+ 2\Big[(1 - 2e(\psi))(\lambda^2|grad_M \ln\lambda|^2 + |grad_N \ln\lambda|^2) \\
&+ d\psi(grad_N \ln\lambda)\ln\lambda + h(\tau(\psi), grad_N \ln\lambda) - \Delta^M \ln\lambda \\
&- |grad_M \ln\lambda|^2 - d\psi(\Delta_N \ln\lambda)\Big](0, d\psi(grad_M \ln\lambda)) \\
&+ 2\Big[e(\psi)(\Delta^M \ln\lambda + |grad_M \ln\lambda|^2) + 2e(\psi)(e(\psi) - 1)(\lambda^2|grad_M \ln\lambda|^2) \\
&+ e(\psi)d\psi(\Delta_N \ln\lambda) + 2(e(\psi) - 1)d\psi(grad_N \ln\lambda)\ln\lambda - e(\psi)|grad_N \ln\lambda|^2 \\
&- e(\psi)(h(\tau(\psi), grad_N \ln\lambda) - d\psi(grad_N \ln\lambda)\ln\lambda)\Big](0, grad_N \ln\lambda) \\
&+ 4(e(\psi) - 1)\Big[e(\psi)(0, grad_N(|grad_N \ln\lambda|^2)) - (\nabla^{\psi}_{grad_N \ln\lambda} grad_N \ln\lambda)\Big] \\
&+ 2(0, R(d\psi(grad_N \ln\lambda), d\psi(e_j))d\psi(e_j) \\
&- 2e(\psi)(0, R(grad_N \ln\lambda, d\psi(e_j))d\psi(e_j). \qquad (4.23)
\end{aligned}
$$

Putting (4.22) and (4.23) together, we have

$$\begin{aligned}
\tau_2(\phi_2) = &-e(\psi)(tr_g\nabla^2 grad_M\lambda^2, 0) - d\psi(\Delta_N e(\psi))(grad_M\lambda^2, 0)\\
&- \frac{1}{\lambda^2}|grad_M\lambda^2|^2(0, d\psi(grad_N e(\psi))) - e(\psi)(Ricci_M(grad_M\lambda^2), 0)\\
&+ \frac{1}{2}e(\psi)^2(grad_M(|grad_M\lambda^2|^2), 0)\\
&+ \Big[-2d\psi(\Delta_N e(\psi)) - (3e(\psi)+1)d\psi(grad_N \ln\lambda)\ln\lambda\\
&+ e(\psi)(2e(\psi)-1)(\lambda^2|grad_M \ln\lambda|^2)\\
&+ e(\psi)d\psi(\Delta_N \ln\lambda) - e(\psi)|grad_N \ln\lambda|^2\Big](0, grad_N \ln\lambda)\\
&+ \Big[10d\psi(grad_N \ln\lambda)\ln\lambda + 2(1-2e(\psi))(\lambda^2|grad_M \ln\lambda|^2 + |grad_N \ln\lambda|^2)\\
&+ e(\psi)|grad_N \ln\lambda|^2(\frac{1}{2\lambda^2} - 8 - 4\lambda)\Big](0, d\psi(grad_N \ln\lambda))\\
&- 2e(\psi)(0, J_\psi(grad_N \ln\lambda)) + 2(0, J_\psi(d\psi(grad_N \ln\lambda)))\\
&+ 4e(\psi)(e(\psi)-1)(0, grad_N(|grad_N \ln\lambda|^2))\\
&+ 2(2+e(\psi))(0, \nabla^\psi_{grad_N \ln\lambda} grad_N \ln\lambda) - 4(0, \nabla^\psi_{grad_N e(\psi)} grad_N \ln\lambda)\\
&- 4|grad_N \ln\lambda|^2(0, d\psi(grad_N e(\psi))) + (1-2e(\psi))(0, d\psi(grad_N(|grad_N \ln\lambda|^2)))\\
&+ \Big[2e(\psi)^2|grad_N \ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N \ln\lambda) - |d\psi(grad_N \ln\lambda)|^2\\
&- 2d\psi(grad_N e(\psi))\ln\lambda\Big](grad_M\lambda^2, \frac{1}{\lambda^2}grad_N\lambda^2)\\
&+ \Big[2d\psi(grad_N e(\psi))\ln\lambda + e(\psi), d\psi(\Delta^N \ln\lambda)\Big](grad_M\lambda^2, 0)\\
&+ 4(0, \nabla^\psi_{grad_N \ln\lambda} grad_N \ln\lambda) + (0, \nabla_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda))\\
&+ (0, d\psi(grad_N \ln\lambda)(d\psi(grad_N \ln\lambda)\ln\lambda)).
\end{aligned}$$

$$\begin{aligned}
\tau_{2,f}(\phi_2) &= f(x,y)\tau_2(\phi_2) - (\Delta_{M\times N} f(x,y))\tau(\phi_2) - 2\nabla^{\phi_2}_{grad_{M\times N} f(x,y)}\tau(\phi_2)\\
&= \Big(-f(x,y)e(\psi)tr_g\nabla^2 grad_M\lambda^2 - f(x,y)d\psi(\Delta_N e(\psi))grad_M\lambda^2\\
&- f(x,y)e(\psi)(Ricci_M(grad_M\lambda^2) + \frac{f(x,y)}{2}e(\psi)^2 grad_M(|grad_M\lambda^2|^2)\\
&+ f(x,y)\Big[2d\psi(grad_N e(\psi))\ln\lambda + e(\psi), d\psi(\Delta^N \ln\lambda)\Big]grad_M\lambda^2\\
&- e(\psi)\Delta_{M\times N} f(x,y)grad_M\lambda^2 + 2grad_N f(e(\psi))grad_M\lambda^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2e(\psi)\nabla^{M}_{grad_M f} grad_M\lambda^2 + 2grad_N \ln\lambda)(f)grad_M\lambda^2 + 2\Big[2e(\psi)^2|grad_N \ln\lambda|^2 \\
&- \frac{1}{2}d\psi(\Delta_N \ln\lambda) - |d\psi(grad_N \ln\lambda)|^2 - 2d\psi(grad_N e(\psi))\ln\lambda\Big] grad_M\lambda^2, 0\Big) \\
&+ \Big(0, -\frac{f(x,y)}{\lambda^2}|grad_M\lambda^2|^2 d\psi(grad_N e(\psi)) + f(x,y)\Big[-2d\psi(\Delta_N e(\psi)) \\
&- (3e(\psi)+1)d\psi(grad_N \ln\lambda)\ln\lambda + e(\psi)(2e(\psi)-1)(\lambda^2|grad_M \ln\lambda|^2) \\
&+ e(\psi)d\psi(\Delta_N \ln\lambda) - e(\psi)|grad_N \ln\lambda|^2\Big] grad_N \ln\lambda \\
&+ f(x,y)\Big[10d\psi(grad_N \ln\lambda)\ln\lambda + 2(1-2e(\psi))(\lambda^2|grad_M \ln\lambda|^2 \\
&+ |grad_N \ln\lambda|^2) + e(\psi)|grad_N \ln\lambda|^2(\frac{1}{2\lambda^2} - 8 - 4\lambda)\Big] d\psi(grad_N \ln\lambda) \\
&- 2e(\psi)f(x,y)J_\psi(grad_N \ln\lambda) + 2f(x,y)J_\psi(d\psi(grad_N \ln\lambda)) \\
&+ 4e(\psi)f(x,y)(e(\psi)-1)grad_N(|grad_N \ln\lambda|^2) \\
&+ 2(2+e(\psi))f(x,y)\nabla^{\psi}_{grad_N \ln\lambda} grad_N \ln\lambda - 4f(x,y)\nabla^{\psi}_{grad_N e(\psi)} grad_N \ln\lambda \\
&- 4|grad_N \ln\lambda|^2 f(x,y)d\psi(grad_N e(\psi)) \\
&+ f(x,y)(1-2e(\psi))d\psi(grad_N(|grad_N \ln\lambda|^2)) \\
&+ \frac{f(x,y)}{\lambda^2}\Big[2e(\psi)^2|grad_N \ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N \ln\lambda) - |d\psi(grad_N \ln\lambda)|^2 \\
&- 2d\psi(grad_N e(\psi))\ln\lambda\Big] grad_N\lambda^2 + 4f(x,y)\nabla^{\psi}_{grad_N \ln\lambda} grad_N \ln\lambda \\
&+ f(x,y)\nabla_{grad_N \ln\lambda} d\psi(grad_N \ln\lambda) \\
&+ f(x,y)d\psi(grad_N \ln\lambda)(d\psi(grad_N \ln\lambda)\ln\lambda) \\
&- 4(\Delta_{M\times N} f(x,y))d\psi(grad_N \ln\lambda) - \frac{2}{\lambda^2}(\Delta_{M\times N} f(x,y))grad_N\lambda^2 \\
&+ 2\frac{e(\psi)}{2\lambda^2}|grad_M\lambda^2|^2 grad_N f - 8(grad_M f)(\ln\lambda)d\psi(grad_N(\ln\lambda\circ\psi)) \\
&- 4\nabla_{grad_N f} d\psi(grad_N(\ln\lambda\circ\psi)) - 2(d\psi(grad_N(\ln\lambda\circ\psi)))(\ln\lambda)grad_N f \\
&+ \frac{1}{\lambda^2}(d\psi(grad_N(\ln\lambda\circ\psi)))(f)grad_N\lambda^2 + 4e(\psi)\nabla^{N}_{grad_N f} grad_N \ln\lambda \\
&- 2(grad_N \ln\lambda)(f)grad_N \ln\lambda - 2|grad_N \ln\lambda|^2 grad_N + \frac{2}{\lambda^2}grad_N\lambda^2\Big).
\end{aligned}
$$

□

Corollary 13. *[38] Suppose that $\psi : M \longrightarrow N$ is a harmonic map and $f \in C^\infty(M \times N)$ is a positive function. For the product map $\phi_2 : (M \times N, G) \longrightarrow (M \times_\lambda N, G_\lambda)$ is a proper* *bi-f-harmonic if and only if the warping function λ is a non constant solution to*

$$\begin{aligned}
&- f(x,y)e(\psi)tr_g\nabla^2 grad_M\lambda^2 - f(x,y)d\psi(\Delta_N e(\psi))grad_M\lambda^2\\
&- f(x,y)e(\psi)(Ricci_M(grad_M\lambda^2) + \frac{f(x,y)}{2}e(\psi)^2 grad_M(|grad_M\lambda^2|^2)\\
&+ f(x,y)\Big[2d\psi(grad_N e(\psi))\ln\lambda\\
&+ e(\psi), d\psi(\Delta^N\ln\lambda)\Big]grad_M\lambda^2 - e(\psi)\Delta_{M\times N}f(x,y)grad_M\lambda^2\\
&+ 2grad_N f(e(\psi))grad_M\lambda^2 + 2e(\psi)\nabla^M_{grad_M f}grad_M\lambda^2 + 2grad_N\ln\lambda)(f)grad_M\lambda^2\\
&+ 2\Big[2e(\psi)^2|grad_N\ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N\ln\lambda) - |d\psi(grad_N\ln\lambda)|^2\\
&- 2d\psi(grad_N e(\psi))\ln\lambda\Big]grad_M\lambda^2 = 0.\\
&- \frac{f(x,y)}{\lambda^2}|grad_M\lambda^2|^2 d\psi(grad_N e(\psi))\\
&+ f(x,y)\Big[-2d\psi(\Delta_N e(\psi)) - (3e(\psi)+1)d\psi(grad_N\ln\lambda)\ln\lambda\\
&+ e(\psi)(2e(\psi)-1)(\lambda^2|grad_M\ln\lambda|^2)\\
&+ e(\psi)d\psi(\Delta_N\ln\lambda) - e(\psi)|grad_N\ln\lambda|^2\Big]grad_N\ln\lambda\\
&+ f(x,y)\Big[10d\psi(grad_N\ln\lambda)\ln\lambda + 2(1-2e(\psi))(\lambda^2|grad_M\ln\lambda|^2 + |grad_N\ln\lambda|^2)\\
&+ e(\psi)|grad_N\ln\lambda|^2(\frac{1}{2\lambda^2} - 8 - 4\lambda)\Big]d\psi(grad_N\ln\lambda) - 2e(\psi)f(x,y)J_\psi(grad_N\ln\lambda)\\
&+ 2f(x,y)J_\psi(d\psi(grad_N\ln\lambda)) + 4e(\psi)f(x,y)(e(\psi)-1)grad_N(|grad_N\ln\lambda|^2)\\
&+ 2(2+e(\psi))f(x,y)\nabla^\psi_{grad_N\ln\lambda}grad_N\ln\lambda - 4f(x,y)\nabla^\psi_{grad_N e(\psi)}grad_N\ln\lambda\\
&- 4|grad_N\ln\lambda|^2 f(x,y)d\psi(grad_N e(\psi)) + f(x,y)(1-2e(\psi))d\psi(grad_N(|grad_N\ln\lambda|^2))\\
&+ \frac{f(x,y)}{\lambda^2}\Big[2e(\psi)^2|grad_N\ln\lambda|^2 - \frac{1}{2}d\psi(\Delta_N\ln\lambda) - |d\psi(grad_N\ln\lambda)|^2\\
&- 2d\psi(grad_N e(\psi))\ln\lambda\Big]grad_N\lambda^2 + 4f(x,y)\nabla^\psi_{grad_N\ln\lambda}grad_N\ln\lambda\\
&+ f(x,y)\nabla_{grad_N\ln\lambda}d\psi(grad_N\ln\lambda) + f(x,y)d\psi(grad_N\ln\lambda)(d\psi(grad_N\ln\lambda)\ln\lambda)\\
&- 4(\Delta_{M\times N}f(x,y))d\psi(grad_N\ln\lambda) - \frac{2}{\lambda^2}(\Delta_{M\times N}f(x,y))grad_N\lambda^2\\
&+ 2\frac{e(\psi)}{2\lambda^2}|grad_M\lambda^2|^2 grad_N f - 8(grad_M f)(\ln\lambda)d\psi(grad_N(\ln\lambda\circ\psi))\\
&- 4\nabla_{grad_N f}d\psi(grad_N(\ln\lambda\circ\psi)) - 2(d\psi(grad_N(\ln\lambda\circ\psi)))(\ln\lambda)grad_N f\\
&+ \frac{1}{\lambda^2}(d\psi(grad_N(\ln\lambda\circ\psi)))(f)grad_N\lambda^2 + 4e(\psi)\nabla^N_{grad_N f}grad_N\ln\lambda\\
&- 2(grad_N\ln\lambda)(f)grad_N\ln\lambda - 2|grad_N\ln\lambda|^2 grad_N + \frac{2}{\lambda^2}grad_N\lambda^2 = 0.
\end{aligned}$$

CONCLUSION

Let (M, g) be an n-dimensional Riemannian manifold and T^*M be its cotangent bundle equipped with a Riemannian metric of Sasaki type which rescale the vertical part by a positive differentiable function. The main purpose of the present work is to discuss curvature properties of T^*M and on the other hand we give the definition and some geometric properties of generalized warped product manifolds and we present some new properties for bi-f-harmonic map between generalized warped product manifolds and from here we find if N is a surface of dimension 2 then the map $\phi_1 : M \times_\lambda N \longrightarrow M \times N$ is biharmonic if and only if the projection maps $\pi_1 : M \times N \longrightarrow N$ is biharmonic.

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