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By :

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Properties of the Solutions of Complex Differential and Difference Equations.

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Abstract:

This thesis investigates meromorphic solutions of linear differential and differential-difference equations in the complex plane using Nevanlinna theory. We study the growth, order, and fixed points of solutions to equations whose coefficients are entire or meromorphic functions of $[p, q]$ -order. In particular, for differential-difference equations, we analyze the growth and the zero distribution of meromorphic solutions. For differential equations, we extend previous results from the unit disc to the entire complex plane and provide generalizations for both homogeneous and non-homogeneous equations, offering a unified framework for understanding the behavior and oscillation of meromorphic solutions.

Keywords. Nevanlinna theory, Linear delay-differential equations, linear differential equations, entire functions, meromorphic functions, growth of solutions, $[p, q]$ -order.

Résumé :

Cette thèse est consacrée à l'étude des solutions méromorphes d'équations différentielles linéaires et d'équations différentielles aux différences linéaires dans le plan complexe à l'aide de la théorie de Nevanlinna. Nous étudions la croissance, l'ordre et les points fixes des solutions d'équations dont les coefficients sont des fonctions entières ou des fonctions méromorphes d'ordre $[p, q]$. En particulier, pour les équations différentielles aux différences, nous analysons la croissance et la distribution des zéros des solutions méromorphes. En ce qui concerne les équations différentielles, nous étendons les résultats précédents obtenus dans le disque unité à l'ensemble du plan complexe et proposons des généralisations pour les équations homogènes et non homogènes, offrant ainsi un cadre unifié pour comprendre le comportement et l'oscillation des solutions méromorphes.

Mots-Clés. Théorie de Nevanlinna, équations différentielles aux différences linéaires, équations différentielles linéaires, fonctions entières, fonctions méromorphes, croissance des solutions, ordre $[p, q]$.

المخلص:

تدرس هذه الأطروحة الحلول الميرومورفية للمعادلات التفاضلية والخطية التفاضلية-الفرقية في المستوى العقدي باستخدام نظرية نيفانلينا. ندرس النمو والرتبة والنقاط الثابتة للحلول للمعادلات ذات المعاملات دوال شاملة، الميرومورفية من رتبة $[p, q]$. وعلى وجه الخصوص، بالنسبة للمعادلات التفاضلية-الفرقية، نقوم بتحليل نمو وتوزيع أصفار الحلول الميرومورفية. أما بالنسبة للمعادلات التفاضلية، فنمّد النتائج السابقة من قرص الوحدة إلى كامل المستوى العقدي، ونقدّم تعميمات لكلٍ من المعادلات المتجانسة وغير المتجانسة، مما يوفر إطارًا موحدًا لفهم سلوك وتذبذب الحلول الميرومورفية.

الكلمات المفتاحية:

نظرية نيفانلينا، معادلات الفروق التفاضلية الخطية، المعادلات التفاضلية الخطية، الدوال الشاملة، الدوال الميرومورفية، نمو الحلول، الرتبة $[p, q]$.

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Introduction

In 1929, the Finnish mathematician Rolf Nevanlinna developed his celebrated theory of value distribution [51], which laid the foundation of modern Nevanlinna theory. Its origins can be traced back to the classical theorems of Sokhotskii-Casorati (1868), Weierstrass (1876), Picard (1879), Hadamard (1892), and Borel (1897), among others. These pioneering results were later unified and significantly refined within the framework of Nevanlinna theory (see [18], [20], [24], [27] [53], [62], [63], [65]). This brief historical overview illustrates the broad scope of the theory, whose importance has become increasingly evident through its applications to functional equations, differential equations, complex dynamics, and Diophantine equations.

In particular, Nevanlinna theory has been extensively applied to the study of complex differential equations and their meromorphic solutions (see [30], [34], [35]). Early contributions in this area were made by F. Nevanlinna [51], R. Nevanlinna [52] and K. Yosida [64]. However, the first systematic application of Nevanlinna theory to differential equations was carried out during the 1940s and 1950s by H. Wittich (see, for example, [59], [60], [61]), whose pioneering work laid the foundation for subsequent developments. His results have since been generalized and extended in numerous studies, particularly concerning the growth and oscillation of meromorphic solutions. These investigations established important connections between the growth of the coefficients and that of the solutions, as well as between the exponent of convergence of their zeros, a line of research initiated by Bank and Laine (see [1], [2]).

Nevanlinna theory has also been successfully extended to several settings, including the unit disc $\mathbb{D}$ (see [27], [23], [35], [55]), the punctured complex plane $\mathbb{C} \setminus \{z_0\}$, and the entire complex plane $\mathbb{C}$ (see [4], [9], [58], [38]). The latter setting, in particular, plays a central role in this thesis.

Bank and Laine were among the first mathematicians to investigate the oscillation of solutions of second-order linear differential equations with entire coefficients. In 1982, they studied in [36] the zero distribution of solutions of the differential equation

$$f'' + Af = 0,$$

where A is either a polynomial or a transcendental entire function. Later, In 1996, Kwon

[33] investigated the hyper-order of solutions of the differential equation

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z)$ are non-zero entire functions of finite order.

More recently, Li and Wang [45], as well as Cao, Xu, and Chen [12] investigated higher-order linear differential equations, while Li and Cao [44], Liu, Tu, and Shi [49] and Belaïdi and Belmiloud [9] studied the $[p, q]$ -order of solutions of equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad (1)$$

and showed that, under suitable conditions on the coefficients, the solutions may have either finite or infinite order. In recent years, remarkable progress has also been achieved in the study of complex difference equations, with many important problems being solved. A major breakthrough was the establishment of the difference analogue of the lemma on the logarithmic derivative, obtained independently by Halburd and Korhonen ([28], [29]) and Chiang and Feng [19]. Consequently, considerable attention has been devoted to the study of meromorphic solutions of complex linear difference equations of the form

$$A_n(z)f(z + c_n) + A_{n-1}(z)f(z + c_{n-1}) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0,$$

where $c_n, \dots, c_1$ are distinct non-zero complex constants. Significant results have been obtained, particularly in the case where one coefficient has maximal order or, among the coefficients of maximal order, exactly one possesses strictly larger type than the others (see, [19], [37], [38], [47], [57], [58] and [67]).

This thesis consists of an introduction and six chapters.

The first chapter provides a concise overview of the fundamental concepts and basic definitions of Nevanlinna theory that are required through the thesis.

The second chapter is devoted to the study of the growth of meromorphic solutions of the homogeneous linear difference equation

$$A_n(z)f^{(n)}(z + c_n) + \cdots + A_1(z)f'(z + c_1) + A_0(z)f(z + c_0) = 0, \quad (2)$$

where the coefficients $A_j(z)$ ($j = 0, \dots, n$) are entire functions of the same order and c_j ($j = 0, \dots, n$) are complex constants. Under suitable assumptions on the coefficients, we extend the results obtained by Lan and Chen [38].

The third chapter investigates meromorphic solutions of the homogeneous and non-

homogeneous linear delay-differential equations

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) f^{(j)}(z + c_s) = 0, \quad (3)$$

and

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) f^{(j)}(z + c_s) = F(z), \quad (4)$$

where $c_s (s = 0, \dots, n)$ are complex constants, and $A_{sj}(z) (s = 0, \dots, n, j = 0, \dots, m)$ together with $F(z)$ are entire or meromorphic functions of the same order. Under appropriate assumptions on the coefficients, we extend both the results established in Chapter 2 and those of Wu and Zheng [58].

The fourth chapter studies the growth of meromorphic solutions to the homogeneous linear differential equation (2) and the differential-difference equation (3), where the coefficients $A_j(z) (j = 0, \dots, n)$ are entire functions of the same order, while $A_{ij}(z) (i = 0, \dots, n, j = 0, \dots, m)$ are meromorphic functions of finite order. Our results improve and extend those obtained by Lan and Chen [38].

The fifth chapter investigates the growth and fixed points of solutions of the homogeneous linear differential equations (1) and

$$A_k(z) f^{(k)}(z) + A_{k-1}(z) f^{(k-1)}(z) + \dots + A_1(z) f'(z) + A_0(z) f(z) = 0,$$

where the coefficients $A_j(z) (j = 0, \dots, n)$ are entire functions of order $[p, q]$. We extend the results of Belaïdi and Belmiloud from the unit disc to the entire complex plane.

Finally, the sixth chapter generalizes and extends several results of Belaïdi and Belmiloud [9] concerning equation (1) by considering meromorphic coefficients of $[p, q]$ -order and transferring the corresponding results from the unit disc to the entire complex plane.

Chapter 1

Notations in Nevanlinna Theory of Meromorphic Functions

1 The Characteristic Function of Nevanlinna

We recall some definitions, basic notations, and main results of Nevanlinna's theory of meromorphic functions that will be need later, see ([6], [27], [34]).

1.1 Jensen's Formula

Theorem 1.1.1 ([34]) *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ denote the zeros and poles of f , respectively, where each zero and pole is counted according to its multiplicity. Then, we have*

$$\ln |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(r \exp i\varphi)| d\varphi + \sum_{|b_j| < r} \ln \frac{r}{|b_j|} - \sum_{|a_j| < r} \ln \frac{r}{|a_j|}.$$

Definition 1.1.2 ([34]) *For all real $x \geq 0$, we define*

$$\ln^+ x := \max(\ln x, 0) := \begin{cases} \ln x & \text{if } x > 1, \\ 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Lemma 1.1.3 ([6], [34]) *We have the following properties:*

- (i) $\ln x \leq \ln^+ x$, $x \geq 0$.
- (ii) $\ln^+ x \leq \ln^+ y$ ($0 \leq x \leq y$).
- (iii) $\ln x = \ln^+ x - \ln^+ \frac{1}{x}$, $x \geq 0$.
- (iv) $|\ln x| = \ln^+ x + \ln^+ \frac{1}{x}$, $x \geq 0$.

$$(v) \ln^+ \left(\prod_{j=1}^n x_j \right) \leq \sum_{j=1}^n \ln^+ x_j, \quad x_j \geq 0 (j = 0, \dots, n).$$

$$(vi) \ln^+ \left(\sum_{j=1}^n x_j \right) \leq \sum_{j=1}^n \ln^+ x_j + \ln n, \quad x_j \geq 0 (j = 0, \dots, n).$$

1.2 Counting Function

Definition 1.1.4 (function a - points, see([27], [34])) Let f be a meromorphic function. For any complex number a , we denote by $n(t, a, f)$ the number of zeros of the equation $f(z) = a$ in the disc $|z| \leq t$ (each zero being counted with its order of multiplicity). Also, let $n(t, \infty, f)$ be the number of poles of f in the disc $|z| \leq t$ (each pole being counted with its order of multiplicity). The counting function of f is defined by

$$N(r, a, f) := N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \ln r, \quad f \neq a \in \mathbb{C}$$

and

$$N(r, \infty, f) := N(r, f) = \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \ln r.$$

$N(r, a, f)$ is called the counting function or function a -points of f in the disc $|z| \leq r$.

Similarly, the distinct counting function of f is defined as:

$$\bar{N}(r, a, f) := \bar{N}\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{\bar{n}(t, a, f) - \bar{n}(0, a, f)}{t} dt + \bar{n}(0, a, f) \ln r, \quad f \neq a \in \mathbb{C}$$

and

$$\bar{N}(r, \infty, f) := \bar{N}(r, f) = \int_0^r \frac{\bar{n}(t, \infty, f) - \bar{n}(0, \infty, f)}{t} dt + \bar{n}(0, \infty, f) \ln r,$$

where $\bar{n}(t, a, f)$ denotes the number of distinct zeros of the equation $f(z) = a$ and $\bar{n}(t, \infty, f)$ denotes the number of distinct poles of f in the disc $|z| \leq t$.

Example 1.1.5 Let $f(z) = \frac{\exp\{az\}}{z}$, where $a \in \mathbb{C}^*$. Then $z = 0$ is a simple pole of f . Hence, the counting function is given by

$$\begin{aligned} N(r, f) &= \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \ln r \\ &= \int_0^r \frac{1-1}{t} dt + \ln r \\ &= \ln r. \end{aligned}$$

Lemma 1.1.6 ([6], [34]) Let f be a meromorphic function with a -points $\alpha_1, \alpha_2, \dots, \alpha_n$ such

that $0 < |\alpha_1| \leq |\alpha_2| \cdots \leq |\alpha_n| \leq r$. Then

$$\int_0^r \frac{n(t, a, f)}{t} dt = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt = \sum_{0 < |\alpha_j| \leq r} \ln \frac{r}{|\alpha_j|}, j = 1, \dots, n.$$

Proposition 1.1.7 ([34]) *Let f be a meromorphic function with the Laurent series expansion around the origin*

$$f(z) = \sum_{j=m}^{+\infty} c_j z^j, (c_m \in \mathbb{C}^*, m \in \mathbb{Z}).$$

Then

$$\ln |c_m| = \frac{1}{2\pi} \int_0^{2\pi} |f(r \exp\{i\varphi\})| d\varphi + N(r, f) - N\left(r, \frac{1}{f}\right).$$

1.3 Proximity Function

Definition 1.1.8 ([34]) *For a meromorphic function f , we define the proximity function by*

$$m(r, a, f) := m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{1}{|f(r \exp\{i\varphi\})|} d\varphi, a \neq \infty,$$

and

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(r \exp\{i\varphi\})| d\varphi.$$

Example 1.1.9 *The proximity function of Nevanlinna of the meromorphic function $f(z) = \frac{\exp az}{z}$ ($a \in \mathbb{C}^*$) is given by*

$$\begin{aligned} m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(r \exp\{i\varphi\})| d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| \frac{\exp\{ar \exp\{i\varphi\}\}}{r \exp\{i\varphi\}} \right| d\varphi \\ &= \frac{|a|r}{\pi} - \frac{\ln r}{2}. \end{aligned}$$

1.4 Characteristic Function

Definition 1.1.10 ([34]) *The characteristic function of a meromorphic function f will be defined as*

$$T(r, f) := N(r, f) + m(r, f).$$

Example 1.1.11 *From Examples 1.1.5 and 1.1.9, we have*

$$N(r, f) = \ln r, m(r, f) = \frac{|a|r}{\pi} - \frac{\ln r}{2},$$

so

$$T(r, f) = \frac{|a|r}{\pi} + \frac{\ln r}{2}.$$

1.5 First Main Theorem of Nevanlinna

Theorem 1.1.12 ([27], [34]) Let $a \in \mathbb{C}$ and let f be a meromorphic function. Suppose that $f - a$ has the Laurent expansion around the origin, of the form

$$f(z) - a = \sum_{j=n}^{+\infty} c_j z^j, c_n \neq 0, n \in \mathbb{Z}.$$

Then

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) - \ln |c_n| + \varphi(r, a)$$

where

$$|\varphi(r, a)| \leq \ln^+ |a| + \ln 2.$$

Remark 1.1.13 The first main theorem can be formulated as

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1), r \rightarrow +\infty$$

for all $a \in \mathbb{C}$.

Proposition 1.1.14 ([27]) Let $f_1, f_2, \dots, f_n$ be meromorphic functions and let $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Then, for all $r \geq 1$, we have

$$(i) \quad T\left(r, \sum_{i=1}^n f_i\right) \leq \sum_{i=1}^n T(r, f_i) + \ln n.$$

$$(ii) \quad T\left(r, \prod_{i=1}^n f_i\right) \leq \sum_{i=1}^n T(r, f_i).$$

$$(iii) \quad T(r, f^n) = nT(r, f), n \in \mathbb{N}.$$

$$(iv) \quad T(r, af) = T(r, f) + O(1), a \in \mathbb{C}^*.$$

$$(v) \quad T(r, f + a) = T(r, f) + O(1).$$

$$(vi) \quad T\left(r, \frac{af+b}{cf+d}\right) = T(r, f) + O(1), \text{ with assumption } f \neq \frac{-d}{c}.$$

Example 1.1.15 Let $f(z) = \frac{2 \exp\{z\} + 3}{5 \exp\{z\} + 7}$. Then

$$\begin{aligned} T(r, f) &= T\left(r, \frac{2 \exp\{z\} + 3}{5 \exp\{z\} + 7}\right) \\ &= T(r, \exp z) + O(1) \\ &= \frac{r}{\pi} + O(1). \end{aligned}$$

Theorem 1.1.16 ([34]) *let f be a meromorphic function and let*

$$R(z) = \frac{a_n f^n(z) + a_{n-1} f^{n-1}(z) + \cdots + a_0}{b_m f^m(z) + b_{m-1} f^{m-1}(z) + \cdots + b_0},$$

where $n, m \geq 1$ are integers, $a_0, \dots, a_n, b_0, \dots, b_m \in \mathbb{C}$, and $a_n b_m \neq 0$. Then

$$T(r, R(z)) = \max\{n, m\} T(r, f(z)) + O(1).$$

Example 1.1.17 *We compute the characteristic function of $f(z) = \tanh z$. We have*

$$\begin{aligned} f(z) = \tanh z &= \frac{\sinh(z)}{\cosh(z)} \\ &= \frac{\exp\{z\} - \exp\{-z\}}{\exp\{z\} + \exp\{-z\}} \\ &= \frac{\exp\{2z\} - 1}{\exp\{2z\} + 1}. \end{aligned}$$

Thus,

$$T(r, f) = T(r, \exp\{2z\}) + O(1) = \frac{2r}{\pi} + O(1).$$

2 Growth and value Distribution of Meromorphic Function

2.1 The Order of Meromorphic Functions:

Definition 1.2.1 ([20], [27], [63]) *The order of a meromorphic function f is defined by*

$$\rho(f) := \limsup_{r \rightarrow +\infty} \frac{\ln T(r, f)}{\ln r},$$

where $T(r, f)$ is the characteristic function of Nevanlinna.

If $f(z)$ is an entire function, then the order of f is defined as

$$\rho(f) := \limsup_{r \rightarrow +\infty} \frac{\ln \ln M(r, f)}{\ln r} = \limsup_{r \rightarrow +\infty} \frac{\ln T(r, f)}{\ln r}$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Example 1.2.2 *Let $f(z) = \frac{\exp\{az\}}{z}$ ($a \in \mathbb{C}^*$ be a meromorphic function. From Example 1.1.11, we have $T(r, f) = \frac{|a|r}{\pi} + \frac{\ln r}{2}$. Then*

$$\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\ln \left(\frac{|a|r}{\pi} + \frac{\ln r}{2} \right)}{\ln r} = 1.$$

Proposition 1.2.3 *Let f and g be two meromorphic functions. We have*

$$(i) \quad \rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

$$(ii) \quad \rho(fg) \leq \max\{\rho(f), \rho(g)\}.$$

In particular, if $\rho(g) < \rho(f)$, then

$$\rho(f + g) = \rho(fg) = \rho(f).$$

2.2 The Type of Meromorphic Functions:

Definition 1.2.4 ([20], [27]) Let f be a meromorphic function of order $(0 < \rho(f) < +\infty)$. Then, the type of f is defined by

$$\tau(f) := \limsup_{r \rightarrow +\infty} \frac{T(r, f)}{r^{\rho(f)}}.$$

If f is an entire function, then the type of f with $(0 < \rho(f) < +\infty)$ is defined as

$$\tau_M(f) := \limsup_{r \rightarrow +\infty} \frac{\ln M(r, f)}{r^{\rho(f)}}.$$

Example 1.2.5 From Example 1.2.2, the order of the meromorphic function $f(z) = \frac{\exp az}{z}$ ($a \in \mathbb{C}^*$) is equal $\rho(f) = 1$ and therefore its type is given by

$$\tau(f) = \limsup_{r \rightarrow +\infty} \frac{\left(\frac{|a|r}{\pi} + \frac{\ln r}{2}\right)}{r} = \frac{|a|}{\pi}.$$

2.3 The Exponent of Convergence:

Definition 1.2.6 ([34]) Let f be a meromorphic function and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of zeros of f such that $a_n \neq 0$ and

$$\lim_{n \rightarrow +\infty} |a_n| = +\infty.$$

The exponent of convergence of the sequence of zeros $(a_n)_{n \in \mathbb{N}}$ of f is defined by

$$\lambda(f) = \inf \left\{ \alpha > 0 : \sum_{n=1}^{+\infty} |a_n|^{-\alpha} < +\infty \right\}.$$

Example 1.2.7 Consider the entire function $f(z) = \exp z - 1$. We have

$$f(z) = 0 \iff z_n = 2n\pi, (n \in \mathbb{Z}),$$

and

$$\lim_{n \rightarrow +\infty} |z_n| = +\infty.$$

Then, the exponent of convergence of f is given by

$$\begin{aligned}\lambda(f) &= \inf \left\{ \alpha > 0 : \sum_{n=1}^{+\infty} \frac{1}{|z_n|^\alpha} < +\infty \right\} \\ &= \inf \left\{ \alpha > 0 : \sum_{n=1}^{+\infty} \frac{1}{|2n\pi|^\alpha} < +\infty \right\} \\ &= \inf (]1, +\infty[) \\ &= 1.\end{aligned}$$

Definition 1.2.8 ([34]) Let f be a meromorphic function. Then, the exponent of convergence of zeros $\lambda(f)$ of f is defined by

$$\lambda(f) := \limsup_{r \rightarrow +\infty} \frac{\ln N\left(r, \frac{1}{f}\right)}{\ln r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \leq r\}$. Similarly, the exponent of convergence of zeros distinct of f is defined by

$$\bar{\lambda}(f) := \limsup_{r \rightarrow +\infty} \frac{\ln \bar{N}\left(r, \frac{1}{f}\right)}{\ln r},$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of f of zeros distinct in $\{z : |z| \leq r\}$.

Example 1.2.9 Let $f(z) = \exp\{z\}$. Then, f has no zeros, so $N\left(r, \frac{1}{f}\right) \equiv 0$. Hence,

$$\lambda(\exp\{z\}) = \bar{\lambda}(\exp\{z\}) = 0.$$

Definition 1.2.10 ([34]) Let f be a meromorphic function. Then, the exponent of convergence of pôle of f is defined by

$$\lambda\left(\frac{1}{f}\right) := \limsup_{r \rightarrow +\infty} \frac{\ln N(r, f)}{\ln r},$$

where $N(r, f)$ is the integrated counting function of pôle of f in $\{z : |z| \leq r\}$. Similarly, the exponent of convergence of pôle distinct of f is defined by

$$\bar{\lambda}\left(\frac{1}{f}\right) := \limsup_{r \rightarrow +\infty} \frac{\ln \bar{N}(r, f)}{\ln r},$$

where $\bar{N}(r, f)$ is the integrated counting function of f of pôle distinct in $\{z : |z| \leq r\}$.

Example 1.2.11 Let $f(z) = \frac{\exp\{5z-1\}}{z}$. Then, f has a simple pôle and $N(r, f) \equiv \ln r$. Hence

$$\lambda\left(\frac{1}{f}\right) = \bar{\lambda}\left(\frac{1}{f}\right) = 0.$$

Proposition 1.2.12 ([34]) For every meromorphic function f , we have

$$\lambda(f) \leq \rho(f).$$

3 The notion of iterated p -order and $[p, q]$ -order of a meromorphic function

To generalize certain results concerning the properties of solutions of differential equations, we first introduce the notions of iterated p -order and $[p, q]$ -order for meromorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and in the complex plane $\mathbb{C}$. We recall some fundamental definitions that will be used later. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all $r > 0$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = \log_0 r = r$; $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see ([25]).

Definition 1.3.1 [11] Let $p \geq 1$ be an integer. Then, the iterated p -order of a meromorphic function f in $\mathbb{D}$ is defined by

$$\rho_p(f) = \limsup_{r \rightarrow 1^-} \frac{\log_p^+ T(r, f)}{\log\left(\frac{1}{1-r}\right)}.$$

If f is an analytic function in $\mathbb{D}$, the iterated p -order of f is defined by

$$\rho_{M,p}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log\left(\frac{1}{1-r}\right)}, \text{ where } M(r, f) = \max_{|z|=r} |f(z)|.$$

Definition 1.3.2 [3] Let $p \geq q \geq 1$ be integers. Then, the $[p, q]$ -order of a meromorphic function f in $\mathbb{D}$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_p^+ T(r, f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

If f is an analytic function in $\mathbb{D}$, the $[p, q]$ -order of f is defined by

$$\rho_{M,[p,q]}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log_q\left(\frac{1}{1-r}\right)}, \text{ where } M(r, f) = \max_{|z|=r} |f(z)|.$$

Definition 1.3.3 ([44]) Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in $\mathbb{C}$. Then, the $[p, q]$ -order of f is defined by

$$\rho_{[p,q]}(f) := \limsup_{r \rightarrow +\infty} \frac{\log_p^+ T(r, f)}{\log_q r}.$$

For an entire function f , we also define

$$\rho_{[p,q]}(f) := \limsup_{r \rightarrow +\infty} \frac{\log_{p+1}^+ M(r, f)}{\log_q r} = \limsup_{r \rightarrow +\infty} \frac{\log_p^+ T(r, f)}{\log_q r}, \text{ where } M(r, f) = \max_{|z|=r} |f(z)|.$$

Example 1.3.4 Let $f(z) = \exp(\exp 2z^3)$. Then $\rho_{[3,1]}(f) = 3$.

Remark 1.3.5 It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$ for any $p \geq q \geq 1$. By Definition 1.3.3, we have that $\rho_{[1,1]}(f) = \rho(f)$ is the classical order, $\rho_{[2,1]} = \rho_2(f)$ is the hyper-order, and $\rho_{[p,1]}(f) = \rho_p(f)$ is the iterated p -order.

Proposition 1.3.6 ([4], [5]) Let $p \geq q \geq 1$ be integers. If f_1 and f_2 are non-constant meromorphic functions of $[p, q]$ -order, then we have

- (i) $\rho_{[p,q]}(f_1) = \rho_{[p,q]}(\frac{1}{f_1})$ ($f_1 \not\equiv 0$), $\rho_{[p,q]}(af_1) = \rho_{[p,q]}(f_1)$ and $\rho_{[p,q]}(f_1 + a) = \rho_{[p,q]}(f_1)$ ($a \in \mathbb{C}^*$),
- (ii) $\rho_{[p,q]}(f_1') = \rho_{[p,q]}(f_1)$,
- (iii) $\rho_{[p,q]}(f_1 + f_2) \leq \max\{\rho_{[p,q]}(f_1), \rho_{[p,q]}(f_2)\}$,
- (iv) $\rho_{[p,q]}(f_1 f_2) \leq \max\{\rho_{[p,q]}(f_1), \rho_{[p,q]}(f_2)\}$, if $\rho_{[p,q]}(f_1) > \rho_{[p,q]}(f_2)$, then we obtain $\rho_{[p,q]}(f_1 + f_2) = \rho_{[p,q]}(f_1 f_2) = \rho_{[p,q]}(f_1)$.

3.1 The $[p, q]$ -exponent of convergence of a meromorphic function:

Definition 1.3.7 ([44]) Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function. The $[p, q]$ -exponent of convergence of the sequence of zeros of f is defined by

$$\lambda_{[p,q]}(f) := \limsup_{r \rightarrow +\infty} \frac{\log_p^+ N(r, \frac{1}{f})}{\log_q r},$$

where $N(r, \frac{1}{f})$ is the integrated counting function of zeros of f in $\{z, |z| \leq r\}$.

Similarly, the $[p, q]$ -exponent of convergence of the sequence of distinct zeros of f is defined by

$$\bar{\lambda}_{[p,q]}(f) := \limsup_{r \rightarrow +\infty} \frac{\log_p^+ \bar{N}(r, \frac{1}{f})}{\log_q r},$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \leq r\}$.

Definition 1.3.8 [43] Let $p \geq q \geq 1$ be integers and f be a meromorphic function. Then, the $[p, q]$ -exponent of convergence of the sequence of fixed points of f is defined by

$$\lambda_{[p,q]}(f - z) := \limsup_{r \rightarrow +\infty} \frac{\log_p^+ N\left(r, \frac{1}{f-z}\right)}{\log_q r}.$$

Similarly, the $[p, q]$ -exponent of convergence of the sequence of distinct fixed points of f is defined by

$$\bar{\lambda}_{[p,q]}(f - z) := \limsup_{r \rightarrow +\infty} \frac{\log_p^+ \bar{N}\left(r, \frac{1}{f-z}\right)}{\log_q r}.$$

4 The Measure and Density of Sets

4.1 Linear and Logarithmic Measure:

Definition 1.4.1 ([62]) The linear measure of a set $G \subset (0, +\infty)$ is defined as

$$m(G) := \int_0^{+\infty} \chi_G(t) dt,$$

and the logarithmic measure of a set $G \subset (1, +\infty)$ is defined by

$$lm(G) := \int_1^{+\infty} \frac{\chi_G(t)}{t} dt,$$

where $\chi_G(t)$ is the characteristic function of a set G .

Example 1.4.2 The linear measure of the set $E = [1, 6] \subset (0, +\infty)$ is

$$m(E) = \int_0^{+\infty} \chi_E(t) dt = \int_1^6 dt = 5.$$

The logarithmic measure of the set $F = [e^2, e^5] \subset (1, +\infty)$ is

$$lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt = \int_{e^2}^{e^5} \frac{dt}{t} = 3.$$

4.2 The density linear and logarithmic:

Definition 1.4.3 ([62]) The upper density of the set $E \subset (0, +\infty)$ is defined as

$$\overline{dens}(E) = \limsup_{r \rightarrow +\infty} \frac{m(E \cap [0, r))}{r}.$$

The lower density of the set $E \subset (0, +\infty)$ is defined by

$$\underline{dens}(E) = \liminf_{r \rightarrow +\infty} \frac{m(E \cap [0, r))}{r}.$$

Example 1.4.4 The upper and lower densities of the set $E = [2, 5] \subset (0, +\infty)$ are

$$\overline{dens}(E) = \limsup_{r \rightarrow +\infty} \frac{m(E \cap [0, r))}{r} = \underline{dens}(E) = \liminf_{r \rightarrow +\infty} \frac{m(E \cap [0, r))}{r} = 0.$$

Definition 1.4.5 The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log dens}(E) = \limsup_{r \rightarrow +\infty} \frac{lm(F \cap [1, r))}{\log r}.$$

The lower logarithmic density of the set $F \subset (1, +\infty)$ is defined by

$$\underline{\log dens}(E) = \liminf_{r \rightarrow +\infty} \frac{lm(F \cap [1, r))}{\log r}.$$

Proposition 1.4.6 ([4]) For all $H \subset (1, +\infty)$ the following statements hold:

- (i) If $lm(H) = \infty$, then $m(H) = \infty$.
- (ii) If $\overline{dens}(H) > 0$, then $m(H) = \infty$.
- (iii) If $\overline{\log dens}(H) > 0$, then $lm(H) = \infty$.

Chapter 2

Growth of Solutions of the Homogeneous Linear Differential–Difference Equations

1 Introduction and Statement of Main Results

In this chapter, we use the standard notations of the value distribution theory of meromorphic functions developed by Nevanlinna (see [27], [34], [62]). This theory provides a fundamental framework for studying the growth and distribution of meromorphic functions and serves as the basis for our discussion of difference equations and their solutions.

Recently, many results have been obtained for complex difference equations, motivated by the development of difference analogues of Nevanlinna theory. A key result in this direction is the difference analogue of the lemma on the logarithmic derivative, obtained independently by Halburd–Korhonen [28] and Chiang–Feng [19]. In particular, the study of linear difference equations plays an important role, especially for equations of the form

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \quad (2.1)$$

and

$$A_n(z)f(z+c_n) + \cdots + A_1(z)f(z+c_1) = 0, \quad (2.2)$$

where $n \in \mathbb{N}^+$, $A_j(z)$ ($j = 0, \dots, n$) are entire functions, and c_j ($j = 1, \dots, n$) are distinct complex numbers.

Chiang and Feng [19] investigated the growth of solutions of homogeneous linear difference equations with polynomial coefficients and established the following result.

Theorem 2.1.1 [19] *Let $A_0(z), \dots, A_n(z)$ be polynomials. If there exists an integer l ($0 \leq l \leq n$)*

such that

$$\deg(A_l) > \max_{0 \leq j \leq n, j \neq l} \{\deg(A_j)\}$$

holds, then every meromorphic solution $f(\neq 0)$ of equation (2.1) satisfies $\rho(f) \geq 1$, where $\deg(A_l)$ denotes the degree of the polynomial A_l .

In [46], Liu and Mao investigated the growth of meromorphic solutions of equation (2.1) under the condition that most of its coefficients share the same order, there by extending the earlier result of Chiang and Feng. Their main conclusion is summarized in the following theorem.

Theorem 2.1.2 [46] Let $A_j(z) = P_j(z) \exp\{h_j(z)\} (j = 0, 1, \dots, n)$, where $h_j(z) = a_{jk}z^k + \dots + a_{j0}$ are polynomials with degree $k(\geq 1)$, $P_j(z)(\neq 0)$ are entire functions of $\rho(P_j) < k$. If $a_{jk} (j = 0, 1, \dots, n)$ are distinct complex numbers, then every meromorphic solution $f(\neq 0)$ of equation (2.1) satisfies $\rho(f) \geq \max_{0 \leq j \leq n} \{\rho(A_j)\} + 1$.

By imposing certain conditions on the coefficients in the case where there is no dominant coefficient, Lan and Chen [38] studied the growth and oscillation of meromorphic solutions of a homogeneous complex linear difference equation (2.2). Under suitable conditions on the coefficients, they obtained estimates for the order of growth of meromorphic solutions and studied the relationship between the exponent of convergence of zeros and the order of growth of entire solutions of the above linear difference equation and obtained following results.

Theorem 2.1.3 [38] Let $c_j, j = 1, \dots, n$ be different constants and let

$$A_j(z) = P_j(z) \exp\{h_j(z)\} + Q_j(z), (j = 1, \dots, n),$$

where $h_j(z)$ are polynomials of degree $k \geq 1$ and $P_j(z)(\neq 0)$ and $Q_j(z)$ are entire functions whose order is lower than k . Among the leading coefficients of $h_j(z), j \in 1, \dots, n$, with the maximal modulus, there exists a term unequal to the other terms. If $f(z)(\neq 0)$ is a meromorphic solution of equation (2.2), then $\rho(f) \geq k + 1$.

Corollary 2.1.4 [38] Let k and $A_j(z), j = 1, \dots, n$ be defined as in Theorem 2.1.3, let $B_i(z), i = 1, \dots, m$ be entire functions whose order is lower than k , and let $c_j, j = 1, \dots, n + m$ be different constants. If $f(z)(\neq 0)$ is a meromorphic solution of the equation

$$B_m(z)f(z + c_{n+m}) + \dots + B_1(z)f(z + c_{n+1}) + A_n(z)f(z + c_n) + \dots + A_1(z)f(z + c_1) = 0,$$

then $\rho(f) \geq k + 1$.

The aim of this chapter is to improve and extend the results of Theorem 2.1.3 and Corollary 2.1.4 by studying the growth of meromorphic solutions of the homogeneous linear differential-difference equation

$$A_n(z)f^{(n)}(z+c_n)+\cdots+A_1(z)f'(z+c_1)+A_0(z)f(z+c_0)=0, \quad (2.3)$$

where c_j , ($j=0, \dots, n$) are distinct complex constants and $A_j(z)$ ($j=0, \dots, n$) are entire functions. We present the following two results.

Theorem 2.1.5 [39] *Let c_j , $j=0, \dots, n$ be complex constants and let*

$$A_j(z) = P_j(z) \exp\{h_j(z)\} + Q_j(z), \quad (2.4)$$

where $h_j(z) = a_{jk}z^k + a_{j,k-1}z^{k-1} + \cdots + a_{j0}$ are polynomials of degree $k \geq 1$ and $P_j(z) (\neq 0)$ and $Q_j(z)$ are entire functions whose order is lower than k . Suppose that $|a_{0k}| > \max_{1 \leq j \leq n} \{|a_{jk}|\}$. If $f(z) (\neq 0)$ is a meromorphic solution of equation (2.3), then $\rho(f) \geq k+1$.

Corollary 2.1.6 [39] *Let $k, A_j(z) (\neq 0)$, $j=0, \dots, n$ satisfy the assumptions of Theorem 2.1.5, let $B_i(z)$, $i=1, \dots, m$ be entire functions whose order is lower than k , and let c_j , $j=0, \dots, n+m$ be complex constants. If $f(z) (\neq 0)$ is a meromorphic solution of equation*

$$B_m(z)f^{(n+m)}(z+c_{n+m}) + \cdots + B_1(z)f^{(n+1)}(z+c_{n+1}) + A_n(z)f^{(n)}(z+c_n) + \cdots + A_1(z)f'(z+c_1) + A_0(z)f(z+c_0) = 0, \quad (2.5)$$

then $\rho(f) \geq k+1$.

2 Some Auxiliary Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.2.1 [13] *Suppose that $f(z)$ is a meromorphic function with $\rho(f) = \rho < +\infty$. Then, for any given $\varepsilon > 0$, one can find a set $E \in (1, +\infty)$ of finite linear measure or finite logarithmic measure such that*

$$|f(z)| \leq \exp\{r^{\rho+\varepsilon}\}$$

holds for all z satisfying $|z| = r \notin [0, 1] \cup E$ as $r \rightarrow +\infty$.

Lemma 2.2.2 [19] *Let η_1, η_2 be two arbitrary complex numbers and let $f(z)$ be a meromorphic function of finite order ρ . For any given $\varepsilon > 0$, there exists a subset $E \in (0, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, the following double inequality holds:*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left| \frac{f(z+\eta_1)}{f(z+\eta_2)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

Lemma 2.2.3 [21] *Let $f(z)$ be a transcendental meromorphic function of finite order ρ , and let $\varepsilon > 0$ be a given constant. Then, there exists a subset $E \in (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, and for all $k, j, 0 \leq j < k$, we have:*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq r^{(k-j)(\rho-1+\varepsilon)}.$$

3 Proof of Theorem

Proof of Theorem 2.1.5: Contrary to our assertion, we assume that $\rho(f) < k + 1$. Let

$$h_j(z) = a_{jk}z^k + h_j^*(z), \quad (2.6)$$

where $a_{jk} \neq 0$ are complex constants and $h_j^*(z)$ are polynomials with $\deg(h_j^*) \leq k - 1, j = 0, \dots, n$. We set

$$|a_{0k}| > |a_{jk}|, \theta_0 \neq \theta_j, \theta_j = \arg(a_{jk}) \in [0, 2\pi), 1 \leq j \leq n.$$

We now choose θ such that

$$\cos(k\theta + \theta_0) = 1. \quad (2.7)$$

Thus, by $\theta_j \neq \theta_0, 1 \leq j \leq n$, we find

$$\cos(k\theta + \theta_j) < 1, 1 \leq j \leq n. \quad (2.8)$$

Denote

$$a = |a_{0k}|, b = \max_{1 \leq j \leq n} \{|a_{jk}|\}, c = \max_{1 \leq j \leq n} \{b \cos(k\theta + \theta_j)\} < a \quad (2.9)$$

and

$$\beta = \max_{0 \leq j \leq n} \{\rho(P_j), \rho(Q_j)\} < k. \quad (2.10)$$

Clearly,

$$\rho\left(\frac{P_j}{P_0}\right) \leq \max_{1 \leq j \leq n} \{\rho(P_j), \rho(P_0)\} \leq \beta, \quad \rho\left(\frac{Q_j}{P_0}\right) \leq \max_{0 \leq j \leq n} \{\rho(P_0), \rho(Q_j)\} \leq \beta.$$

By Lemma 2.2.1, for any given ε satisfying

$$0 < 2\varepsilon < \min\{1, k + 1 - \rho, k - \beta, a - c\},$$

there is a set $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{P_j(z)}{P_0(z)} \right| \leq e^{r^{\beta+\varepsilon}}, 1 \leq j \leq n, \quad \left| \frac{Q_j(z)}{P_0(z)} \right| \leq e^{r^{\beta+\varepsilon}}, 0 \leq j \leq n. \quad (2.11)$$

By the definition of the order of entire function, for any given $\varepsilon > 0$ and all sufficiently large $z, |z| = r$, we get

$$\left| e^{-h_0^*(z)} \right| \leq e^{r^{k-1+\varepsilon}}, \left| e^{h_j^*(z)} \right| \leq e^{r^{k-1+\varepsilon}}, 1 \leq j \leq n. \quad (2.12)$$

Applying Lemma 2.2.2 and Lemma 2.2.3 to $f(z)$, we conclude that there is a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, we have for $1 \leq j \leq n$

$$\left| \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \right| = \left| \frac{f^{(j)}(z + c_j)}{f(z + c_j)} \frac{f(z + c_j)}{f(z + c_0)} \right| \leq r^{j(\rho-1+\varepsilon)} e^{r^{\rho-1+\varepsilon}}. \quad (2.13)$$

By substituting (2.4) into equation (2.3), we obtain

$$\left| -e^{a_0 k z^k} \right| \leq \left| \sum_{j=1}^n e^{-h_0^*(z)} \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \frac{P_j(z)}{P_0(z)} e^{a_{jk} z^k + h_j^*(z)} + \frac{Q_j(z)}{P_0(z)} \right| + \left| e^{-h_0^*(z)} \frac{Q_0(z)}{P_0(z)} \right|. \quad (2.14)$$

Let $z = r e^{i\theta}$, where $r \in [0, 1] \cup E_1 \cup E_2$. Substituting (2.7)-(2.9) and (2.11)-(2.13) into (2.14), we find

$$e^{ar^k} \leq \sum_{j=1}^n r^{j(\rho-1+\varepsilon)} e^{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}} \left(e^{b \cos(k\theta + \theta_j) r^k + r^{k-1+\varepsilon}} + 1 \right) + e^{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}}.$$

Thus for $0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c\}$, we obtain

$$e^{ar^k} \leq (n+1) r^{n(\rho-1+\varepsilon)} e^{(c+\varepsilon)r^k + 2r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}} \leq (n+1) r^{n(\rho-1+\varepsilon)} e^{(c+2\varepsilon)r^k}. \quad (2.15)$$

Dividing both sides of (2.15) by $(n+1) r^{n(\rho-1+\varepsilon)} e^{(c+2\varepsilon)r^k}$ and letting $r \rightarrow +\infty$, since $0 < 2\varepsilon < a-c$, we get $+\infty \leq 1$. This is a contradiction, hence $\rho(f) \geq k+1$.

Proof of Corollary 2.1.6 Assume that $\rho = \rho(f) < k+1$. By using the similar steps as in the proof of Theorem 2.1.5, we also obtain (2.6)-(2.12). By Lemma 2.2.1, there is a set $E_3 \subset (1, +\infty)$ with finite logarithmic measure such that, for any given $\varepsilon > 0$ and all z satisfying $|z| = r \notin [0, 1] \cup E_3$, we obtain

$$|B_j(z)| \leq e^{r^{\beta_1+\varepsilon}}, 1 \leq j \leq m,$$

where

$$\beta_1 = \max_{1 \leq j \leq m} \{\rho(B_j)\} < k.$$

We have

$$\gamma = \max_{1 \leq j \leq m} \{\rho(B_j(z), \rho(P_0))\} < k, \quad \rho\left(\frac{B_j(z)}{P_0(z)}\right) \leq \max_{1 \leq j \leq m} \{\rho(B_j), \rho(P_0)\}.$$

By applying Lemma 2.2.2 and Lemma 2.2.3 to $f(z)$ we conclude that there is a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we have for $1 \leq j \leq n + m$

$$\left| \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \right| = \left| \frac{f^{(j)}(z + c_j)}{f(z + c_j)} \frac{f(z + c_j)}{f(z + c_0)} \right| \leq r^{j(\rho-1+\varepsilon)} e^{r^{\rho-1+\varepsilon}}, \quad (2.16)$$

and

$$\left| \frac{B_j(z)}{P_0(z)} \right| \leq e^{r^{\gamma+\varepsilon}}, \quad n+1 \leq j \leq n+m. \quad (2.17)$$

By substituting (2.4) into (2.5), we find

$$\begin{aligned} \left| -e^{a_{0k}z^k} \right| \leq & \left| \sum_{j=1}^n e^{-h_0^*(z)} \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \frac{P_j(z)}{P_0(z)} e^{a_{jk}z^k + h_j^*(z)} + \frac{Q_j(z)}{P_0(z)} \right| \\ & + \left| \sum_{j=n+1}^{m+n} e^{-h_0^*(z)} \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \frac{B_{j-n}(z)}{P_0(z)} \right| + \left| e^{-h_0^*(z)} \frac{Q_0(z)}{P_0(z)} \right|. \end{aligned} \quad (2.18)$$

Let $z = re^{i\theta}$, where $r \notin [0, 1] \cup E_1 \cup E_2 \cup E_3 \cup E_4$. Substituting (2.7)-(2.9), (2.11)-(2.12), (2.16) and (2.17) into (2.18) we obtain

$$\begin{aligned} e^{ar^k} \leq & \sum_{j=1}^n r^{j(\rho-1+\varepsilon)} e^{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}} \left(e^{b \cos(k\theta + \theta_j)} r^k + r^{k-1+\varepsilon} + 1 \right) \\ & + \sum_{j=n+1}^{m+n} r^{j(\rho-1+\varepsilon)} e^{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}}, \end{aligned}$$

thus,

$$\begin{aligned} e^{ar^k} & \leq nr^{n(\rho-1+\varepsilon)} e^{(c+\varepsilon)r^k + 2r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}} + mr^{(m+n)(\rho-1+\varepsilon)} e^{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}} \\ & \leq nr^{n(\rho-1+\varepsilon)} e^{(c+2\varepsilon)r^k} + mr^{(m+n)(\rho-1+\varepsilon)} e^{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}}. \end{aligned} \quad (2.19)$$

Dividing both sides of (2.19) by e^{ar^k} and letting $r \rightarrow +\infty$, we obtain $1 \leq 0$ since $0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c, k-\gamma\}$. This is a contradiction, thus $\rho(f) \geq k+1$.

4 Examples

Example 2.4.1 The entire function $f(z) = e^{z^2}$ is a solution of the following differential-difference equation

$$A_2(z)f''(z+i) + A_1(z)f'(z+i) + A_0(z)f(z-4i) = 0,$$

where $A_0(z) = -(4iz^3 - 4i\pi z^2 + 2z(4\pi + i) + 2i\pi)e^{8iz+16}$,
 $A_1(z) = -[(2z^3 - z)e^i - 2iz + 1] - (2z^2 + 4iz - 1)$,
 $A_2(z) = (z^2 - i\pi)e^{-2iz+1} - (z + i)$,
 $P_0(z) = -(4iz^3 - 4i\pi z^2 + 2z(4\pi + i) + 2i\pi)$, $h_0(z) = 8iz + 16$, $Q_0(z) \equiv 0$,
 $P_1(z) = -(2z^3 - z)$, $h_1(z) = -2iz + 1$, $Q_1(z) = -(2z^2 + 4iz - 1)$,
 $P_2(z) = z^2 - i\pi$, $h_2(z) = -2iz + 1$, $Q_2(z) = -(z + i)$.
Furthermore, $\rho(P_j(z)) = 0$, $j = 0, 1, 2$ and

$$|a_{01}| = |8i| = 8 > \max\{|a_{11}|, |a_{21}|\} = \max\{|-2i|, |-2i|\} = 2.$$

Hence, the conditions of Theorem 2.1.5 are satisfied. We see that for $j = 0, 1, 2$

$$\rho(f) = 2 = \rho(A_j) + 1 = \deg(h_j) + 1 = 1 + 1 = 2.$$

Example 2.4.2 The entire function $f(z) = e^{z^2}$ is a solution of the following differential-difference equation

$$B_1(z)f''(z+2) + A_1(z)f'(z-1) + A_0(z)f(z+2) = 0,$$

where $B_1(z) = z^4 - i\pi$,
 $A_1(z) = -(4z^5 + i\pi z^2)e^{4z-2}$,
 $A_0(z) = 4z^2(4z^4 - 4z^3 + i\pi(z-1))e^{-8z-8} - 4[4z^6 + 16z^5 + 17z^4 - i\pi(4z^2 + 16z + 17)]$,
 $P_0(z) = 4z^2(4z^4 - 4z^3 + i\pi(z-1))$, $h_0(z) = -8(z-1)$, $Q_0(z) = -4[4z^6 + 16z^5 + 17z^4 - i\pi(4z^2 + 16z + 17)]$,
 $P_1(z) = -(4z^5 + i\pi z^2)$, $h_1(z) = 4z - 2$, $Q_1(z) \equiv 0$.
Moreover, $\rho(P_0) = \rho(P_1) = 0$ and $\rho(B_1) = 0 < \rho(A_1) = \rho(A_2) = 1$,

$$|a_{01}| = |-8| = 8 > |a_{11}| = |4| = 4.$$

Thus, the conditions of Corollary 2.1.6 are satisfied. We see that for $j = 0, 1$

$$\rho(f) = 2 = \rho(A_j) + 1 = \deg(h_j) + 1 = 2.$$

Chapter 3

Growth of Meromorphic Solutions to Homogeneous Complex Linear Delay Differential Equations

1 Introduction and Statement of Main Results

In this chapter, we study the growth and oscillation of meromorphic solutions of complex homogeneous linear delay-differential equations of the form

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) f^{(j)}(z + c_s) = 0, \quad (3.1)$$

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) f^{(j)}(z + c_s) = F, \quad (3.2)$$

where $c_s, s = 0, \dots, n$ are complex numbers, and $A_{sj}(z), s = 0, \dots, n, j = 0, \dots, m, F(z)$ are entire or meromorphic functions of the same order.

Historically, Naftalevič [50] was among the first to apply operator theory and the iteration method to the study of meromorphic solutions of complex differential-difference equations. Moreover, only a few studies have investigated complex differential-difference equations from the viewpoint of Nevanlinna theory. Recently, Wu-Zheng [57], Zhou-Zheng [68], and Chen-Zheng [17] have studied the growth of solutions of homogeneous and non-homogeneous linear differential-difference equations (3.1) and (3.2) in the case where only one coefficient is a dominant, and have estimated lower bounds for the order of meromorphic solutions.

The study of the properties of meromorphic solutions of complex difference equa-

tions has become an important topic in Nevanlinna's theory and its difference analogues. Several authors have examined the growth properties of meromorphic solutions of complex linear difference equations (2.2). Chiang-Feng [19] studied the growth of meromorphic solutions of homogeneous linear difference equations in the case where there exists only one coefficient with the maximal order. After that, Liu and Mao [46] investigated the growth of meromorphic solutions of difference equation when the coefficients may have the same order and the same type, and derived the following theorem.

Theorem 3.1.1 [46] *Let $A_j(z) = h_j(z) \exp\{P_j(z)\} + D_j(z)$, $j = 0, 1, \dots, k$ where $P_j(z) = a_{jn}z^n + \dots + a_{j0}$ are polynomials with degree $n(\geq 1)$, $h_j(z) (\neq 0)$, $D_j(z)$ are entire functions with order less than n . If a_{jn} , $j = 0, 1, \dots, k$ are distinct complex numbers, then every meromorphic solution $f(z) (\neq 0)$ of equation*

$$A_k(z)f(z+k) + A_{k-1}(z)f(z+k-1) + \dots + A_1(z)f(z+1) + A_0(z)f(z) = 0$$

satisfies $\rho(f) \geq \max_{0 \leq j \leq k} \{\rho(A_j)\} + 1$.

In 2015, Wu and Zheng [58] have investigated a more specific case of the coefficients $A_j(z)$, $j = 0, \dots, n$, than those considered in Theorem 3.1.1, and obtained the following result, which extends and improves the previous result for equation (2.2).

Theorem 3.1.2 ([58]) *Let $n, k \in \mathbb{N}^+$ and $A_j(z) = B_j(z) \exp\{P_j(z)\} + D_j(z) \exp\{Q_j(z)\} + R_j(z)$, $j = 0, \dots, n$, where $P_j(z) = a_{jk}z^k + \dots + a_{j0}$, $Q_j(z) = b_{jk}z^k + \dots + b_{j0}$, $j = 0, \dots, n$ are polynomials with degree k and satisfy $|a_{jk}| \geq |b_{jk}| > 0$, $B_j(z), D_j(z), R_j(z)$, $j = 0, \dots, n$, are meromorphic functions and satisfy $\max_{0 \leq j \leq n} \{\rho(B_j), \rho(D_j), \rho(R_j)\} = \sigma < k$, $A_j(z) - R_j(z) \neq 0$, $j = 0, \dots, n$. Let c_j , $j = 1, \dots, n$ be distinct non-zero complex constants, such that for all $j \neq i$, $|a_{ik}| \geq |a_{jk}|$, and*

$$\arg(a_{ik}) \neq \arg(a_{jk}), \text{ or } \arg(a_{ik}) = \arg(a_{jk}), |a_{ik}| > |a_{jk}|$$

and

$$\arg(a_{ik}) \neq \arg(b_{jk}), \text{ or } \arg(a_{ik}) = \arg(b_{jk}), |a_{ik}| > |b_{jk}|$$

hold simultaneously. Then every meromorphic solution $f(z) (\neq 0)$ of equation (2.2) satisfies $\rho(f) \geq k + 1$. Further, if $\varphi(z) (\neq 0)$ is a meromorphic function with $\rho(\varphi) < k + 1$, then for every meromorphic solution $f(z) (\neq 0)$ of equation (2.2) with $\rho_2(f) < 1$, we have $\lambda(f - \varphi) = \rho(f) \geq k + 1$.

Very recently in [38], Lan and Chen have investigated the growth properties of meromorphic solutions of equation (2.1), by imposing certain restrictions on the coefficients of the difference equations when there is no dominant coefficient and obtained the Theorem 2.1.3. By combining complex difference and complex differential equations, Liu, Laine

and Yang [47] developed new results on complex differential-difference equations, an area with important applications that is currently attracting increasing attention (see [54], [56], [57]). More recently, in [39] we improved and extended the main results of Lan and Chen [38] for the difference equation (2.1) to linear differential-difference equation (2.3) and achieved Theorem 2.1.5 and Corollary 2.1.6. In relation to Theorem 3.1.1, a natural question arises: What happens if the entire coefficients are replaced by meromorphic coefficients while preserving the restriction on the coefficients in equation (3.1). In this direction, we obtain the following results.

Theorem 3.1.3 [40] *Let $c_s, s = 0, \dots, n$ be distinct complex constants and let*

$$A_{sj}(z) = P_{sj}(z) \exp\{h_{sj}(z)\} + Q_{sj}(z), s = 0, \dots, n, j = 0, \dots, m, \quad (3.3)$$

where $h_{sj}(z) = a_{sjk}z^k + a_{sjk-1}z^{k-1} + \dots + a_{sj0}, (a_{sjk} \neq 0)$ are polynomials of degree $k \geq 1, P_{sj}(z) (\neq 0)$ and $Q_{sj}(z)$ are entire functions whose order is lower than k . Suppose that $|a_{l0k}| > \max\{|a_{sjk}|, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0)\}$. If $f(z) (\neq 0)$ is a meromorphic solution of equation (3.1), then

$$\rho(f) \geq k + 1.$$

Theorem 3.1.4 [40] *Let $k, n, m \in \mathbb{N}^+$,*

$$A_{sj}(z) = P_{sj}(z) \exp\{h_{sj}(z)\} + Q_{sj}(z), s = 0, \dots, n, j = 0, \dots, m$$

where $h_{sj}(z) = a_{sjk}z^k + a_{sjk-1}z^{k-1} + \dots + a_{sj0}, (a_{sjk} \neq 0)$, are polynomials with degree k , and $P_{sj}(z) (\neq 0), Q_{sj}(z)$ are meromorphic functions satisfying $\max\{\rho(P_{sj}), \rho(Q_{sj}), 0 \leq s \leq n, 0 \leq j \leq m\} = \sigma < k$, let $c_s, s = 0, \dots, n$ be distinct non-zero complex constants. Suppose that for $(s, j) \neq (l, 0)$ we have

$$\arg(a_{l0k}) \neq \arg(a_{sjk}), |a_{l0k}| \geq |a_{sjk}|,$$

or

$$\arg(a_{l0k}) = \arg(a_{sjk}), |a_{l0k}| > |a_{sjk}|,$$

then every meromorphic solution $f(z) (\neq 0)$ of equation (3.1) satisfies

$$\rho(f) \geq k + 1.$$

Furthermore, if $\varphi(z) (\neq 0)$ is a meromorphic function with $\rho(\varphi) < k + 1$, then for every meromorphic solution $f(z)$ of equation (3.1), we have

$$\lambda(f - \varphi) = \rho(f) \geq k + 1.$$

2 Some Auxiliary Lemmas

For the proofs of our results, we will use Lemmas 2.2.1-2.2.3, see Chapter 2, as well as the following lemmas.

Lemma 3.2.1 [10] Suppose that $P(z) = (\alpha + i\beta)z^k + \dots$ (α, β are real numbers such that $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $k(\geq 1)$, and $\omega(z)(\neq 0)$ is a meromorphic function with $\rho(\omega) < k$. Set $g(z) = \omega(z) \exp\{P(z)\}$, $z = r \exp\{i\theta\}$, $\delta(P, \theta) = \alpha \cos(n\theta) - \beta \sin(n\theta)$. Then for any given $\varepsilon > 0$, there exists a set $H_0 \subset [0, 2\pi)$ with linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_0 \cup H_1)$, there exists $r_0 = r_0(\theta, \varepsilon)(> 0)$ such that for $|z| = r > r_0$, we have

(i) If $\delta(P, \theta) > 0$, then

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^k\} < |g(r \exp i\theta)| < \exp\{(1 + \varepsilon)\delta(P, \theta)r^k\}.$$

(ii) If $\delta(P, \theta) < 0$, then

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^k\} < |g(r \exp i\theta)| < \exp\{(1 - \varepsilon)\delta(P, \theta)r^k\},$$

where $H_1 = \{\theta \in [0, 2\pi[: \delta(P, \theta) = 0\}$ is a finite set.

Remark 3.2.2 [10] Let $P(z) = a_k z^k + \dots + a_0$ ($a_k \neq 0$) be a polynomial with degree k and $z = r \exp\{i\theta\}$, we denote

$$\delta(P, \theta) = |a_k| \cos(\arg a_k + k\theta).$$

Lemma 3.2.3 [40] Let $F(z), A_{sj}(z), s = 0, \dots, n, j = 0, \dots, m$, be meromorphic functions of finite order such that $A_{nm}(z)F(z) \neq 0$ and let $c_s, s = 0, \dots, n$ be distinct non-zero complex constants. Suppose that $f(z)(\neq 0)$ is a meromorphic solution of equation (3.2). If

$$\rho(f) > \max\{\rho(F), \rho(A_{sj}), s = 0, \dots, n, j = 0, \dots, m\}, \quad (3.4)$$

then we have $\lambda(f) = \rho(f)$.

Proof. Suppose that $\lambda(f) < \rho(f)$. From Hadamard's theorem, we can write f in the form

$$f(z) = \frac{H_1(z)}{H_2(z)} \exp\{h(z)\}, \quad (3.5)$$

where $H_1, H_2(\neq 0)$ are the canonical product formed by zeros (poles) of f such that

$$\begin{aligned} \lambda(H_1) = \rho(H_1) &= \lambda(f) < \rho(f), \\ \lambda(H_2) = \rho(H_2) &= \lambda\left(\frac{1}{f}\right) < \rho(f). \end{aligned} \quad (3.6)$$

In addition, we have

$$h(z) = a_p z^p + \cdots + a_1 z + a_0,$$

is a polynomial with $\deg h(z) = p \geq 1$, where $a_p, a_{p-1}, \dots, a_0$ are complex constants such that $a_p \neq 0$. Furthermore, from (3.5) we can rewrite $f^{(j)}(z), j = 1, \dots, m$ in the following form:

$$f^{(j)}(z) = \left(\frac{H_1(z)}{H_2(z)} \exp\{h(z)\} \right)^{(j)} = \psi_j(z) \exp\{h(z)\}, \quad (3.7)$$

where $\psi_j(z) (j = 1, \dots, m)$ are meromorphic functions formed by $H_1, H_2, h(z)$ and their derivatives. Substituting (3.7) into (3.2), we obtain

$$\sum_{s=0}^n \sum_{j=0}^m \psi_j(z + c_s) \exp\{h(z + c_s)\} = F(z). \quad (3.8)$$

By dividing both sides of (3.8) by $\exp\{a_p z^p\}$, we get

$$\sum_{s=0}^n \sum_{j=0}^m \psi_j(z + c_s) \exp\{(p a_p c_s + a_{p-1}) z^{p-1} + \cdots\} = F(z) \exp\{-a_p z^p\} \quad (3.9)$$

By (3.4) and (3.6), from the left side of (3.9), we obtain

$$\begin{aligned} & \rho \left(\sum_{s=0}^n \sum_{j=0}^m \psi_j(z + c_s) \exp\{(p a_p c_s + a_{p-1}) z^{p-1} + \cdots\} \right) \\ & \leq \max_{0 \leq s \leq n, 0 \leq j \leq m} \{ \rho(A_{sj}), \rho(H_1), \rho(H_2), \rho(\exp(p a_p c_s + a_{p-1}) z^{p-1} + \cdots) \} \\ & = \deg((p a_p c_s + a_{p-1}) z^{p-1} + \cdots) = p - 1, 0 \leq s \leq n, \end{aligned}$$

and on the other hand, from the right side of (3.9), we have

$$\rho(F(z) \exp\{-a_p z^p\}) = p.$$

This is a contradiction, hence

$$\lambda(f) = \rho(f).$$

3 Proof of the Theorems:

Proof of Theorem 3.1.3: Contrary to our assertion, we assume that $\rho = \rho(f) < k + 1$. Let

$$h_{sj}(z) = a_{sjk} z^k + h_{sj}^*(z), \quad (3.10)$$

where $a_{sjk} \neq 0$ are complex constants and $h_{sj}^*(z)$ are polynomials with $\deg(h_{sj}^*) \leq k-1$, $s = 0, \dots, n$, $j = 0, \dots, m$. We set

$$|a_{l0k}| > |a_{sjk}|, \theta_{l0} \neq \theta_{sj}, \theta_{sj} = \arg(a_{sjk}) \in [0, 2\pi), 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0).$$

We now choose θ such that

$$\cos(k\theta + \theta_{l0}) = 1. \quad (3.11)$$

Thus, by $\theta_{sj} = \theta_{l0}$, $(s, j) \neq (l, 0)$, we find

$$\cos(k\theta + \theta_{sj}) < 1, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0). \quad (3.12)$$

Denote

$$a = |a_{l0k}|, b = \max_{(s,j) \neq (l,0)} \{|a_{sjk}|\}, c = \max_{(s,j) \neq (l,0)} \{b \cos(k\theta + \theta_{sj})\} < a, \quad (3.13)$$

and

$$\beta = \max_{0 \leq s \leq n, 0 \leq j \leq m} \{\rho(P_{sj}), \rho(Q_{sj})\} < k.$$

Clearly

$$\rho\left(\frac{P_{sj}}{P_{l0}}\right) \leq \max\{\rho(P_{sj}), \rho(P_{l0})\} = \beta, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0)$$

and

$$\rho\left(\frac{Q_{sj}}{P_{l0}}\right) \leq \max\{\rho(Q_{sj}), \rho(P_{l0})\} = \beta, 0 \leq s \leq n, 0 \leq j \leq m.$$

By Lemma 2.2.1, for any given ϵ , with $0 < 2\epsilon < \min\{1, k+1-\rho, k-\beta, a-c\}$, there is a set $E_1 \subset (1, +\infty)$ with a finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, we obtain

$$\left|\frac{P_{sj}(z)}{P_{l0}(z)}\right| \leq \exp\{r^{\beta+\epsilon}\}, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0), \quad (3.14)$$

$$\left|\frac{Q_{sj}(z)}{P_{l0}(z)}\right| \leq \exp\{r^{\beta+\epsilon}\}, 0 \leq s \leq n, 0 \leq j \leq m.$$

It is clear that the functions $\exp\{-h_{l0}^*(z)\}$ and $\exp\{h_{sj}^*(z)\}$, $0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0)$, are of regular order whose degree is lower than $k-1$ respectively. Thus, for all large z , $|z| = r$, we get

$$|\exp\{-h_{l0}^*(z)\}| \leq \exp\{r^{k-1+\epsilon}\}, \quad (3.15)$$

$$|\exp\{h_{sj}^*(z)\}| \leq \exp\{r^{k-1+\epsilon}\}, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0).$$

Applying Lemma 2.2.2 and Lemma 2.2.3 to f , we conclude that there is a set $E_2 \subset (1, +\infty)$

with finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$, we can write for $(s, j) \in \{0, \dots, n\} \times \{0, \dots, m\}$

$$\left| \frac{f^{(j)}(z + c_s)}{f(z + c_l)} \right| = \left| \frac{f^{(j)}(z + c_s)}{f(z + c_s)} \frac{f(z + c_s)}{f(z + c_l)} \right| \leq \begin{cases} \exp\{r^{\rho-1+\varepsilon}\} r^{j(\rho-1+\varepsilon)}, & s \neq l. \\ r^{j(\rho-1+\varepsilon)}, & s = l. \end{cases} \quad (3.16)$$

Equation (3.1) gives

$$-A_{l0}(z) = \sum_{s=0, s \neq l}^n \sum_{j=0}^m A_{sj}(z) \frac{f^{(j)}(z + c_s)}{f(z + c_l)} + \sum_{j=1}^m A_{lj}(z) \frac{f^{(j)}(z + c_l)}{f(z + c_l)}. \quad (3.17)$$

By substituting (3.3) into equation (3.17), we obtain

$$\begin{aligned} & -\exp\{a_{l0k}z^k\} \\ &= \sum_{s=0, s \neq l}^n \sum_{j=0}^m \frac{f^{(j)}(z + c_s)}{f(z + c_l)} \exp\{-h_{l0}(z)\} \left(\frac{P_{sj}(z)}{P_{l0}(z)} \exp\{a_{sjk}z^k + h_{sj}^*(z)\} + \frac{Q_{sj}(z)}{P_{l0}(z)} \right) \\ &+ \sum_{j=1}^m \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \exp\{-h_{l0}^*(z)\} \left(\frac{P_{lj}(z)}{P_{l0}(z)} \exp\{a_{ljk}z^k + h_{lj}^*(z)\} + \frac{Q_{lj}(z)}{P_{l0}(z)} \right) \\ &+ \exp\{-h_{l0}^*(z)\} \frac{Q_{l0}(z)}{P_{l0}(z)}. \end{aligned} \quad (3.18)$$

Let $z = r \exp\{i\theta\}$, where $r \notin [0, 1] \cup E_1 \cup E_2$. Substituting (3.11)-(3.16) into (3.18), we find

$$\begin{aligned} & \exp\{|a_{l0k}|r^k\} \\ &\leq \sum_{s=0, s \neq l}^n \sum_{j=0}^m r^{j(\rho-1+\varepsilon)} \exp\{r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}\} \left(\exp\{b \cos(k\theta + \theta_{sj})r^k + r^{k-1+\varepsilon}\} + 1 \right) \\ &+ \sum_{j=1}^m r^{j(\rho-1+\varepsilon)} \exp\{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}\} \left(\exp\{b \cos(k\theta + \theta_{lj})r^k + r^{k-1+\varepsilon}\} + 1 \right) + \exp\{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}\}. \end{aligned}$$

Thus

$$\begin{aligned} \exp\{ar^k\} &\leq (n+1)(m+1)r^{m(\rho-1+\varepsilon)} \exp\{cr^k + 2r^{k-1+\varepsilon} + r^{\rho-1+\varepsilon} + r^{\beta+\varepsilon}\} \\ &\leq (n+1)(m+1)r^{m(\rho-1+\varepsilon)} \exp\{(c+2\varepsilon)r^k\}. \end{aligned} \quad (3.19)$$

Dividing both sides of (3.19) by $(n+1)(m+1) \exp(c+2\varepsilon)r^k r^{m(\rho-1+\varepsilon)}$ and letting $r \rightarrow +\infty$, we get $+\infty \leq 1$. Thus, this is a contradiction, hence $\rho(f) \geq k+1$.

Proof of Theorem 3.1.4: If $\rho(f) = +\infty$, then the theorem is hold. We suppose $\rho(f) = \rho < +\infty$. Set

$$I_1 = \{(s, j) \neq (l, 0), \arg(a_{l0k}) \neq \arg(a_{sjk}), |a_{l0k}| \geq |a_{sjk}|\}.$$

$$I_2 = \{(s, j) \neq (l, 0), \arg(a_{l0k}) = \arg(a_{sjk}), |a_{l0k}| > |a_{sjk}|\}.$$

It is clear that I_1, I_2 do not intersect with each other and $I = I_1 \cup I_2 = \{0, \dots, n\} \times \{0, \dots, m\} \setminus$

$\{(l, 0)\}$. Now, we may choose $\theta_0 \in (0, 2\pi)$ such that $\cos(\arg a_{l0k} + k\theta_0) = 1$ (if $k = 1$ and $\arg(a_{l0k}) = 0$, then we replace $[0, 2\pi)$ in Lemma 3.2.1 by $[-\frac{\pi}{2}, \frac{3\pi}{2})$, if $k \geq 2$ this kind of θ_0 can always be chosen). Set $z = re^{i\theta}$, $\theta \in [0, 2\pi)$.

For $(s, j) \in I_1$, there exists sufficiently small $\varepsilon_1 > 0$ such that for all $\theta \in (\theta_0 - \varepsilon_1, \theta_0 + \varepsilon_1) \subset (0, 2\pi)$, we have

$$\cos(\arg(a_{l0k}) + k\theta) > \max_{(s,j) \in I_1} \{\cos(\arg a_{sjk} + k\theta), 0\}.$$

Since $|a_{l0k}| \geq |a_{sjk}| > 0$, $(s, j) \neq (l, 0)$, then we see that

$$\delta(h_{l0}, \theta) > \max_{(s,j) \in I_1} \{\delta(h_{sj}, \theta), 0\}.$$

For $(i, j) \in I_2$, there exists sufficiently small $\varepsilon_2 > 0$ such that for all $\theta \in (\theta_0 - \varepsilon_2, \theta_0 + \varepsilon_2) \subset (0, 2\pi)$, we have

$$\cos(\arg(a_{l0k}) + k\theta) = \cos(\arg(a_{sjk}) + k\theta) > 0, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0).$$

Since $|a_{l0k}| > |a_{sjk}|$, $(s, j) \neq (l, 0)$, we see that

$$\delta(h_{l0}, \theta) > \max_{(s,j) \in I_2} \{\delta(h_{sj}, \theta)\} > 0.$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Then for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \subset (0, 2\pi)$, we have

$$\delta(h_{l0}, \theta) = \delta_1 > \max_{(s,j) \in I} \{\delta(h_{sj}, \theta), 0\} = \delta_2. \quad (3.20)$$

By Lemma 2.2.1, for any given $\varepsilon \left(0 < \varepsilon < \min\left\{1, \frac{\rho-1-k}{2}, \frac{k-\sigma}{2}, \frac{\delta_1-\delta_2}{2\delta_1+\delta_2}\right\}\right)$, there exists a set $E_3 \subset (1, +\infty)$ with a finite linear measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $r \rightarrow +\infty$, we have

$$|P_{sj}(z)| \leq \exp\{r^{\sigma+\varepsilon}\}, |Q_{sj}(z)| \leq \exp\{r^{\sigma+\varepsilon}\}, s = 0, \dots, n, j = 0, \dots, m. \quad (3.21)$$

By Lemma 3.2.1, (3.20) and (3.21), for the above ε , there exists $E_4 \subset [0, 2\pi)$ with linear measure zero and a finite set $E_5 = \{\theta \in [0, 2\pi) : \delta(h_{l0}, \theta) = 0\}$, such that for all $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \setminus (E_4 \cup E_5)$, there exists $r_0 = r_0(\theta, \varepsilon) > 0$, such that for $r > r_0 = r_0(\theta, \varepsilon)$ and $0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0)$ we have

$$\begin{aligned} |A_{sj}(z)| &\leq |P_{sj}(z) \exp\{h_{sj}(z)\}| + |Q_{sj}(z)| \\ &\leq \exp\{(1+\varepsilon)\delta_2 r^k\} + \exp\{r^{\sigma+\varepsilon}\} \\ &\leq 2 \exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon}\}, \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} |A_{l0}(z)| &\geq |P_{l0}(z) \exp\{h_{l0}(z)\}| - |Q_{l0}(z)| \\ &\geq \exp\{(1-\varepsilon)\delta_1 r^k\} - \exp\{r^{\sigma+\varepsilon}\}. \end{aligned} \quad (3.23)$$

By applying Lemma 2.2.2 and Lemma 2.2.3 to f , for the above ε , we conclude that there is a set $E_6 \subset (1, +\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0, 1] \cup E_6$, we can write for $0 \leq s \leq n, 0 \leq j \leq m$,

$$\left| \frac{f^{(j)}(z + c_s)}{f(z + c_l)} \right| = \left| \frac{f^{(j)}(z + c_s)}{f(z + c_s)} \right| \left| \frac{f(z + c_s)}{f(z + c_l)} \right| \leq \begin{cases} \exp\{r^{\rho-1+\varepsilon}\} r^{j(\rho-1+\varepsilon)}, & s \neq l, \\ r^{j(\rho-1+\varepsilon)}, & s = l. \end{cases} \quad (3.24)$$

Equation (3.1) gives

$$|A_{l0}(z)| \leq \sum_{s=0, s \neq l}^n \sum_{j=0}^m |A_{sj}(z)| \left| \frac{f^{(j)}(z + c_s)}{f(z + c_l)} \right| + \sum_{j=0}^m |A_{lj}(z)| \left| \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \right|. \quad (3.25)$$

Let $z = r \exp\{i\theta\}$, where $\arg z = \theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \setminus (E_4 \cup E_5)$, $|z| = r \notin [0, 1] \cup E_3 \cup E_6$. Substituting (3.22)-(3.24) into (3.25), for sufficiently large r , we have

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta_1 r^k\} - \exp\{r^{\sigma+\varepsilon}\} \\ &\leq 2n(m+1)r^{m(\rho-1+\varepsilon)} \exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\} + 2mr^{m(\rho-1+\varepsilon)} \exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon}\} \\ &\leq (2n(m+1) + 2m)r^{m(\rho-1+\varepsilon)} \exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\}. \end{aligned} \quad (3.26)$$

Since $\left(0 < \varepsilon < \min\left\{1, \frac{\rho-1-k}{2}, \frac{k-\sigma}{2}, \frac{\delta_1-\delta_2}{2\delta_1+\delta_2}\right\}\right)$, then we get

$$(1+\varepsilon)\delta_2 < (1-2\varepsilon)\delta_1, \sigma + \varepsilon < k - \varepsilon,$$

and

$$\frac{\exp\{r^{\sigma+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta_1 r^k\}} \rightarrow 0, r \notin [0, 1] \cup E_3 \cup E_6, r \rightarrow +\infty.$$

Then, from (3.26) for sufficiently large r , we obtain

$$\frac{1}{2} \exp\{(1-\varepsilon)\delta_1 r^k\} \leq (2n(m+1) + 2m)r^{m(\rho-1+\varepsilon)} \exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\},$$

i.e.,

$$\exp\{(1-\varepsilon)\delta_1 r^k - (1+\varepsilon)\delta_2 r^k - r^{\sigma+\varepsilon}\} \leq 2(2n(m+1) + 2m)r^{m(\rho-1+\varepsilon)} \exp\{r^{\rho-1+\varepsilon}\},$$

which gives

$$\exp\left\{\frac{\varepsilon}{2}\delta_1 r^k\right\} \leq 2(2n(m+1) + 2m)r^{m(\rho-1+\varepsilon)} \exp\{r^{\rho-1+\varepsilon}\} \leq \exp\{r^{\rho-1+2\varepsilon}\}. \quad (3.27)$$

Thus, (3.27) implies $k \leq \rho - 1 + 2\varepsilon$, and by $\varepsilon > 0$ is arbitrary, we obtain

$$\rho(f) \geq k + 1.$$

Set $g(z) = f(z) - \varphi(z)$, by substituting g into equation (3.1), then g solve the equation

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) g^{(j)}(z + c_s) = - \sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) \varphi^{(j)}(z + c_s).$$

Since $\rho(\varphi) < k + 1$, then $\varphi(z) (\neq 0)$ does not solve (3.1), that is,

$$\sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) \varphi^{(j)}(z + c_s) \neq 0,$$

and

$$\begin{aligned} \rho\left(- \sum_{s=0}^n \sum_{j=0}^m A_{sj}(z) \varphi^{(j)}(z + c_s)\right) &\leq \max\{\rho(A_{sj}), 0 \leq s \leq n, 0 \leq j \leq m, \rho(\varphi)\} \\ &< k + 1 \leq \rho(f). \end{aligned}$$

Therefore, by Lemma 3.2.3, we obtain $\lambda(g) = \rho(g)$. Hence

$$\lambda(g) = \lambda(f - \varphi) = \rho(f).$$

4 Examples

Example 3.4.1 *The entire function $f(z) = \exp\{z^2\}$ is a solution of delay differential equation*

$$A_{12}(z)f''(z+i) + A_{02}(z)f''(z) + A_{11}(z)f'(z+i) + A_{01}(z)f'(z) + A_{10}(z)f(z+i) + A_{00}(z)f(z) = 0,$$

where

$$A_{00}(z) = -2(2z^2 + 4iz - 1) \exp\{(3+4i)z - 1\}, \quad A_{10}(z) = 2(z+i) \exp\{4iz\} - (2z^3 + 2iz^2 - 2z - 2i),$$

$$A_{01}(z) = (2z^2 + 1) \exp\{-iz + 1\} - (2\pi z^2 + \pi), \quad A_{11}(z) = -(\exp\{4iz\} - (z^2 - 1)),$$

$$A_{02}(z) = -(z \exp\{-iz + 1\} - \pi z), \quad A_{12}(z) = \exp\{(3+2i)z\},$$

In this equation, we have $\rho(P_{sj}(z)) = \rho(Q_{sj}(z)) = 0$, $(s = 0, 1, j = 0, 1, 2)$ and

$$\begin{aligned} |a_{001}| &= |3 + 4i| = 5 > \max\{|a_{sjk}|, (s, j) \neq (0, 0)\}, s = 0, 1, j = 0, 1, 2 \\ &= \max\{|-i|, |4i|, |-i|, |4i|, |3 + 2i|\} \\ &= 4. \end{aligned}$$

Thus, the conditions of Theorem 3.1.3 are verified. We see that for $s = 0, 1, j = 0, 1, 2$

$$\rho(f) = 2 = \rho(A_{sj}) + 1 = \deg(h_{sj}) + 1 = 1 + 1 = 2,$$

we have

Example 3.4.2 The entire function $f(z) = \exp\{z^2\}$ is a solution of delay differential equation

$$\begin{aligned} A_{12}(z)f''(z - 2i) + A_{02}(z)f''(z + 1 + 2i) + A_{11}(z)f'(z - 2i) + A_{01}(z)f'(z + 1 + 2i) \\ + A_{10}(z)f(z - 2i) + A_{00}(z)f(z + 1 + 2i) = 0, \end{aligned}$$

where

$$A_{00}(z) = 2 \frac{z^2 - 1}{z^2 + 2i\pi z - 1} \exp\{-3iz + 3 - 4i\} + \frac{2(z^2 - 1)}{iz^2 - 4\pi}, \quad A_{10}(z) = - \left(\frac{2z^3}{z^2 - i\pi} \exp\{(4 + 4i)z + 1\} + \frac{2iz^3}{z - 2i} \right),$$

$$A_{01}(z) = - \left(\frac{z + i\pi}{z^2 + (1 + 2i)z} \exp\{(-2 - 4i)z - 4i\} + \frac{2(z^2 - 1)}{(iz^2 - 4\pi)(z + 1 + 2i)} \right),$$

$$A_{11}(z) = - \left(\frac{z^2 - 1}{(z - 2i)(z^2 + 2i\pi z - 1)} \exp\{(2 + 5i)z + 4\} + \frac{2i}{z^2 - 4iz - 4} \right),$$

$$A_{02}(z) = \frac{z^3}{(z^2 - i\pi)(2z^2 + (4 + 8i)z + 8i - 5)} \exp\{(2 - 4i)z - 4i\} + \frac{z^2 - 1}{(iz^2 - 4\pi)(2z^2 + (4 + 8i)z + 8i - 5)},$$

$$A_{12}(z) = \frac{z + i\pi}{2z^3 - 8iz^2 - 7z} \exp\{4iz + 1\} + \frac{iz^3 + 2i}{(z - 2i)(2z^2 - 8iz - 7)}.$$

We have

$$\begin{aligned} |a_{101}| &= |4 + 4i| = 4\sqrt{2} > \max\{|a_{sjk}|, (s, j) \neq (1, 0)\} \\ &= \max\{|-3i|, |-2 - 4i|, |2 + 5i|, |2 - 4i|, |4i|\} \\ &= |2 + 5i| \\ &= \sqrt{29}, \end{aligned}$$

and

$$\arg(a_{101}) \neq \arg(a_{sjk}), (s, j) \neq (1, 0).$$

Moreover $\rho(P_{sj}(z)) = \rho(Q_{sj}(z)) = 0$, $s = 0, 1$, $j = 0, 1, 2$. Hence, the conditions of Theorem 3.1.4 are satisfied, we see that

$$\rho(f) = 2 = \rho(A_{sj}) + 1 = 1 + 1, \quad 0 \leq s \leq 1, 0 \leq j \leq 2.$$

Set $\varphi(z) = \exp\{z\} + 1$, with $\rho(\varphi) = 1 < 2$. Then

$$\lambda(f - \varphi) = \rho(f) = 2.$$

Example 3.4.3 The function $f(z) = \exp\{\exp z\}$ is a solution of the following delay differential equation:

$$A_{11}(z)f'(z+2i) + A_{01}(z)f'(z-i) + A_{10}(z)f(z+2i) + A_{00}(z)f(z-i) = 0,$$

where

$$A_{00}(z) = \frac{z}{iz^2 - 12z + 6i} \exp\{(4+4i)z\}, \quad A_{10}(z) = -\frac{2z^3 + 2iz}{4z^4 + i\pi} \exp\{3z + 2i\},$$

$$A_{01}(z) = -\frac{z}{iz^2 - 12z + 6i} \exp\{(3+4i)z + i\}, \quad A_{11}(z) = \frac{2z^3 + 2iz}{4z^4 + i\pi} \exp\{2z\},$$

We have $\rho(P_{ij}(z)) = 0$, $s = 0, 1$, $j = 0, 1$ and

$$\begin{aligned} |a_{001}| &= |4+4i| = 4\sqrt{2} > \max\{|a_{sjk}|, (s, j) \neq (0, 0)\} \\ &= \max\{|3+2i|, |3+4i|, |2|\} \\ &= 5, \end{aligned}$$

and

$$\arg(a_{001}) \neq \arg(a_{sjk}), (s, j) \neq (0, 0).$$

Hence, the conditions of Theorem 3.1.4 are satisfied, we see that

$$\rho(f) = +\infty > \rho(A_{sj}) + 1 = 1 + 1 = 2, \quad 0 \leq s \leq 1, 0 \leq j \leq 1.$$

Set $\varphi(z) = \cos(2z)$, with $\rho(\varphi) = 1 < 2$. Then

$$\lambda(f - \varphi) = \rho(f) = +\infty.$$

Example 3.4.4 The entire function $f(z) = \exp\{z^2\} - 1$ which is a solution to the following

delay-differential equation

$$A_{11}(z)f'(z+i) + A_{01}(z)f'(z+1) + A_{10}(z)f(z+i) + A_{00}(z)f(z+1) = F(z),$$

where

$$A_{00}(z) = \frac{z^3 - \pi}{z+1}, A_{01}(z) = -\frac{z^3 - \pi}{2(z+1)^2}, A_{10}(z) = -\frac{2iz^2}{z+i} \exp\{-2iz\},$$

$$A_{11}(z) = \frac{iz^2}{(z+i)^2} \exp\{-2iz\}, F(z) = \frac{2iz^2}{z+i} \exp\{-2iz\} - \frac{z^3 - \pi}{z+1},$$

and

$$\max_{0 \leq s \leq 1, 0 \leq j \leq 1} \{\rho(A_{sj}), \rho(F)\} = 1 < \rho(f) = 2.$$

Hence, the conditions of Lemma 3.2.3 are satisfied. We see that

$$\lambda(f) = \rho(f) = 2.$$

Chapter 4

Properties of Solutions of Certain Complex Delay-Differential Equations

1 Introduction and Statement of Main Results:

In this chapter, we investigate the growth properties of solutions to homogeneous linear differential–difference equations (2.3) and (3.1). Our study extends and further develops the results previously established by Lan and Chen. In the recent ten years, complex difference and difference equations have attracted increasing attention. Many results on the complex difference equations have been obtained, see, for example ([7], [8], [14], [15], [38], [37], [67]) for further details.

Indeed, the investigation of meromorphic solutions to complex linear difference equations has become an important topic in complex analysis and in the theory of difference equations. Consider the homogeneous form

$$A_n(z)f(z + c_n) + A_{n-1}(z)f(z + c_{n-1}) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0,$$

together with its non-homogeneous counterpart

$$A_n(z)f(z + c_n) + A_{n-1}(z)f(z + c_{n-1}) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = F(z),$$

where $A_j(z)$ ($j = 0, \dots, n$) and $F(z)$ are entire or meromorphic functions, and c_j ($j = 1, \dots, n$) are distinct constants.

In 2008, Chiang-Feng [19] investigated the growth of meromorphic solutions to certain homogeneous linear difference equations in the case where only one coefficient attains the maximal order. They established the following result.

Theorem 4.1.1 [19] *Let $A_0(z), \dots, A_n(z)$ be entire functions for which there exists an integer*

$l, 0 \leq l \leq n$, such that

$$\rho(A_l) > \max_{1 \leq j \leq n, j \neq l} \{\rho(A_j)\}.$$

If $f(z)$ is a meromorphic solution to the difference equation

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0,$$

then $\rho(f) \geq \rho(A_l) + 1$.

Laine and Yang [37] studied the growth of meromorphic solutions of homogeneous linear difference equation in which the leading coefficients depends on the type but not on the order, and obtained the following result.

Theorem 4.1.2 [37] *Let $c_j, j = 1, \dots, n$ be distinct complex constants. Let $A_0(z), \dots, A_n(z)$ be entire functions of finite order such that among those having the maximal order $\rho = \max\{\rho(A_j), 0 \leq j \leq n\}$, exactly one has its type strictly greater than the other. If $f(z) \not\equiv 0$ is a meromorphic solution of the equation*

$$A_n(z)f(z+c_n) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = 0,$$

then $\rho(f) \geq \rho + 1$.

In this chapter, we combine the theories of differential and difference equations to investigate the properties of solutions of the complex linear delay-differential equations (2.3) and (3.1). We extend previous results for equation (2.3) and establish the following results.

Theorem 4.1.3 [42] *Let $c_j, j = 0, \dots, n$ be distinct constants and let $A_0(z), \dots, A_n(z)$ be entire functions such that*

$$\rho(A_0) > \max_{1 \leq j \leq n} \{\rho(A_j)\}.$$

If $f(z)$ is a meromorphic solution of the equation (2.3), then

$$\rho(f) \geq \rho(A_0) + 1.$$

Theorem 4.1.4 [42] *Let $A_0(z), \dots, A_n(z)$ be entire functions of finite order such that*

$$\rho(A_0) = \max\{\rho(A_j), 0 \leq j \leq n\},$$

and

$$\tau_M(A_0) > \max\{\tau_M(A_j), 1 \leq j \leq n\}.$$

If $f(z) (\neq 0)$ is a meromorphic solution of the equation (2.3), then

$$\rho(f) \geq \rho(A_0) + 1.$$

In [38], Lan and Chen investigated the growth of meromorphic solutions of difference equations when there is no leading coefficient. They established the following result.

Theorem 4.1.5 [38] *Let $c_j, j = 1, \dots, n$ be different constants and let $A_j(z) (\neq 0), j = 1, \dots, n$ be entire functions of finite order. Suppose that $f(z)$ is a finite-order entire solution of equation*

$$A_n(z)f(z + c_n) + \dots + A_1(z)f(z + c_1) = 0,$$

such that

$$\rho(f) > \max\{\rho(A_j) : 1 \leq j \leq n\} + 1.$$

Then $f(z)$ takes every finite value d infinitely often and $\lambda(f - d) = \rho(f)$.

In this chapter, we study the complex oscillation of certain complex delay-differential equations by establishing the following results.

Theorem 4.1.6 [42] *Let $c_j, j = 1, 2$ be different non-zero constants, let $h_j(z), j = 0, 2$, be polynomials, and let $A_j(z) (\neq 0), j = 0, 1, 2$, be entire functions such that*

$$\max\{\rho(A_j), 0 \leq j \leq 2\} < \max\{\deg(h_0), \deg(h_2)\}.$$

If $f(z) (\neq 0)$ is a meromorphic solution of the equation

$$A_2(z) \exp\{h_2(z)\} f''(z + c_2) + A_1(z) f'(z + c_1) + A_0(z) \exp\{h_0(z)\} f(z) = 0, \quad (4.1)$$

then

$$\rho(f) \geq \max\{\deg h_0, \deg h_2\} + 1.$$

Theorem 4.1.7 [42] *Let $c_j, j = 0, \dots, n$, be distinct complex constants and let $A_j(z) (\neq 0), j = 0, \dots, n$, be entire functions of finite order. Suppose that $f(z)$ is a finite-order entire solution of equation (2.3) such that*

$$\rho(f) \geq \max\{\rho(A_j), j = 0, \dots, n\} + 1.$$

Then $f(z)$ takes every finite value $d \neq 0$ infinitely often and

$$\lambda(f - d) = \rho(f).$$

Theorem 4.1.8 [42] Let $c_s, s = 0, \dots, n$ be distinct complex constants, and let $A_{sj}(z) (\neq 0), s = 0, \dots, n; j = 0, \dots, m$ be meromorphic functions of finite order. If $f(z) (\neq 0)$ is entire solution of finite order of equation (3.1) such that

$$\rho(f) > \max\{\rho(A_{sj}), s = 0, \dots, n; j = 0, \dots, m\},$$

then $f(z)$ takes every finite value $d (\neq 0)$ infinitely often and

$$\lambda(f - d) = \rho(f).$$

2 Some Auxiliary Lemmas:

In this section we use Lemmas 2.2.2, 2.2.3 (see chapter 2), Lemma 3.2.1, Remark 3.2.2 (see chapter 3) as well as the following lemmas

Lemma 4.2.1 [19] Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let f be a finite order meromorphic function. Let ρ be the order of $f(z)$, then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

Lemma 4.2.2 [27] Let f be a meromorphic function and let $k \geq 1$ is an integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad r \notin E,$$

where $S(r, f) = O(\log T(r, f) + \log r)$ outside a set $E \subset (0, +\infty)$ of finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

3 Proofs of the Theorems and Corollary

Proof of Theorem 4.1.3. Suppose that $f(z)$ is a finite order meromorphic solution of equation (2.3) such that

$$\rho(f) < \rho(A_0) + 1. \quad (4.2)$$

From $\rho(A_0) > \max_{1 \leq j \leq n} \{\rho(A_j)\}$, let us choose ρ such that

$$\max_{1 \leq j \leq n} \{\rho(A_j)\} < \rho < \rho(A_0). \quad (4.3)$$

Since (4.2) and (4.3) hold, then we may choose $\varepsilon > 0$ such that

$$\rho(f) + 2\varepsilon < \rho(A_0) + 1 \text{ and } \rho + 2\varepsilon < \rho(A_0). \quad (4.4)$$

Equation (2.3) gives

$$-A_0(z) = \sum_{j=1}^n A_j(z) \frac{f^{(j)}(z + c_j)}{f(z + c_0)}. \quad (4.5)$$

By Lemma 4.2.1 and Lemma 4.2.2, for any given $\varepsilon > 0$ and for all z satisfying $|z| = r$, we have

$$\begin{aligned} m\left(r, \frac{f^{(j)}(z + c_j)}{f(z + c_0)}\right) &= m\left(r, \frac{f^{(j)}(z + c_j)}{f(z + c_j)} \frac{f(z + c_j)}{f(z + c_0)}\right) \\ &\leq m\left(r, \frac{f^{(j)}(z + c_j)}{f(z + c_j)}\right) + m\left(r, \frac{f(z + c_j)}{f(z + c_0)}\right) = O(r^{\rho(f)-1+\varepsilon}) + O(\log r). \end{aligned} \quad (4.6)$$

Then we deduce from (4.5) and (4.6) that

$$\begin{aligned} m(r, A_0) &\leq \sum_{j=1}^n m\left(r, \frac{f^{(j)}(z + c_j)}{f(z + c_0)}\right) + \sum_{j=1}^n m(r, A_j) + \ln n \\ &\leq \sum_{j=1}^n m\left(r, \frac{f^{(j)}(z + c_j)}{f(z + c_0)}\right) + \sum_{j=1}^n T(r, A_j) + \ln n \\ &\leq O(r^{\rho(f)-1+\varepsilon}) + O(\log r) + nr^{\rho+\varepsilon} + \ln n \\ &\leq o(r^{\rho(A_0)-\varepsilon}). \end{aligned}$$

This is a contradiction. Hence, $\rho(f) \geq \rho(A_0) + 1$.

Proof of Theorem 4.1.4. Suppose that $f(z) (\neq 0)$ is a finite order meromorphic solution of equation (2.3) such that $\rho(f) < \rho(A_0) + 1$ and set

$$\tau = \max_{1 \leq j \leq n} \{\tau_M(A_j)\} < \tau_M(A_0).$$

By equation (2.3), we can write

$$|A_0(z)| \leq \sum_{j=1}^n |A_j(z)| \left| \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \right|. \quad (4.7)$$

Now take $\varepsilon > 0$ small enough to satisfy $\rho(f) + 2\varepsilon < \rho(A_0) + 1$. Fix also σ such that

$$\tau < \sigma < \sigma + 3\varepsilon < \tau_M(A_0).$$

Applying Lemma 2.2.2 and Lemma 2.2.3 to $f(z)$, we conclude that there exists a set $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, we

can write for $1 \leq j \leq n$:

$$\left| \frac{f^{(j)}(z + c_j)}{f(z + c_0)} \right| = \left| \frac{f^{(j)}(z + c_j)}{f(z + c_j)} \right| \left| \frac{f(z + c_j)}{f(z + c_0)} \right| \leq r^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\}. \quad (4.8)$$

By the definition of type and substituting (4.8) into (4.7), we find

$$\begin{aligned} M(r, A_0) &\leq nr^{n(\rho(f)-1+\varepsilon)} \exp\left\{(\tau + \varepsilon) r^{\rho(A_0)} + r^{\rho(f)-1+\varepsilon}\right\} \\ &\leq O\left(\exp\left\{(\sigma + 2\varepsilon) r^{\rho(A_0)}\right\}\right), \end{aligned}$$

hence $\tau_M(A_0) \leq \sigma + 2\varepsilon$. A contradiction. Therefore, $\rho(f) \geq \rho(A_0) + 1$.

Proof of Theorem 4.1.6. Suppose that $f(z)$ is a meromorphic solution of equation (2.3), and that $f(z)$ is not identically zero.

Equation (2.3) gives

$$-A_0(z) \exp\{h_0(z)\} = A_2(z) \exp\{h_2(z)\} \frac{f''(z + c_2)}{f(z)} + A_1(z) \frac{f'(z + c_1)}{f(z)}. \quad (4.9)$$

For any given $\varepsilon > 0$ and sufficiently large r , we have

$$|A_1(z)| \leq \exp\{r^{\rho(A_1)+\varepsilon}\}. \quad (4.10)$$

By Lemma 2.2.2 and Lemma 2.2.3 to f , we conclude that there is a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$, we can write for $j = 1, 2$

$$\left| \frac{f^{(j)}(z + c_j)}{f(z)} \right| = \left| \frac{f^{(j)}(z + c_j)}{f(z + c_j)} \frac{f(z + c_j)}{f(z)} \right| \leq r^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\}. \quad (4.11)$$

Case 1: Suppose that $\deg(h_0) = n \geq \deg(h_2) = m$.

By Lemma 3.2.1, there exists a set $E_3 \subset [0, 2\pi)$ with linear measure zero and a finite set $E_4 = \{\theta \in [0, 2\pi) : \delta(h_0, \theta) = 0\}$, such that for all $\theta \in (0, 2\pi) \setminus (E_3 \cup E_4)$, there exists $r_0 = r_0(\theta, \varepsilon) > 0$, such that for $r > r_0 = r_0(\theta, \varepsilon)$, $\delta(h_0, \theta) = \delta_0 > 0$ and $\delta(h_2, \theta) = \delta_2 > 0$

$$|A_0(z) \exp\{h_0(z)\}| \geq \exp\{(1 - \varepsilon)\delta(h_0, \theta)r^n\} \quad (4.12)$$

and

$$|A_2(z) \exp\{h_2(z)\}| \leq \exp\{(1 + \varepsilon)\delta(h_2, \theta)r^m\} \quad (4.13)$$

Let $z = r \exp\{i\theta\}$, where $\arg z = \theta \in (0, 2\pi) \setminus (E_3 \cup E_4)$, $|z| = r \notin [0, 1] \cup E_1 \cup E_2$. By substituting

(4.10)-(4.13) into (4.9), we obtain

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(h_0, \theta)r^n\} &\leq \exp\{(1+\varepsilon)\delta(h_2, \theta)r^m + r^{\rho(f)-1+\varepsilon}\}r^{2(\rho(f)-1+\varepsilon)} \\ &\quad + \exp\{r^{\rho(A_0)+\varepsilon} + r^{\rho(f)-1+\varepsilon}\}r^{\rho(f)-1+\varepsilon} \\ &\leq \exp\{(1+\varepsilon)\delta(h_2, \theta)r^m + r^{\rho(f)-1+\varepsilon} + r^{\rho(A_0)+\varepsilon}\}r^{2(\rho(f)-1+\varepsilon)}. \end{aligned} \quad (4.14)$$

If $n > m$, then for $0 < \varepsilon < n - \rho(A_0)$ we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(h_0, \theta)r^n - (1+\varepsilon)\delta(h_2, \theta)r^m - r^{\rho(A_0)+\varepsilon}\} &= \exp\{(1-o(1))(1-\varepsilon)\delta(h_0, \theta)r^n\} \\ &\leq \exp\{r^{\rho(f)-1+\varepsilon}\}r^{2(\rho(f)-1+\varepsilon)}, \end{aligned} \quad (4.15)$$

which implies $n \leq \rho(f) - 1 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\rho(f) \geq n + 1.$$

If $n = m$ and for any given $0 < \varepsilon < \min\{n - \rho(A_0), \frac{\delta_0 - \delta_2}{\delta_0 + \delta_2}\}$, the inequality (4.14) again implies $n \leq \rho(f) - 1 + \varepsilon$, and by $\varepsilon > 0$ is arbitrary, we obtain

$$\rho(f) \geq n + 1.$$

Case 2: Suppose that $\deg(h_0) = n < \deg(h_2) = m$.

By Lemma 3.2.1, there exists a set $E_5 \subset [0, 2\pi)$ with linear measure zero and a finite set $E_6 = \{\theta \in [0, 2\pi) : \delta(h_0, \theta) = 0\}$, such that for all $\theta \in (0, 2\pi) \setminus (E_5 \cup E_6)$, there exists $r_0 = r_0(\theta, \varepsilon) > 0$, such that for $r > r_0 = r_0(\theta, \varepsilon)$, $\delta(h_0, \theta) > 0$ and $\delta(h_2, \theta) < 0$ we have (4.12) and

$$|A_2(z) \exp\{h_2(z)\}| \leq \exp\{(1-\varepsilon)\delta(h_2, \theta)r^m\}. \quad (4.16)$$

Let $z = r \exp\{i\theta\}$, where $\arg z = \theta \in (0, 2\pi) \setminus (E_5 \cup E_6)$, $|z| = r \notin [0, 1] \cup E_1 \cup E_2$. By substituting (4.10)-(4.12) and (4.16) into (4.9), we obtain for $0 < \varepsilon < m - \rho(A_0)$

$$\begin{aligned} \exp\{-(1-\varepsilon)\delta(h_2, \theta)r^m + (1-\varepsilon)\delta(h_0, \theta)r^n - r^{\rho(A_0)+\varepsilon}\} &= \exp\{-(1-o(1))(1-\varepsilon)\delta(h_2, \theta)r^m\} \\ &\leq \exp\{r^{\rho(f)-1+\varepsilon}\}r^{2(\rho(f)-1+\varepsilon)}, \end{aligned}$$

which implies $m \leq \rho(f) - 1 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\rho(f) \geq m + 1.$$

Proof of Theorem 4.1.7. Suppose that $\lambda(f - d) < \rho(f)$. Then $f(z)$ can be represented

as

$$f(z) = H(z) \exp\{h(z)\} + d, \quad (4.17)$$

where $H(z) (\neq 0)$ is the canonical product (or polynomial) formed by the roots of $f(z)$ such that

$$\rho(H) = \lambda(H) = \lambda(f - d) < \rho(f), \quad (4.18)$$

and

$$h(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0, \quad (4.19)$$

is a polynomial with $\deg\{h(z)\} = k = \rho(f) \geq 1$, where $a_k (\neq 0)$, a_j , $(j = 0, \dots, k-1)$ are constants. So from (4.17), we can write $f^{(j)}(z) (j = 1, \dots, n)$ in the following form

$$f^{(j)}(z) = (H(z) \exp\{h(z)\})^{(j)} = \phi_j(z) \exp\{h(z)\}, \quad (4.20)$$

where $\phi_j(z)$ are entire functions formed by $H(z)$, $h(z)$ and their derivatives. Substituting (4.17) and (4.20) into (2.3), we obtain

$$\begin{aligned} & A_n(z) \phi_n(z + c_n) \exp\{h(z + c_n)\} + \cdots + A_1(z) \phi_1(z + c_1) \exp\{h(z + c_1)\} \\ & + A_0(z) H(z + c_0) \exp\{h(z + c_0)\} + d A_0(z) = 0, \end{aligned}$$

that is,

$$\begin{aligned} & A_n(z) \phi_n(z + c_n) \exp\{a_k z^k + (k a_k c_n + a_{k-1}) z^{k-1} + \cdots\} \\ & + \cdots + A_1(z) \phi_1(z + c_1) \exp\{a_k z^k + (k a_k c_1 + a_{k-1}) z^{k-1} + \cdots\} \\ & + A_0(z) H(z + c_0) \exp\{a_k z^k + (k a_k c_0 + a_{k-1}) z^{k-1} + \cdots\} = -d A_0(z), \end{aligned} \quad (4.21)$$

dividing both sides of (4.21) by $\exp\{a_k z^k\}$, we find

$$\begin{aligned} & A_n(z) \phi_n(z + c_n) \exp\{(k a_k c_n + a_{k-1}) z^{k-1} + \cdots\} \\ & + \cdots + A_0(z) H(z + c_0) \exp\{(k a_k c_0 + a_{k-1}) z^{k-1} + \cdots\} = -d A_0(z) \exp\{-a_k z^k\}. \end{aligned} \quad (4.22)$$

We are comparing the order of two sides of equation (4.22). The order of the left-hand side of (4.22) is

$$\begin{aligned} & \rho \left(\sum_{j=0}^n A_j(z) \phi_j(z + c_j) \exp\{(k a_k c_j + a_{k-1}) z^{k-1} + \cdots\} \right) \\ & \leq \max_{0 \leq j \leq n} \left\{ \rho(A_j), \rho(H), \rho \left(\exp\{(k a_k c_j + a_{k-1}) z^{k-1} + \cdots\} \right) \right\} \\ & = \deg \left((k a_k c_j + a_{k-1}) z^{k-1} + \cdots \right) = k-1, \quad j = 0, \dots, n, \end{aligned}$$

while the order of the right-hand side of (4.22) is equal

$$\rho\left(-dA_0(z)\exp\{-a_k z^k\}\right) = k.$$

Thus a contradiction arises. Hence

$$\lambda(f - d) = \rho(f).$$

Proof of Theorem 4.1.8. Suppose that $\lambda(f - d) < \rho(f)$. By using the same steps as in the proof of Theorem 4.1.7, we get (4.17)-(4.19). Substituting (4.17) and (4.20) into (3.1), we obtain

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=1}^m A_{ij}(z) \phi_j(z + c_i) \exp\{a_k z^k + (ka_k c_i + a_{k-1})z^{k-1} + \dots\} \\ & + \sum_{i=0}^n A_{i0}(z) H(z + c_i) \exp\{a_k z^k + (ka_k c_i + a_{k-1})z^{k-1} + \dots\} = -d \sum_{i=0}^n A_{i0}(z). \end{aligned} \quad (4.23)$$

Dividing both sides of (4.23) by $\exp\{a_k z^k\}$, we have

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=1}^m A_{ij}(z) \phi_j(z + c_i) \exp\{(ka_k c_i + a_{k-1})z^{k-1} + \dots\} \\ & + \sum_{i=0}^n A_{i0}(z) H(z + c_i) \exp\{(ka_k c_i + a_{k-1})z^{k-1} + \dots\} \\ & = -d \exp\{-a_k z^k\} \sum_{i=0}^n A_{i0}(z). \end{aligned} \quad (4.24)$$

We are comparing the order of two sides of equation (4.24). The order of the left-hand side of (4.24) is

$$\begin{aligned} & \rho\left(\sum_{i=0}^n \sum_{j=1}^m A_{ij}(z) \phi_j(z + c_i) e^{(ka_k c_i + a_{k-1})z^{k-1} + \dots} \right. \\ & \left. + \sum_{i=0}^n A_{i0}(z) H(z + c_i) e^{(ka_k c_i + a_{k-1})z^{k-1} + \dots} \right) \\ & \leq \max_{0 \leq i \leq n, 0 \leq j \leq m} \{\rho(A_{ij}), \rho(H), \rho(e^{(ka_k c_i + a_{k-1})z^{k-1} + \dots})\} \\ & = \deg\left((ka_k c_i + a_{k-1})z^{k-1} + \dots + a_0\right) = k - 1, 0 \leq i \leq n, \end{aligned}$$

while the order of the right-hand side of (4.24) is equal

$$\rho\left(-d \exp\{-a_k z^k\} \sum_{i=0}^n A_{i0}(z)\right) = k.$$

Thus a contradiction arises. Hence

$$\lambda(f - d) = \rho(f).$$

4 Examples:

Example 4.4.1 *The entire function $f(z) = \exp\{z^2\}$ is a solution of the following complex delay-differential equation*

$$A_1(z)f'(z+1) + A_0(z)f(z+i) = 0,$$

where

$$A_0(z) = -2(z^3 + (1-i)z^2 + (1-i)z + 1) \exp\{2(1-i)z + 2\},$$

$$A_1(z) = z^2 - iz + 1.$$

We note that $\rho(A_0) = 1 > \rho(A_1) = 0$. Hence, the conditions of Theorem 4.1.3 are verified. We see that $\rho(f) = 2 = \rho(A_0) + 1$.

Example 4.4.2 *The entire function $f(z) = \exp\{z^3\}$ is a solution of complex delay-differential equation*

$$A_1(z)f'(z) + A_0(z)f(z+2i) = 0,$$

where

$$A_0(z) = -3(z^4 + 4iz^2) \exp\{-6iz^2 + 8z\},$$

$$A_1(z) = (z^2 + 4i) \exp\{-4z - 8i\}.$$

We have

$$\max\{\rho(A_j), j = 0, 1\} = \rho(A_0) = 2,$$

and

$$\tau_M(A_1) = 4 < \tau_M(A_0) = 6.$$

Hence, the conditions of Theorem 4.1.4 are satisfied. Moreover, we see that

$$\rho(f) = \rho(A_0) + 1 = 3.$$

Example 4.4.3 *The entire function $f(z) = \exp\{z^3\}$ satisfies the complex delay-differential equation*

$$A_2(z) \exp\{h_2(z)\} f''(z-i) + A_1(z) f'(z+1+i) + A_0 \exp\{h_0(z)\} f(z) = 0,$$

where

$$A_0(z) = -(2z^5 - 2(1 + 7i)z^4 - 2(9 + i)z^3 + 2(1 + 6i)z^2 + (3 - 2i)z),$$

$$h_0(z) = 3(1 + i)z^2 - 2(1 - i), \quad A_1(z) = -\frac{1}{3}z^3 \exp\{-6iz\},$$

$$A_2(z) = \frac{1}{3}z \exp\{3z\}, \quad h_2(z) = 3(1 + 2i)z^2 - 2 + i,$$

and we have

$$\max\{\rho(A_j), j = 0, 1, 2\} = 1 < \deg(h_0(z)) = \deg(h_2(z)) = 2.$$

Hence, the conditions of Theorem 4.1.6 are satisfied. We see that

$$\rho(f) = \deg(h_0(z)) + 1 = \deg(h_2(z)) + 1 = 3.$$

Example 4.4.4 The entire function $f(z) = \exp\{z^3\}$ is a solution to differential-difference equation

$$A_2(z)f''(z) + A_1(z)f'(z - i) + A_0(z)f(z + 3i) = 0,$$

where

$$A_0(z) = -3(3z^6 - z^5 - (3\pi - 2i)z^4 + 3z^3 - 2iz^2 - 2(1 + \pi)z + 2i) \exp\{-9iz^2 + 27z + 27i\},$$

$$A_1(z) = -(z^3 + 2i) \exp\{3iz^2 + 3z - i\}, \quad A_2(z) = z^2 - \pi.$$

We have

$$\max\{\rho(A_0), \rho(A_1), \rho(A_2)\} = 2 = \rho(f) - 1.$$

Hence, the conditions of Theorem 4.1.7 are satisfied. Then, for every finite value $d \neq 0$, we have

$$\lambda(f - d) = \rho(f) = 3.$$

Example 4.4.5 The entire function $f(z) = \exp\{z^2\}$ is a solution of homogeneous complex delay-differential equation

$$A_{11}(z)f'(z + 2i) + A_{01}(z)f'(z) + A_{10}(z)f(z + 2i) + A_{00}(z)f(z) = 0,$$

where

$$A_{00}(z) = -\frac{2z^2}{z^3 - i\pi}, A_{10}(z) = -\frac{z^2 + \pi}{z + 2i} \exp\{-2iz + 1\},$$

$$A_{01}(z) = \frac{z}{z^3 - i\pi}, A_{11}(z) = \frac{z^2 + \pi}{2(z + 2i)^2} \exp\{-2iz + 1\}$$

and $\max\{\rho(A_{ij}(z)), i = 0, 1, j = 0, 1\} = 1 < \rho(f) = 2$. Then, by Theorem [4.1.8](#) for every finite value $d \neq 0$, we have

$$\lambda(f - d) = \rho(f) = 2.$$

Chapter 5

Growth and fixed points of solutions of linear differential equations with entire coefficients of $[p, q]$ -order

1 Introduction and Statement of Main Results:

In this chapter, we study the growth and fixed points of solutions to complex higher-order linear differential equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad (5.1)$$

and

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad (5.2)$$

where $k \geq 2$, and the coefficients $A_i(z)$, for $i = 0, \dots, k$, are entire functions of finite $[p, q]$ -order.

Historically, Juneja, Kapoor, and Bajpai [31] introduced the concept of $[p, q]$ -order to study properties of entire functions. Building on this foundation, Liu, Tu, and Shi [49] made a slight modification to the original definition, which enabled a more detailed investigation of the growth behavior of entire solutions to higher-order complex linear differential equations such as (5.1).

More recently, Li and Cao [44], Hu and Zheng [25], Belaïdi [4], and Chen et al. [16] have investigated the growth of solutions to equations (5.1) and (5.2) with meromorphic and analytic coefficients, both in the complex plane and in the unit disc $D = \{z : |z| < 1\}$, using the concept of $[p, q]$ -order.

In 2015, Belaïdi [4] whose studied the growth of solutions to homogeneous linear differential equations in which the coefficients are meromorphic functions of finite $[p, q]$ -order. For a particular case of equation (5.1), he obtained a significant result.

Theorem 5.1.1 [4] Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z), \dots, A_{k-1}(z)$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_i), i = 0, \dots, k-1\} \leq \rho$ ($0 < \rho < \infty$). Suppose that there exist two real numbers α and β satisfying $0 \leq \beta < \alpha$ such that, we have

$$|A_0(z)| \geq \exp_p \left\{ \alpha \left(\log_{q-1} r \right)^\rho \right\}$$

and

$$|A_i(z)| \leq \exp_p \left\{ \beta \left(\log_{q-1} r \right)^\rho \right\}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then the following statements hold:

- (i) If $p \geq q \geq 1$ or $3 \leq q = p + 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.
- (ii) If $p = 1, q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (5.1) satisfies $\rho_{[2,2]}(f) \geq \rho$.

Recently, Chen et al.[16] studied the growth and complex oscillation of solutions, as well as their derivatives of arbitrary order, for linear differential equations with analytic coefficients in the unit disc. They established several significant results.

Theorem 5.1.2 [16] Let $p \geq 1$ be an integer. Let H be a set of complex numbers satisfying $\overline{\text{dens}}_D\{|z| : z \in H \subseteq D\} > 0$, and let $A_0, \dots, A_{k-1}$ be analytic functions in the unit disc D such that

$$\max\{\rho_{M,p}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{M,p}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z| \rightarrow 1^-, z \in H} \left((1 - |z|)^\mu \log_p |A_0(z)| \right) > \alpha$$

and

$$|A_i(z)| \leq \exp_p \left\{ \alpha \left(\frac{1}{1-|z|} \right)^\mu \right\} \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (5.1) satisfies $\rho_p(f) = \rho_{M,p}(f) = \infty$ and $\rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A_0) = \mu$.

Theorem 5.1.3 ([16]) Let $p \geq 1$ be an integer. Let H be a set of complex numbers satisfying $\overline{\text{dens}}_D\{|z| : z \in H \subseteq D\} > 0$, and let $A_0, \dots, A_{k-1}$ be analytic functions in the unit disc D such that

$$\max\{\rho_{M,p}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{M,p}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and

$$\limsup_{|z| \rightarrow 1^-, z \in H} \left((1 - |z|)^\mu \log_p |A_i(z)| \right) < \liminf_{|z| \rightarrow 1^-, z \in H} \left((1 - |z|)^\mu \log_p |A_0(z)| \right) \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (5.1) satisfies $\rho_p(f) = \rho_{M,p}(f) = \infty$ and $\rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A_0) = \mu$.

Very recently, Belaïdi and Belmiloud [9] generalized the results of Chen et al. [16] by replacing the notion of iterated p -order with the more general notion of $[p, q]$ -order, and obtained the following result.

Theorem 5.1.4 ([9]) *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{\text{dens}}_D \{ |z| : z \in H \subseteq D \} > 0$, and let $A_0, \dots, A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max \left\{ \sigma_{M,[p,q]}(A_i) : i = 1, 2, \dots, k-1 \right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z| \rightarrow 1^-, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^\mu} > \alpha$$

and

$$|A_i(z)| \leq \exp_p \left\{ \alpha \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^\mu \right\} \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (5.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Building upon the work of Belaïdi and Belmiloud [9], we extend their results from the unit disc to the whole complex plane. This chapter is devoted to the study of the growth of solutions of complex differential equations whose coefficients are entire functions of finite $[p, q]$ -order. The main results of our investigation are presented below.

Theorem 5.1.5 [43] *Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{ |z| : z \in H \} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max \left\{ \rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1 \right\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} > \alpha \quad (5.3)$$

and

$$|A_i(z)| \leq \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \quad (i = 1, \dots, k-1) \quad (5.4)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every solution $f \neq 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

By Theorem 5.1.5, we easily obtain the following corollary.

Corollary 5.1.6 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that

$$\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for some real constants α, β where $0 \leq \beta < \alpha$, we have

$$|A_0(z)| \geq \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

and

$$|A_i(z)| \leq \exp_p \left\{ \beta \left(\log_{q-1} |z| \right)^\mu \right\}, \quad i = 1, \dots, k-1$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every solution $f \neq 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 5.1.7 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that

$$\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z| \right)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} \quad (i = 1, \dots, k-1) \quad (5.5)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every solution $f \neq 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 5.1.8 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that

$$\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, if $p \geq q \geq 2$ we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z| \right)^\mu} > \alpha \quad (5.6)$$

and

$$T(r, A_i) \leq \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}, \quad (i = 1, \dots, k-1) \quad (5.7)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every solution $f \neq 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$. If $p = q = 1$, we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} > (k-1)\alpha \quad (5.8)$$

and

$$T(r, A_i) \leq \alpha |z|^\mu, \quad (i = 1, \dots, k-1) \quad (5.9)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every nontrivial solution f of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

By Theorem 5.1.8, we easily obtain the following corollary.

Corollary 5.1.9 [43] *Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max \left\{ \rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1 \right\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for some real constants α, β , where $0 \leq \beta < \alpha$, we have

$$T(r, A_0) \geq \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

and

$$T(r, A_i) \leq \exp_{p-1} \left\{ \beta \left(\log_{q-1} |z| \right)^\mu \right\}, \quad (i = 1, \dots, k-1)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$. Then the following statements hold:

- (i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every nontrivial solution f of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.
- (ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every nontrivial solution f of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 5.1.10 [43] *Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max \left\{ \rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1 \right\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and if $p \geq q \geq 2$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1} |z| \right)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z| \right)^\mu}, \quad (i = 1, \dots, k-1) \quad (5.10)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every nontrivial solution f of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

If $p = q = 1$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{(k-1)T(r, A_i)}{|z|^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu}, \quad (i = 1, \dots, k-1) \quad (5.11)$$

as $|z| \rightarrow +\infty$ for $z \in H$, then every nontrivial solution f of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Theorem 5.1.11 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_k$ be entire functions such that for some constants $\alpha \geq 0$ and $\mu > 0$, we have (5.3) and

$$|A_i(z)| \leq \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}, \quad (i = 1, \dots, k)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Theorem 5.1.12 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_k$ be entire functions such that for a constant $\mu > 0$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z| \right)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu}, \quad (i = 1, \dots, k)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Theorem 5.1.13 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_k$ be entire functions such that for some constants $\alpha \geq 0$ and $\mu > 0$, if $p \geq q \geq 2$ we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z| \right)^\mu} > \alpha \quad (5.12)$$

and

$$T(r, A_i) \leq \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}, \quad (i = 1, \dots, k), \quad (5.13)$$

as $|z| \rightarrow +\infty$ for $z \in H$, then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

If $p = q = 1$, we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} > k\alpha \quad (5.14)$$

and

$$T(r, A_i) \leq \alpha |z|^\mu, \quad (i = 1, \dots, k) \quad (5.15)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \mu$.

Theorem 5.1.14 [43] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \dots, A_k(z)$ be entire functions such that for a constant $\mu > 0$, if $p \geq q \geq 2$ we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_i)}{(\log_{q-1} |z|)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\log_{q-1} |z|^\mu}, \quad (i = 1, \dots, k) \quad (5.16)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

If $p = q = 1$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{kT(|z|, A_i)}{|z|^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(|z|, A_0)}{|z|^\mu}, \quad (i = 1, \dots, k) \quad (5.17)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$, then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \mu$.

In the same paper, Belaïdi and Belmiloud [9] obtained several results on the fixed points of solutions of equations (5.1) and (5.2). In this chapter, we extend these results and establish the following theorems.

Theorem 5.1.15 [43] Assume that the assumptions of Theorem 5.1.5 or Theorem 5.1.7 hold. Then every solution $f \neq 0$ of equation (5.1) satisfies

$$\bar{\lambda}_{[p,q]}(f^{(j)} - z) = \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f^{(j)} - z) = \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu, \quad (j = 1, 2, \dots).$$

Theorem 5.1.16 [43] Assume that the assumptions of Theorem 5.1.8 or Theorem 5.1.10 hold. Then every solution $f \neq 0$ of equation (5.1) satisfies

$$\bar{\lambda}_{[p,q]}(f^{(j)} - z) = \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f^{(j)} - z) = \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu, \quad (j = 1, 2, \dots).$$

Theorem 5.1.17 [43] Assume that the assumptions of one of Theorem 5.1.11 to Theorem 5.1.14 hold. Then every meromorphic (or entire) solution $f \neq 0$ of equation (5.2) satisfies

$$\bar{\lambda}_{[p,q]}(f^{(j)} - z) = \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f^{(j)} - z) = \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) \geq \mu, \quad (j = 1, 2, \dots).$$

2 Some Auxiliary Lemmas

In this section, we present some lemmas that will be used in the proofs of our main theorems.

Lemma 5.2.1 [21] *Let $f(z)$ be a transcendental meromorphic function, and let $\eta > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on η and j ($0 \leq i < j \leq k$), such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B \left[\frac{T(\eta r, f)}{r} (\log^\eta r) T(\eta r, f) \right]^{j-i}.$$

Lemma 5.2.2 [22] *Let $g : (0, +\infty) \rightarrow \mathbb{R}$ and $h : (0, +\infty) \rightarrow \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ for all $r \notin E_3 \cup [0, 1]$ where $E_3 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $d > 1$ be a given constant. Then there exists an $r_0 = r_0(d) > 0$ such that $g(r) \leq h(dr)$ for all $r > r_0$.*

Lemma 5.2.3 [66] *Let $p \geq q \geq 1$ be integers. If $A_0(z), \dots, A_{k-1}(z)$ are entire functions of finite $[p, q]$ -order, then every solution $f \neq 0$ of equation (5.1) satisfies*

$$\rho_{[p+1, q]}(f) \leq \max\{\rho_{[p, q]}(A_j) : j = 0, 1, \dots, k-1\}.$$

Lemma 5.2.4 [5] *Let $p \geq q \geq 1$ be integers. Let $A_0, \dots, A_{k-1}$ and $F \neq 0$ be finite $[p, q]$ -order meromorphic functions. If f is a meromorphic solution with $\rho_{[p, q]}(f) = \infty$ and $\rho_{[p+1, q]}(f) = \rho < \infty$ of equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f = F,$$

then

$$\begin{aligned} \bar{\lambda}_{[p, q]}(f) &= \lambda_{[p, q]}(f) = \rho_{[p, q]}(f) = \infty, \\ \bar{\lambda}_{[p+1, q]}(f) &= \lambda_{[p+1, q]}(f) = \rho_{[p+1, q]}(f) = \rho. \end{aligned}$$

3 Proofs of Theorems :

Proof of Theorem 5.1.5. Suppose that every solution f of equation (5.1) not being identically equal to 0. From the conditions of Theorem 5.1.5, there exists a set H of complex numbers satisfying $\overline{\log dens} \{|z| = r : z \in H\} > 0$. Set $H_1 = \{|z| = r : z \in H\}$. Then by Proposition 1.4.6, H_1 is a set with $\int_{H_1} \frac{dr}{r} = +\infty$, such that for $z \in H$ we have (5.3) and (5.4) as $|z| \rightarrow +\infty$. By Lemma 5.2.1 for $\eta = 2$, there exists a set $E_1 \subset (1, +\infty)$ with $\int_{E_1} \frac{dr}{r} < \infty$ such

that for $|z| \notin [0, 1] \cup E_1$, we have for $j = 1, \dots, k$

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B \left[\frac{T(2r, f)}{r} (\log^2 r) \log T(2r, f) \right]^j \leq B [T(2r, f)]^{j+1}. \quad (5.18)$$

From (5.1), we get

$$|A_0(z)| \leq \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \dots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right|. \quad (5.19)$$

By (5.3), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} > \gamma > \alpha.$$

Obviously

$$\frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} > \gamma > \alpha \geq 0 \quad (5.20)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (5.3) and (5.20), we obtain

$$\begin{aligned} |A_0(z)| &> \exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \\ &> \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq |A_i(z)| \quad (i = 1, 2, \dots, k-1) \end{aligned} \quad (5.21)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Applying (5.18) and (5.21) into (5.19), we obtain

$$\begin{aligned} \exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} &\leq |A_0(z)| \\ &\leq kB [T(2|z|, f)]^{k+1} \times \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \end{aligned}$$

holds for all z satisfying $|z| \in H_1 \setminus ([0, 1] \cup E_1)$ as $|z| \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, for all $r = |z| \in H_1$, we have

$$\exp \left((1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \right) \leq kB [T(2dr, f)]^{k+1}. \quad (5.22)$$

Therefore, from (5.22) we obtain $\rho_{[p, q]}(f) = \infty$ and

$$\rho_{[p+1, q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1} T(r, f)}{\log_q r} \geq \mu. \quad (5.23)$$

By Lemma 5.2.3, we get

$$\rho_{[p+1, q]}(f) \leq \max \left\{ \rho_{[p, q]}(A_i) : i = 0, 1, \dots, k-1 \right\} = \rho_{[p, q]}(A_0) = \mu. \quad (5.24)$$

Therefore, by (5.23) and (5.24), we obtain $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0) = \mu.$$

Proof of Theorem 5.1.7. Set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z|\right)^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z|\right)^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (5.5), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z|\right)^\mu} < \alpha < \gamma < \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z|\right)^\mu}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Hence, we have (5.21) as $|z| \rightarrow +\infty$ for $z \in H$. Then, by using the same proof of Theorem 5.1.5, we get

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) \geq \mu$$

and by Lemma 5.2.3 we obtain the conclusion of Theorem 5.1.7.

Proof of Theorem 5.1.8. Suppose that every solution f of equation (5.1) not being identically equal to 0. By (5.1), we can write

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z) \frac{f^{(k-1)}(z)}{f(z)} + \dots + A_1(z) \frac{f'(z)}{f(z)}. \quad (5.25)$$

From (5.25), we obtain

$$\begin{aligned} T(r, A_0) &= m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1) \\ &= \sum_{i=1}^{k-1} T(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1). \end{aligned} \quad (5.26)$$

If $p \geq q \geq 2$, then by (5.6), we know that

$$\exists \gamma \in \mathbb{R}: \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z|\right)^\mu} > \gamma > \alpha.$$

Obviously

$$\frac{\log_{p-1} T(r, A_0)}{(\log_{q-1} |z|)^\mu} > \gamma > \alpha \geq 0 \quad (5.27)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (5.7) and (5.27), we obtain

$$\begin{aligned} T(r, A_0) &> \exp_{p-1} \left\{ \gamma \left(\log_{q-1} (|z|) \right)^\mu \right\} \\ &> \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1) \end{aligned} \quad (5.28)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (5.28) into (5.26), we obtain

$$\exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \leq (k-1) \exp_{p-1} \left\{ \alpha \left(\log_{q-1} r \right)^\mu \right\} + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, we have

$$(1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \leq O(\log T(dr, f) + \log dr) \quad (5.29)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (5.29) we obtain

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) \geq \mu. \quad (5.30)$$

By Lemma 5.2.3, we get

$$\begin{aligned} \rho_{[p+1,q]}(f) &\leq \max \left\{ \rho_{[p,q]}(A_i) : i = 0, 1, \dots, k-1 \right\} \\ &= \rho_{[p,q]}(A_0) = \mu. \end{aligned} \quad (5.31)$$

Therefore, by (5.30) and (5.31), we obtain

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) = \mu.$$

If $p = q = 1$, then by (5.8), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} > \gamma > (k-1)\alpha.$$

Obviously

$$\frac{T(r, A_0)}{|z|^\mu} > \gamma > (k-1)\alpha \geq 0 \quad (5.32)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (5.9) and (5.32), we obtain

$$T(r, A_0) > \gamma |z|^\mu > (k-1)\alpha |z|^\mu \geq \alpha |z|^\mu \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1) \quad (5.33)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (5.33) into (5.26), we obtain

$$\gamma|z|^\mu \leq (k-1)\alpha|z|^\mu + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > (k-1)\alpha$, by the last inequality and combining with Lemma 5.2.2, we have

$$(\gamma - (k-1)\alpha)r^\mu \leq O(\log T(dr, f) + \log dr) \quad (5.34)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (5.34) we obtain

$$\rho(f) = \infty \text{ and } \rho_2(f) \geq \mu. \quad (5.35)$$

By Lemma 5.2.3, we get

$$\rho_2(f) \leq \max\{\rho(A_i) : i = 0, 1, \dots, k-1\} = \rho(A_0) = \mu. \quad (5.36)$$

Therefore, by (5.35) and (5.36), we obtain $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Proof of Theorem 5.1.10. If $p \geq q \geq 2$, we set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{(\log_{q-1} |z|)^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_i)}{(\log_{q-1} |z|)^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (5.10), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{\log_{p-1} T(r, A_i)}{(\log_{q-1} |z|)^\mu} < \alpha < \gamma < \frac{\log_{p-1} T(r, A_0)}{(\log_{q-1} |z|)^\mu}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Hence, we have

$$T(r, A_0) > \exp_{p-1} \left\{ \gamma (\log_{q-1} |z|)^\mu \right\} > \exp_{p-1} \left\{ \alpha (\log_{q-1} |z|)^\mu \right\} \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then, by using the same proof of Theorem 5.1.8, we get

$$\rho_{[p, q]}(f) = \infty \text{ and } \rho_{[p+1, q]}(f) \geq \mu,$$

and by Lemma 5.2.3 we obtain the conclusion of Theorem 5.1.10.

If $p = q = 1$, we set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{(k-1)T(r, A_i)}{|z|^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (5.11), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{(k-1)T(r, A_i)}{|z|^\mu} < \alpha < \gamma < \frac{T(r, A_0)}{|z|^\mu} \quad (5.37)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (5.37), we obtain

$$T(r, A_0) > \gamma|z|^\mu > \alpha|z|^\mu \geq (k-1)T(r, A_i), \quad (i = 1, 2, \dots, k-1) \quad (5.38)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (5.38) into (5.26), we obtain

$$\gamma r^\mu \leq \alpha r^\mu + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, we have

$$(\gamma - \alpha) r^\mu \leq O(\log T(dr, f) + \log dr) \quad (5.39)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (5.39) we obtain

$$\rho(f) = \infty \text{ and } \rho_2(f) \geq \mu. \quad (5.40)$$

By Lemma 5.2.3, we get

$$\rho_2(f) \leq \max\{\rho(A_i) : i = 0, 1, \dots, k-1\} = \rho(A_0) = \mu. \quad (5.41)$$

Therefore, by (5.40) and (5.41), we obtain $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Proof of Theorems 5.1.11 and 5.1.12. Suppose that every meromorphic (or entire) solution f of equation (5.2) not being identically equal to 0. From (5.2), we get

$$|A_0(z)| \leq |A_k(z)| \left| \frac{f^{(k)}}{f} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f} \right| + \dots + |A_1(z)| \left| \frac{f'}{f} \right|. \quad (5.42)$$

By using a similar proof as in Theorem 5.1.5 or Theorem 5.1.7, we obtain

$$\begin{aligned} |A_0(z)| &> \exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \\ &> \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq |A_i(z)| \quad (i = 1, 2, \dots, k) \end{aligned} \quad (5.43)$$

as $|z| \rightarrow +\infty$, for $z \in H$. Applying (5.18) and (5.43) into (5.42), we obtain

$$\exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \leq |A_0(z)| \leq k B [T(2|z|, f)]^{k+1} \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

holds for all z satisfying $|z| \in H_1 \setminus ([0, 1] \cup E_1)$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, for all $r = |z| \in H_1$, we get

$$\exp \left((1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \right) \leq k B [T(2dr, f)]^{k+1}. \quad (5.44)$$

Therefore, from (5.44) we obtain $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p+1,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1} T(r, f)}{\log_q r} \geq \mu.$$

Proof of Theorems 5.1.13 and 5.1.14. Suppose that every meromorphic (or entire) solution f of equation (5.2) not being identically equal to 0. By (5.2), we can write

$$-A_0(z) = A_k(z) \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z) \frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z) \frac{f'(z)}{f(z)}. \quad (5.45)$$

From (5.45), we obtain

$$\begin{aligned} T(r, A_0) &= m(r, A_0) \leq \sum_{i=1}^k m(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1) \\ &= \sum_{i=1}^k T(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1). \end{aligned} \quad (5.46)$$

If $p \geq q \geq 2$, then by using a similar proof as in Theorem 5.1.8 or Theorem 5.1.10, we obtain

$$\begin{aligned} T(r, A_0) &> \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \\ &> \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq T(r, A_i), \quad (i = 1, 2, \dots, k) \end{aligned} \quad (5.47)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (5.47) into (5.46), we obtain

$$\exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \leq k \exp_{p-1} \left\{ \alpha \left(\log_{q-1} r \right)^\mu \right\} + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, we have

$$(1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \leq O(\log T(dr, f) + \log dr) \quad (5.48)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (5.48) we obtain

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) \geq \mu.$$

If $p = q = 1$, then by using a similar proof as in Theorem 5.1.8 or Theorem 5.1.10, we have

$$T(r, A_0) > \gamma |z|^\mu > k\alpha |z|^\mu > \alpha |z|^\mu \geq T(r, A_i), \quad (i = 1, 2, \dots, k) \quad (5.49)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (5.49) into (5.46), we obtain

$$\gamma r^\mu \leq k\alpha r^\mu + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > k\alpha$, by the last inequality and combining with Lemma 5.2.2, we have

$$(\gamma - k\alpha) r^\mu \leq O(\log T(dr, f) + \log dr) \quad (5.50)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (5.50) we obtain

$$\rho(f) = \infty \text{ and } \rho_2(f) \geq \mu.$$

Proof of Theorem 5.1.15. Suppose that every solution f of equation (5.1) not being identically equal to 0.

First step: We consider the fixed points of f . Define the function g by setting

$$g(z) := f(z) - z, \quad z \in \mathbb{C}.$$

It follows from (5.1) that

$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_1g' + A_0g = -A_1 - zA_0 \quad (5.51)$$

and by Theorem 5.1.5 or Theorem 5.1.7, we have

$$\begin{aligned} \rho_{[p,q]}(g) &= \rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g) &= \bar{\lambda}_{[p+1,q]}(f - z). \end{aligned} \quad (5.52)$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$. Clearly $A_0 \not\equiv 0$. Then

$\lim_{|z| \rightarrow +\infty, z \in H} \left| \frac{A_1}{A_0} \right| = +\infty$ and by (5.21), we have

$$\begin{aligned} \left| \frac{A_1(z)}{A_0(z)} \right| &< \frac{\exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}}{\exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \\ &= \frac{1}{\exp \left\{ (1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \right\}} \rightarrow 0 \end{aligned}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then $\lim_{|z| \rightarrow +\infty, z \in H} \left| \frac{A_1}{A_0} \right| = 0$. It is easy to see the contradiction. Hence, $-A_1 - zA_0 \neq 0$. Next by Proposition 1.3.6, we get

$$\max \left\{ \rho_{[p,q]}(A_i) \ (i = 0, 1, \dots, k-1), \rho_{[p,q]}(-A_1 - zA_0) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce, by using (5.51), (5.52) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g) = \rho_{[p,q]}(g) = \infty, \bar{\lambda}_{[p+1,q]}(g) = \rho_{[p+1,q]}(g) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f - z) = \bar{\lambda}_{[p,q]}(g) = \rho_{[p,q]}(g) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f - z) = \bar{\lambda}_{[p+1,q]}(g) = \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f) = \mu.$$

Second step. For the following proof, we use the principle of mathematical induction. Set $A_k(z) \equiv 1$, then

$$|A_k(z)| \leq \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

and equation (5.1) becomes (5.2). We consider the fixed points of $f^{(j)}(z)$ ($j = 1, 2, \dots$). Define the function g_1 by setting

$$g_1(z) := f'(z) - z, \ z \in \mathbb{C}.$$

Then, by Lemma 5.2.4 and (5.52), we obtain

$$\begin{aligned} \rho_{[p,q]}(g_1) &= \rho_{[p,q]}(f) = \infty, \rho_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_1) &= \bar{\lambda}_{[p+1,q]}(f' - z). \end{aligned} \tag{5.53}$$

Dividing both sides of (5.2) by A_0 , we obtain

$$\frac{A_k}{A_0} f^{(k)} + \frac{A_{k-1}}{A_0} f^{(k-1)} + \dots + \frac{A_1}{A_0} f' + f = 0. \tag{5.54}$$

It follows, by differentiating both sides of equation (5.54) that

$$\begin{aligned} \frac{A_k}{A_0} f^{(k+1)} + \left(\left(\frac{A_k}{A_0} \right)' + \frac{A_{k-1}}{A_0} \right) f^{(k)} + \cdots + \left(\left(\frac{A_2}{A_0} \right)' + \frac{A_1}{A_0} \right) f'' \\ + \left(\left(\frac{A_1}{A_0} \right)' + 1 \right) f' = 0. \end{aligned} \quad (5.55)$$

Multiplying (5.55) by A_0 , we obtain

$$A_{k,1} f^{(k+1)} + A_{k-1,1} f^{(k)} + \cdots + A_{1,1} f'' + A_{0,1} f' = 0. \quad (5.56)$$

Substituting $f' = g_1 + z$ into (5.56), we obtain

$$A_{k,1} g_1^{(k)} + A_{k-1,1} g_1^{(k-1)} + \cdots + A_{1,1} g_1' + A_{0,1} g_1 = F_1, \quad (5.57)$$

where

$$A_{k,1} = A_k = 1, \quad A_{i,1} = A_0 \left(\left(\frac{A_{i+1}}{A_0} \right)' + \frac{A_i}{A_0} \right) \quad (i = 1, 2, \dots, k-1), \quad (5.58)$$

$$A_{0,1} = A_0 \left(\left(\frac{A_1}{A_0} \right)' + 1 \right), \quad (5.59)$$

$$F_1 = - (A_{1,1} + z A_{0,1}). \quad (5.60)$$

Next, we prove that $A_{0,1} \neq 0$ and $F_1 \neq 0$. Assume that $A_{0,1} \equiv 0$. So, by $A_0 \neq 0$ we get $\left(\frac{A_1}{A_0} \right)' + 1 = 0$, and then $\frac{A_1}{A_0} = -z + C_1$, where C_1 is an arbitrary constant. Hence, we have $A_1 + (z - C_1) A_0 = 0$. Then, $f = z - C_1$ is a solution of (5.1) and $\rho_{[p,q]}(f) = 0 < \infty$. This contradicts (5.52), hence $A_{0,1} \neq 0$. Now, assume that $F_1 \equiv 0$. Then

$$A_{1,1} + z A_{0,1} \equiv 0. \quad (5.61)$$

By (5.56) and (5.61), we know that the function f such that $f' = z$ is a solution of equation (5.56). Hence equation (5.1) has a solution f_1 satisfying $f_1' = z$ and $\rho_{[p,q]}(f_1) = 0 < \infty$. This contradicts (5.52). Therefore, $F_1 \neq 0$. It follows by (5.58)–(5.60) and Proposition 1.3.6 that

$$\max \left\{ \rho_{[p,q]}(A_{i,1}) \quad (i = 0, 1, \dots, k), \rho_{[p,q]}(F_1) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce by using (5.53), (5.57) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g_1) = \rho_{[p,q]}(g_1) = \infty, \quad \bar{\lambda}_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(g_1) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f' - z) = \bar{\lambda}_{[p,q]}(g_1) = \rho_{[p,q]}(g_1) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f' - z) = \bar{\lambda}_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(f) = \mu.$$

Set $g_2(z) = f''(z) - z$, $z \in \mathbb{C}$. Then, by Proposition 1.3.6 and (5.52), we obtain

$$\begin{aligned} \rho_{[p,q]}(g_2) &= \rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_2) &= \bar{\lambda}_{[p+1,q]}(f'' - z). \end{aligned} \quad (5.62)$$

Then, by using a similar discussion as in the case of the function g_1 , we can get

$$A_{k,2}f^{(k+2)} + A_{k-1,2}f^{(k+1)} + \cdots + A_{1,2}f^{(3)} + A_{0,2}f'' = 0 \quad (5.63)$$

and

$$A_{k,2}g_2^{(k)} + A_{k-1,2}g_2^{(k-1)} + \cdots + A_{1,2}g_2' + A_{0,2}g_2 = F_2, \quad (5.64)$$

where

$$A_{k,2} = 1, \quad A_{i,2} = A_{0,1} \left(\left(\frac{A_{i+1,1}}{A_{0,1}} \right)' + \frac{A_{i,1}}{A_{0,1}} \right) \quad (i = 1, 2, \dots, k-1), \quad (5.65)$$

$$A_{0,2} = A_{0,1} \left(\left(\frac{A_{1,1}}{A_{0,1}} \right)' + 1 \right), \quad (5.66)$$

$$F_2 = -(A_{1,2} + zA_{0,2}). \quad (5.67)$$

Next, we prove that $A_{0,2} \neq 0$ and $F_2 \neq 0$. Assume that $A_{0,2} \equiv 0$. So, by $A_{0,1} \neq 0$ we have $\left(\frac{A_{1,1}}{A_{0,1}} \right)' + 1 \equiv 0$, then $\frac{A_{1,1}}{A_{0,1}} = -z + C_2$, where C_2 is an arbitrary constant. Hence, we have $A_{1,1} + (z - C_2)A_{0,1} = 0$. Then, $f' = z - C_2$ is a solution of (5.56). Thus, equation (5.1) has a solution f satisfying $f' = z - C_2$ and $\rho_{[p,q]}(f) = 0 < \infty$. This contradicts (5.52), hence $A_{0,2} \neq 0$. Now, assume that $F_2 \equiv 0$. Then

$$A_{1,2} + zA_{0,2} \equiv 0. \quad (5.68)$$

By (5.63) and (5.68), we know that the function f such that $f'' = z$ is a solution of equation (5.63). Hence equation (5.1) has a solution f_2 satisfying $f_2'' = z$ and $\rho_{[p,q]}(f_2) = 0 < \infty$. This contradicts (5.52). Therefore, $F_2 \neq 0$. It follows by (5.65)-(5.67) and Proposition 1.3.6 that

$$\max \left\{ \rho_{[p,q]}(A_{i,2}) \quad (i = 0, 1, \dots, k), \rho_{[p,q]}(F_2) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce by using (5.62), (5.64) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g_2) = \rho_{[p,q]}(g_2) = \infty, \quad \bar{\lambda}_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(g_2) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f'' - z) = \bar{\lambda}_{[p,q]}(g_2) = \rho_{[p,q]}(g_2) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f'' - z) = \bar{\lambda}_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \mu.$$

Now, assume that

$$\begin{aligned} A_{0,s} &\neq 0, \\ \bar{\lambda}_{[p,q]}(f^{(s)} - z) &= \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(s)} - z) &= \rho_{[p+1,q]}(f) = \mu \end{aligned} \quad (5.69)$$

for all $s = 0, 1, \dots, j-1$, and we prove that for $s = j$ we have (5.69) holds. Set $g_j(z) = f^{(j)}(z) - z$, $z \in \mathbb{C}$. Then, by using (5.52) and Proposition 1.3.6, we obtain

$$\begin{aligned} \rho_{[p,q]}(g_j) &= \rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(g_j) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_j) &= \bar{\lambda}_{[p+1,q]}(f^{(j)} - z). \end{aligned} \quad (5.70)$$

By following the same procedure as before, we have

$$A_{k,j}f^{(k+j)} + A_{k-1,j}f^{(k+j-1)} + \dots + A_{1,j}f^{(j+1)} + A_{0,j}f^{(j)} = 0 \quad (5.71)$$

and

$$A_{k,j}g_j^{(k)} + A_{k-1,j}g_j^{(k-1)} + \dots + A_{1,j}g_j' + A_{0,j}g_j = F_j, \quad (5.72)$$

where

$$\begin{aligned} A_{k,j} &= 1, \quad A_{i,j} = A_{0,j-1} \left(\left(\frac{A_{i+1,j-1}}{A_{0,j-1}} \right)' + \frac{A_{i,j-1}}{A_{0,j-1}} \right) \quad (i = 1, 2, \dots, k-1), \\ A_{0,j} &= A_{0,j-1} \left(\left(\frac{A_{1,j-1}}{A_{0,j-1}} \right)' + 1 \right) \neq 0 \quad (A_{0,0} = A_0, \quad A_{1,0} = A_1), \\ F_j &= -(A_{1,j} + zA_{0,j}) \neq 0. \end{aligned}$$

We deduce by applying Lemma 5.2.4 in (5.72) that

$$\begin{aligned} \bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p,q]}(g_j) = \rho_{[p,q]}(g_j) = \rho_{[p,q]}(f^{(j)}) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(g_j) = \rho_{[p+1,q]}(g_j) = \rho_{[p+1,q]}(f^{(j)}) = \mu \quad (j = 1, 2, \dots). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu \quad (j = 1, 2, \dots). \end{aligned}$$

Proof of Theorem 5.1.16. Suppose that every solution f of equation (5.1) not being identically equal to 0. By applying Theorem 5.1.8 or Theorem 5.1.10, we get

$$\rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(f) = \mu.$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$, then we can easily obtain

$$\begin{aligned} T(r, A_1) &= T(r, -zA_0) \leq T(r, A_0) + T(r, z), \\ T(r, A_0) &= T\left(r, \frac{A_1}{-z}\right) \leq T(r, A_1) + T(r, z) + O(1). \end{aligned} \quad (5.73)$$

It follows from (5.73) that

$$1 - \frac{T(r, z) + O(1)}{T(r, A_0)} \leq \frac{T(r, A_1)}{T(r, A_0)} \leq 1 + \frac{T(r, z)}{T(r, A_0)}. \quad (5.74)$$

By following the same reasoning as in the proof of Theorem 5.1.8 or Theorem 5.1.10, we have

$$T(r, A_0) > \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} > \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq T(r, A_1) \quad (5.75)$$

as $r = |z| \rightarrow +\infty$ for $z \in H$. By using (5.75), we have

$$\frac{T(r, z)}{T(r, A_0)} \leq \frac{T(r, z)}{\exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \rightarrow 0 \quad (5.76)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then, by (5.74) and (5.76), we get

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} = 1. \quad (5.77)$$

On the other hand, we have for $p = q = 1$

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\alpha}{\gamma},$$

which implies

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} \leq \frac{\alpha}{\gamma} < 1 \quad (5.78)$$

and for $p \geq q \geq 2$

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}}{\exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \rightarrow 0 \quad (5.79)$$

as $|z| \rightarrow +\infty$ for $z \in H$. It follows by (5.78) and (5.79) that

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} \neq 1. \quad (5.80)$$

Obviously, (5.77) contradicts with (5.80). Hence, $-A_1 - zA_0 \not\equiv 0$. Set $A_k(z) \equiv 1$, then $T(r, A_k) \leq \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$. Clearly, $A_0 \not\equiv 0$. We can get the conclusion of Theorem 5.1.16, by reasoning in the same way as we did in the proof of Theorem 5.1.15.

Proof of Theorem 5.1.17. Suppose that every meromorphic (or entire) solution f of equation (5.2) not being identically equal to 0. By applying one of Theorem 5.1.11 to Theorem 5.1.14, we get

$$\rho_{[p,q]}(f) = \infty, \rho_{[p+1,q]}(f) \geq \mu.$$

Then, we can get the conclusion of Theorem 5.1.17, by reasoning in the same way as we did in the proof of Theorem 5.1.15 and Theorem 5.1.16 by using $\rho_{[p+1,q]}(f) \geq \mu$ instead of $\rho_{[p+1,q]}(f) = \mu$ and $\rho_{[p+1,q]}(f^{(j)}) \geq \mu$ instead of $\rho_{[p+1,q]}(f^{(j)}) = \mu$ ($j = 1, 2, \dots$).

Chapter 6

Complex differential equations with meromorphic coefficients of $[p, q]$ -order

1 Introduction and Statement of Main Results :

In this chapter, we continue to investigate the higher-order linear differential equation (5.1), whose coefficients $A_i(z)$ ($i = 0, \dots, k-1$) are meromorphic functions of $[p, q]$ -order. Using the concept of $[p, q]$ -order, we extend several recent results obtained by Belaïdi and Belmiloud [9] from the unit disc to the whole complex plane.

Considerable attention has been devoted to the study of the growth of solutions to linear differential equations whose coefficients are entire or meromorphic functions of finite $[p, q]$ -order in the complex plane. Notable contribution to this area have been made by Li and Cao, Hu and Zheng, Belaïdi and Zemirni, and Belaïdi.

In 2012, Li and Cao [44] investigated the growth of meromorphic solutions to the linear differential equation (5.1), assuming that all coefficients are meromorphic functions of finite $[p, q]$ -order. They considered the case where the $[p, q]$ -order of a fixed coefficient A_0 dominates the $[p, q]$ -orders of the other coefficients. Under these conditions, they established the following result.

Theorem 6.1.1 ([44]) *Let $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions in the plane satisfying*

$$\max \left\{ \rho_{(p,q)}(A_i), \lambda_{(p,q)} \left(\frac{1}{A_0} \right) : i = 1, 2, \dots, k-1 \right\} < \rho_{(p,q)}(A_0) < +\infty,$$

then every meromorphic solution f whose poles are of uniformly bounded multiplicities of (5.1) satisfies

$$\rho_{(p+1,q)}(f) = \rho_{(p,q)}(A_0).$$

Recently, Belaïdi [4] investigated the growth of meromorphic solutions of the linear differential equation (5.1). They generalized and improved some results of Kinnunen [32],

Liu-Tu-Shi [49], Cao, Xu, and Chen [12], and Li-Cao [44], and established the following result.

Theorem 6.1.2 ([4]) *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho$ ($0 < \rho < +\infty$). Suppose that there exist two real numbers satisfying $0 \leq \beta < \alpha$ such that, we have*

$$|A_0(z)| \geq \exp_p \{\alpha (\log_{q-1} r)^\rho\}$$

and

$$|A_i(z)| \leq \exp_p \{\beta (\log_{q-1} r)^\rho\}, \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then the following statements hold:

- (i) *If $p \geq q \geq 1$ or $3 \leq q = p+1$, then every meromorphic solution $f \neq 0$ whose poles are of uniformly bounded multiplicities or $\delta(1, f) > 0$ of equation (5.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.*
- (ii) *If $p = 1, q = 2$, then every meromorphic solution $f \neq 0$ of equation (5.1) satisfies $\rho_{[2,2]}(f) \geq \rho$.*

The purpose of this chapter is to investigate the growth and oscillation of solutions to complex linear differential equations of the form (5.1), where the coefficients are meromorphic functions of finite $[p, q]$ -order. We extend and improve the results obtained by Belaïdi and Belmiloud [9], originally formulated in the setting of the unit disc to the whole complex plane. The main results of this chapter are presented below.

Theorem 6.1.3 [41] *Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that $\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu$ ($0 < \mu < +\infty$), and for a constant $\alpha \geq 0$, we have*

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{(\log_{q-1} |z|)^\mu} > \alpha \quad (6.1)$$

and

$$|A_i(z)| \leq \exp_p \{\alpha (\log_{q-1} |z|)^\mu\} \quad (i = 1, \dots, k-1) \quad (6.2)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \neq 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies

$$\rho_{[p,q]}(f) = \infty \quad \text{and} \quad \rho_{[p+1,q]}(f) = \mu.$$

By Theorem 6.1.3, we easily obtain the following corollary.

Corollary 6.1.4 [41] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that $\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu$ ($0 < \mu < +\infty$), and for some real constants α, β where $0 \leq \beta < \alpha$, we have

$$|A_0(z)| \geq \exp_p\{\alpha(\log_{q-1}|z|)^\mu\}$$

and

$$|A_i(z)| \leq \exp_p\{\beta(\log_{q-1}|z|)^\mu\}, \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \neq 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 6.1.5 [41] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that

$$\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty),$$

and

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_i(z)|}{(\log_{q-1}|z|)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{(\log_{q-1}|z|)^\mu} \quad (i = 1, \dots, k-1) \quad (6.3)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \neq 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 6.1.6 [41] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that

$$\lambda_{[p,q]}\left(\frac{1}{A_0}\right) < \rho_{[p,q]}(A_0),$$

$$\max\{\rho_{[p,q]}(A_i), i = 1, 2, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, if $p \geq q \geq 2$ we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{(\log_{q-1}|z|)^\mu} > \alpha \quad (6.4)$$

and

$$T(r, A_i) \leq \exp_{p-1}\{\alpha(\log_{q-1}|z|)^\mu\}, \quad (i = 1, \dots, k-1) \quad (6.5)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \neq 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and

$\rho_{[p+1,q]}(f) = \mu$. If $p = q = 1$, we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} > (k-1)\alpha \quad (6.6)$$

and

$$T(r, A_i) \leq \alpha|z|^\mu, (i = 1, \dots, k-1) \quad (6.7)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

By Theorem 6.1.6, we easily obtain the following corollary.

Corollary 6.1.7 [41] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that

$$\lambda_{[p,q]} \left(\frac{1}{A_0} \right) < \rho_{[p,q]}(A_0),$$

$$\max \{ \rho_{[p,q]}(A_i), : i = 1, 2, \dots, k-1 \} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for some real constants α, β where $0 \leq \beta < \alpha$, we have

$$T(r, A_0) \geq \exp_{p-1} \{ \alpha (\log_{q-1} |z|)^\mu \}$$

and

$$T(r, A_i) \leq \exp_{p-1} \{ \beta (\log_{q-1} |z|)^\mu \}, (i = 1, \dots, k-1)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then the following statements hold:

- (i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.
- (ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

Theorem 6.1.8 [41] Let $p \geq q \geq 1$ and $k \geq 2$ be integers. Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be meromorphic functions such that

$$\lambda_{[p,q]} \left(\frac{1}{A_0} \right) < \rho_{[p,q]}(A_0),$$

$$\max \{ \rho_{[p,q]}(A_i), : i = 1, 2, \dots, k-1 \} \leq \rho_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and if $p \geq q \geq 2$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_i)}{(\log_{q-1} |z|)^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{(\log_{q-1} |z|)^\mu} \quad (i = 1, \dots, k-1) \quad (6.8)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \mu$.

If $p = q = 1$, we have

$$\limsup_{|z| \rightarrow +\infty, z \in H} \frac{(k-1)T(r, A_i)}{|z|^\mu} < \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} \quad (i = 1, \dots, k-1) \quad (6.9)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Theorem 6.1.9 [41] Assume that the assumptions of Theorem 6.1.3 or Theorem 6.1.5 hold. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies

$$\begin{aligned} \bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu, \quad (j = 1, 2, \dots). \end{aligned}$$

Theorem 6.1.10 [41] Assume that the assumptions of Theorem 6.1.6 or Theorem 6.1.8 hold. Then every solution $f \not\equiv 0$ of equation (5.1) satisfies

$$\begin{aligned} \bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu, \quad (j = 1, 2, \dots). \end{aligned}$$

2 Some Auxiliary Lemmas :

The proofs of our results rely to the Lemma 4.2.2 (see [Chapter 4]), Lemmas 5.2.1-5.2.2, 5.2.4 (see [Chapter 5]), together with the following lemma.

Lemma 6.2.1 ([4]) Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be non-constant meromorphic functions of $[p, q]$ -order. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1) satisfies

$$\rho_{[p+1,q]}(f) \leq \max\{\rho_{[p,q]}(A_i) : i = 0, 1, \dots, k-1\},$$

3 Proofs of Theorems :

Proof of Theorem 6.1.3. Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1). From the conditions of Theorem 6.1.3, there exists a set H of complex numbers such that the upper logarithmic density satisfies $\overline{\log dens} \{ |z| = r : z \in H \} > 0$. Set $H_1 = \{ |z| = r : z \in H \}$. Then by Proposition 1.4.6, H_1 is a set with $\int_{H_1} \frac{dr}{r} = +\infty$, such that for $z \in H$ we have (6.1) and (6.2) as $|z| \rightarrow +\infty$. By Lemma 5.2.1 for $\eta = 2$, there exists a set $E_1 \subset (1, +\infty)$ with $\int_{E_1} \frac{dr}{r} < \infty$ such that for $|z| = r \notin [0, 1] \cup E_1$, we have for $j = 1, \dots, k$

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B \left[\frac{T(2r, f)}{r} (\log^2 r) \log T(2r, f) \right]^j \leq B [T(2r, f)]^{j+1}. \quad (6.10)$$

From (5.1), we get

$$|A_0(z)| \leq \left| \frac{f^{(k)}}{f} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f} \right| + \dots + |A_1(z)| \left| \frac{f'}{f} \right|. \quad (6.11)$$

By (6.1), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} > \gamma > \alpha.$$

Obviously

$$\frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z| \right)^\mu} > \gamma > \alpha \geq 0 \quad (6.12)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (6.2) and (6.12), we obtain

$$\begin{aligned} |A_0(z)| &> \exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \\ &> \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq |A_i(z)| \quad (i = 1, 2, \dots, k-1) \end{aligned} \quad (6.13)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Applying (6.10) and (6.13) into (6.11), we obtain

$$\exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \leq |A_0(z)| \leq kB [T(2|z|, f)]^{k+1} \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

holds for all z satisfying $|z| \in H_1 \setminus ([0, 1] \cup E_1)$ as $|z| \rightarrow +\infty$. Noting that $\gamma > \alpha$, by the last inequality and combining with Lemma 5.2.2, for all $|z| = r \in H_1$, we have

$$\exp \left\{ (1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \right\} \leq kB [T(2dr, f)]^{k+1}. \quad (6.14)$$

Therefore, from (6.14) we obtain $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p+1,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1} T(r, f)}{\log_q r} \geq \mu. \quad (6.15)$$

By Lemma 6.2.1, we get

$$\rho_{[p+1,q]}(f) \leq \max \left\{ \rho_{[p,q]}(A_i) : i = 0, 1, \dots, k-1 \right\} = \rho_{[p,q]}(A_0) = \mu. \quad (6.16)$$

Therefore, by (6.15) and (6.16), we obtain $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0) = \mu.$$

Proof of Theorem 6.1.5. Set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z|\right)^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z|\right)^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (6.3), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{\log_p |A_i(z)|}{\left(\log_{q-1} |z|\right)^\mu} < \alpha < \gamma < \frac{\log_p |A_0(z)|}{\left(\log_{q-1} |z|\right)^\mu}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Hence, we have (6.13) as $|z| \rightarrow +\infty$ for $z \in H$. Then, by using the same proof of Theorem 6.1.3, we get

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) \geq \mu$$

and by Lemma 6.2.1 we obtain the conclusion of Theorem 6.1.5.

Proof of Theorem 6.1.6. Suppose that $f \neq 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1).

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z) \frac{f^{(k-1)}(z)}{f(z)} + \dots + A_1(z) \frac{f'(z)}{f(z)}. \quad (6.17)$$

From (6.17), we obtain

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1)$$

By adding $N(r, A_0)$ to both sides of the last inequality, we get

$$\begin{aligned} T(r, A_0) &= m(r, A_0) + N(r, A_0) \leq N(r, A_0) + \sum_{i=1}^{k-1} m(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1) \\ &\leq N(r, A_0) + \sum_{i=1}^{k-1} T(r, A_i) + \sum_{i=1}^k m\left(r, \frac{f^{(i)}}{f}\right) + O(1). \end{aligned} \quad (6.18)$$

If $p \geq q \geq 2$, then by (6.4), we know that

$$\exists \gamma \in \mathbb{R}: \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z|\right)^\mu} > \gamma > \alpha.$$

Obviously

$$\frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z|\right)^\mu} > \gamma > \alpha \geq 0 \quad (6.19)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (6.5) and (6.20), we obtain

$$\begin{aligned} T(r, A_0) &> \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \\ &> \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1) \end{aligned} \quad (6.20)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Set $\lambda_{[p,q]} \left(\frac{1}{A_0} \right) = \lambda_0$, then by the definition of $\lambda_{[p,q]} \left(\frac{1}{A_0} \right)$, for any given ε ($0 < \varepsilon < \mu - \lambda_0$), we have

$$N(r, A_0) \leq \exp_{p-1} \left\{ \left(\log_{q-1} |z| \right)^{\lambda_0 + \varepsilon} \right\} \quad (6.21)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (6.20) and (6.21) into (6.18), we obtain

$$\begin{aligned} \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} &\leq (k-1) \exp_{p-1} \left\{ \alpha \left(\log_{q-1} r \right)^\mu \right\} + \exp_{p-1} \left\{ \left(\log_{q-1} r \right)^{\lambda_0 + \varepsilon} \right\} \\ &\quad + O(\log T(r, f) + \log r) \end{aligned}$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$ where $E_2 \subset (0, +\infty)$ is a set of finite linear measure. Noting that $\gamma > \alpha$ and $\lambda_0 + \varepsilon < \mu$, by the last inequality and combining with Lemma 5.2.2, we have for $d > 1$

$$(1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} r \right)^\mu \right\} \leq O(\log T(dr, f) + \log dr) \quad (6.22)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (6.22) we obtain

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) \geq \mu. \quad (6.23)$$

By Lemma 6.2.1, we get

$$\begin{aligned}\rho_{[p+1,q]}(f) &\leq \max\left\{\rho_{[p,q]}(A_i) : i = 0, 1, \dots, k-1\right\} \\ &= \rho_{[p,q]}(A_0) = \mu.\end{aligned}\tag{6.24}$$

Therefore, by (6.23) and (6.24), we obtain

$$\rho_{[p,q]}(f) = \infty \text{ and } \rho_{[p+1,q]}(f) = \mu.$$

If $p = q = 1$, then by (6.6), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu} > \gamma > (k-1)\alpha.$$

Obviously

$$\frac{T(r, A_0)}{|z|^\mu} > \gamma > (k-1)\alpha \geq 0 \tag{6.25}$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (6.18) and (6.25), we obtain

$$T(r, A_0) > \gamma |z|^\mu > (k-1)\alpha |z|^\mu \geq \alpha |z|^\mu \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1) \tag{6.26}$$

as $|z| \rightarrow +\infty$ for $z \in H$. From (6.21) with $p = 1, q = 1$, for any given $0 < \varepsilon < \mu - \lambda_0$ we have

$$N(r, A_0) \leq |z|^{\lambda_0 + \varepsilon}. \tag{6.27}$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (6.26) and (6.27) into (6.18), we obtain

$$\gamma r^\mu \leq (k-1)\alpha r^\mu + r^{\lambda_0 + \varepsilon} + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > (k-1)\alpha$ and $\lambda_0 + \varepsilon < \mu$, by the last inequality and combining with Lemma 5.2.2, we have for $d > 1$

$$(1 - o(1))(\gamma - (k-1)\alpha) r^\mu \leq O(\log T(dr, f) + \log dr) \tag{6.28}$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (6.28) we obtain

$$\rho(f) = \infty \text{ and } \rho_2(f) \geq \mu. \tag{6.29}$$

By Lemma 6.2.1, we get

$$\rho_2(f) \leq \max\{\rho(A_i) : i = 0, 1, \dots, k-1\} = \rho(A_0) = \mu. \tag{6.30}$$

Therefore, by (6.29) and (6.30), we obtain $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Proof of Theorem 6.1.8. If $p \geq q \geq 2$, we set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z|\right)^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1} |z|\right)^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (6.8), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1} |z|\right)^\mu} < \alpha < \gamma < \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1} |z|\right)^\mu}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Hence, we have

$$\begin{aligned} T(r, A_0) &> \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z|\right)^\mu \right\} \\ &> \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z|\right)^\mu \right\} \geq T(r, A_i), \quad (i = 1, 2, \dots, k-1) \end{aligned}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then, by using the same proof of Theorem 6.1.6, we get

$$\rho_{[p,q]}(f) = \infty \quad \text{and} \quad \rho_{[p+1,q]}(f) \geq \mu,$$

and by Lemma 6.2.1 we obtain the conclusion of Theorem 6.1.6.

If $p = q = 1$, we set

$$\alpha_0 = \liminf_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_0)}{|z|^\mu},$$

$$\alpha_i = \limsup_{|z| \rightarrow +\infty, z \in H} \frac{(k-1)T(r, A_i)}{|z|^\mu}, \quad (i = 1, 2, \dots, k-1).$$

By (6.9), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, \dots, k-1$. It yields for $i = 1, 2, \dots, k-1$

$$\frac{(k-1)T(r, A_i)}{|z|^\mu} < \alpha < \gamma < \frac{T(r, A_0)}{|z|^\mu} \quad (6.31)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By (5.31), we obtain

$$T(r, A_0) > \gamma |z|^\mu > \alpha |z|^\mu \geq (k-1)T(r, A_i), \quad (i = 1, 2, \dots, k-1) \quad (6.32)$$

as $|z| \rightarrow +\infty$ for $z \in H$. By applying Lemma 4.2.2 and substituting (6.27) and (6.32) into (6.18), we obtain

$$\gamma r^\mu \leq \alpha r^\mu + r^{\lambda_0 + \varepsilon} + O(\log T(r, f) + \log r)$$

for all z satisfying $|z| = r \in H_1 \setminus E_2$ as $|z| = r \rightarrow +\infty$. Noting that $\gamma > \alpha$ and $\lambda_0 + \varepsilon < \mu$, by the last inequality and combining with Lemma 5.2.2, we have for $d > 1$

$$(1 - o(1)) (\gamma - \alpha) r^\mu \leq O(\log T(dr, f) + \log dr) \quad (6.33)$$

for all z satisfying $|z| = r \in H_1$ as $|z| = r \rightarrow +\infty$. Therefore, from (6.33) we obtain

$$\rho(f) = \infty \text{ and } \rho_2(f) \geq \mu. \quad (6.34)$$

By Lemma 6.2.1, we get

$$\rho_2(f) \leq \max \{ \rho(A_i) : i = 0, 1, \dots, k-1 \} = \rho(A_0) = \mu. \quad (6.35)$$

Therefore, by (6.34) and (6.35), we obtain $\rho(f) = \infty$ and $\rho_2(f) = \mu$.

Proof of Theorem 6.1.9. Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (5.1).

First step. We consider the fixed points of f . Define the function g by setting

$$g(z) := f(z) - z, \quad z \in \mathbb{C}.$$

It follows from (5.1) that

$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_1g' + A_0g = -A_1 - zA_0 \quad (6.36)$$

and by Theorem 6.1.3 or Theorem 6.1.6, we have

$$\begin{aligned} \rho_{[p,q]}(g) &= \rho_{[p,q]}(f) = \infty, \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g) &= \bar{\lambda}_{[p+1,q]}(f - z). \end{aligned} \quad (6.37)$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$. Clearly $A_0 \not\equiv 0$. Then

$\lim_{|z| \rightarrow +\infty, z \in H} \left| \frac{A_1}{A_0} \right| = +\infty$ and by (6.13), we have

$$\begin{aligned} \left| \frac{A_1(z)}{A_0(z)} \right| &< \frac{\exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}}{\exp_p \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \\ &= \frac{1}{\exp \left\{ (1 - o(1)) \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} \right\}} \rightarrow 0 \end{aligned}$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then $\lim_{|z| \rightarrow +\infty, z \in H} \left| \frac{A_1}{A_0} \right| = 0$. It is easy to see the contradiction. Hence,

$-A_1 - zA_0 \neq 0$. Next by Proposition 1.3.6, we get

$$\max \left\{ \rho_{[p,q]}(A_i) \ (i = 0, 1, \dots, k-1), \rho_{[p,q]}(-A_1 - zA_0) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce, by using (6.36), (6.37) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g) = \rho_{[p,q]}(g) = \infty, \bar{\lambda}_{[p+1,q]}(g) = \rho_{[p+1,q]}(g) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f-z) = \bar{\lambda}_{[p,q]}(g) = \rho_{[p,q]}(g) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f-z) = \bar{\lambda}_{[p+1,q]}(g) = \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f) = \mu.$$

Second step. For the following proof, we use the principle of mathematical induction. Set $A_k(z) \equiv 1$, then

$$|A_k(z)| \leq \exp_p \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}$$

and equation (5.1) becomes

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0. \quad (6.38)$$

We consider the fixed points of $f^{(j)}(z)$ ($j = 1, 2, \dots$). Define the function g_1 by setting

$$g_1(z) := f'(z) - z, \ z \in \mathbb{C}.$$

Then, by Lemma 5.2.4 and (6.37), we obtain

$$\begin{aligned} \rho_{[p,q]}(g_1) &= \rho_{[p,q]}(f) = \infty, \rho_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_1) &= \bar{\lambda}_{[p+1,q]}(f' - z). \end{aligned} \quad (6.39)$$

Dividing both sides of (6.38) by A_0 , we obtain

$$\frac{A_k}{A_0}f^{(k)} + \frac{A_{k-1}}{A_0}f^{(k-1)} + \dots + \frac{A_1}{A_0}f' + f = 0. \quad (6.40)$$

It follows, by differentiating both sides of equation (6.40) that

$$\begin{aligned} \frac{A_k}{A_0}f^{(k+1)} + \left(\left(\frac{A_k}{A_0} \right)' + \frac{A_{k-1}}{A_0} \right) f^{(k)} + \dots + \left(\left(\frac{A_2}{A_0} \right)' + \frac{A_1}{A_0} \right) f'' \\ + \left(\left(\frac{A_1}{A_0} \right)' + 1 \right) f' = 0. \end{aligned} \quad (6.41)$$

Multiplying (6.41) by A_0 , we obtain

$$A_{k,1}f^{(k+1)} + A_{k-1,1}f^{(k)} + \cdots + A_{1,1}f'' + A_{0,1}f' = 0. \quad (6.42)$$

Substituting $f' = g_1 + z$ into (6.42), we obtain

$$A_{k,1}g_1^{(k)} + A_{k-1,1}g_1^{(k-1)} + \cdots + A_{1,1}g_1' + A_{0,1}g_1 = F_1, \quad (6.43)$$

where

$$A_{k,1} = A_k = 1, \quad A_{i,1} = A_0 \left(\left(\frac{A_{i+1}}{A_0} \right)' + \frac{A_i}{A_0} \right) \quad (i = 1, 2, \dots, k-1), \quad (6.44)$$

$$A_{0,1} = A_0 \left(\left(\frac{A_1}{A_0} \right)' + 1 \right), \quad (6.45)$$

$$F_1 = -(A_{1,1} + zA_{0,1}). \quad (6.46)$$

Next, we prove that $A_{0,1} \neq 0$ and $F_1 \neq 0$. Assume that $A_{0,1} \equiv 0$. So, by $A_0 \neq 0$ we get $\left(\frac{A_1}{A_0} \right)' + 1 \equiv 0$, and then $\frac{A_1}{A_0} = -z + C_1$, where C_1 is an arbitrary constant. Hence, we have $A_1 + (z - C_1)A_0 = 0$. Then, $f = z - C_1$ is a solution of (6.38) and $\rho_{[p,q]}(f) = 0 < \infty$. This contradicts (6.37), hence $A_{0,1} \neq 0$. Now, assume that $F_1 \equiv 0$. Then

$$A_{1,1} + zA_{0,1} \equiv 0. \quad (6.47)$$

By (6.42) and (6.47), we know that the function f such that $f' = z$ is a solution of equation (6.42). Hence equation (6.38) has a solution f_1 satisfying $f_1' = z$ and $\rho_{[p,q]}(f_1) = 0 < \infty$. This contradicts (6.37). Therefore, $F_1 \neq 0$. It follows by (6.44)-(6.46) and Proposition 1.3.6 that

$$\max \left\{ \rho_{[p,q]}(A_{i,1}) \quad (i = 0, 1, \dots, k), \rho_{[p,q]}(F_1) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce by using (6.39), (6.43) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g_1) = \rho_{[p,q]}(g_1) = \infty, \quad \bar{\lambda}_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(g_1) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f' - z) = \bar{\lambda}_{[p,q]}(g_1) = \rho_{[p,q]}(g_1) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f' - z) = \bar{\lambda}_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(g_1) = \rho_{[p+1,q]}(f) = \mu.$$

Set $g_2(z) = f''(z) - z$, $z \in \mathbb{C}$. Then, by Proposition 1.3.6 and (6.37), we obtain

$$\begin{aligned} \rho_{[p,q]}(g_2) &= \rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_2) &= \bar{\lambda}_{[p+1,q]}(f'' - z). \end{aligned} \quad (6.48)$$

Then, by using a similar discussion as in the case of the function g_1 , we can get

$$A_{k,2}f^{(k+2)} + A_{k-1,2}f^{(k+1)} + \cdots + A_{1,2}f^{(3)} + A_{0,2}f'' = 0 \quad (6.49)$$

and

$$A_{k,2}g_2^{(k)} + A_{k-1,2}g_2^{(k-1)} + \cdots + A_{1,2}g_2' + A_{0,2}g_2 = F_2, \quad (6.50)$$

where

$$A_{k,2} = A_k = 1, \quad A_{i,2} = A_{0,1} \left(\left(\frac{A_{i+1,1}}{A_{0,1}} \right)' + \frac{A_{i,1}}{A_{0,1}} \right) \quad (i = 1, 2, \dots, k-1), \quad (6.51)$$

$$A_{0,2} = A_{0,1} \left(\left(\frac{A_{1,1}}{A_{0,1}} \right)' + 1 \right), \quad (6.52)$$

$$F_2 = -(A_{1,2} + zA_{0,2}). \quad (6.53)$$

Next, we prove that $A_{0,2} \neq 0$ and $F_2 \neq 0$. Assume that $A_{0,2} \equiv 0$. So, by $A_{0,1} \neq 0$ we have $\left(\frac{A_{1,1}}{A_{0,1}} \right)' + 1 \equiv 0$, then $\frac{A_{1,1}}{A_{0,1}} = -z + C_2$, where C_2 is an arbitrary constant. Hence, we have $A_{1,1} + (z - C_2)A_{0,1} = 0$. Then, $f' = z - C_2$ is a solution of (6.42). Thus, equation (5.1) has a solution f satisfying $f' = z - C_2$ and $\rho_{[p,q]}(f) = 0 < \infty$. This contradicts (6.37), hence $A_{0,2} \neq 0$. Now, assume that $F_2 \equiv 0$. Then

$$A_{1,2} + zA_{0,2} \equiv 0. \quad (6.54)$$

By (6.49) and (6.54), we know that the function f such that $f'' = z$ is a solution of equation (6.49). Hence equation (5.1) has a solution f_2 satisfying $f_2'' = z$ and $\rho_{[p,q]}(f_2) = 0 < \infty$. This contradicts (6.37). Therefore, $F_2 \neq 0$. It follows by ((6.51)-(6.53)) and Proposition 1.3.6 that

$$\max \left\{ \rho_{[p,q]}(A_{i,2}) \quad (i = 0, 1, \dots, k), \rho_{[p,q]}(F_2) \right\} = \rho_{[p,q]}(A_0) < \infty.$$

We deduce by using (6.48), (6.50) and Lemma 5.2.4 that

$$\bar{\lambda}_{[p,q]}(g_2) = \rho_{[p,q]}(g_2) = \infty, \quad \bar{\lambda}_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(g_2) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f'' - z) = \bar{\lambda}_{[p,q]}(g_2) = \rho_{[p,q]}(g_2) = \rho_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f'' - z) = \bar{\lambda}_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \mu.$$

Now, assume that

$$\begin{aligned} A_{0,s} &\neq 0, \\ \bar{\lambda}_{[p,q]}(f^{(s)} - z) &= \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(s)} - z) &= \rho_{[p+1,q]}(f) = \mu \end{aligned} \quad (6.55)$$

for all $s = 0, 1, \dots, j-1$, and we prove that for $s = j$ we have (6.55) holds. Set $g_j(z) = f^{(j)}(z) -$

$z, z \in \mathbb{C}$. Then, by using (6.37) and Proposition 1.3.6, we obtain

$$\begin{aligned}\rho_{[p,q]}(g_j) &= \rho_{[p,q]}(f) = \infty, \rho_{[p+1,q]}(g_j) = \rho_{[p+1,q]}(f) = \mu, \\ \bar{\lambda}_{[p+1,q]}(g_j) &= \bar{\lambda}_{[p+1,q]}(f^{(j)} - z).\end{aligned}\tag{6.56}$$

By following the same procedure as before, we have

$$A_{k,j}f^{(k+j)} + A_{k-1,j}f^{(k+j-1)} + \dots + A_{1,j}f^{(j+1)} + A_{0,j}f^{(j)} = 0\tag{6.57}$$

and

$$A_{k,j}g_j^{(k)} + A_{k-1,j}g_j^{(k-1)} + \dots + A_{1,j}g_j' + A_{0,j}g_j = F_j,\tag{6.58}$$

where

$$\begin{aligned}A_{k,j} &= A_k = 1, \quad A_{i,j} = A_{0,j-1} \left(\left(\frac{A_{i+1,j-1}}{A_{0,j-1}} \right)' + \frac{A_{i,j-1}}{A_{0,j-1}} \right) \quad (i = 1, 2, \dots, k-1), \\ A_{0,j} &= A_{0,j-1} \left(\left(\frac{A_{1,j-1}}{A_{0,j-1}} \right)' + 1 \right) \neq 0 \quad (A_{0,0} = A_0, \quad A_{1,0} = A_1), \\ F_j &= -(A_{1,j} + zA_{0,j}) \neq 0.\end{aligned}$$

We deduce by applying Lemma 5.2.4 in (6.58) that

$$\begin{aligned}\bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p,q]}(g_j) = \rho_{[p,q]}(g_j) = \rho_{[p,q]}(f^{(j)}) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(g_j) = \rho_{[p+1,q]}(g_j) \\ &= \rho_{[p+1,q]}(f^{(j)}) = \mu \quad (j = 1, 2, \dots).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\bar{\lambda}_{[p,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \infty, \\ \bar{\lambda}_{[p+1,q]}(f^{(j)} - z) &= \bar{\lambda}_{[p+1,q]}(f - z) = \rho_{[p+1,q]}(f) = \mu \quad (j = 1, 2, \dots).\end{aligned}$$

Proofs of Theorem 6.1.10. Suppose that every solution f of equation (5.1) not being identically equal to 0. By applying Theorem 6.1.6 or Theorem 6.1.8, we get

$$\rho_{[p,q]}(f) = \infty, \quad \rho_{[p+1,q]}(f) = \mu.$$

Now, we prove that $-A_1 - zA_0 \neq 0$. Assume that $-A_1 - zA_0 \equiv 0$, then we can easily obtain

$$\begin{aligned}T(r, A_1) &= T(r, -zA_0) \leq T(r, A_0) + T(r, z), \\ T(r, A_0) &= T\left(r, \frac{A_1}{-z}\right) \leq T(r, A_1) + T(r, z) + O(1).\end{aligned}\tag{6.59}$$

It follows from (6.59) that

$$1 - \frac{T(r, z) + O(1)}{T(r, A_0)} \leq \frac{T(r, A_1)}{T(r, A_0)} \leq 1 + \frac{T(r, z)}{T(r, A_0)}. \quad (6.60)$$

By following the same reasoning as in the proof of Theorem 6.1.6 or Theorem 6.1.8, we have

$$T(r, A_0) > \exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\} > \exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\} \geq T(r, A_1) \quad (6.61)$$

as $|z| = r \rightarrow +\infty$ for $z \in H$. By using (6.61), we have

$$\frac{T(r, z)}{T(r, A_0)} \leq \frac{T(r, z)}{\exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \rightarrow 0 \quad (6.62)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then, by (6.60) and (6.62), we get

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} = 1. \quad (6.63)$$

On the other hand, we have for $p = q = 1$

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\alpha}{\gamma},$$

which implies

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} \leq \frac{\alpha}{\gamma} < 1 \quad (6.64)$$

and for $p \geq q \geq 2$

$$\frac{T(r, A_1)}{T(r, A_0)} < \frac{\exp_{p-1} \left\{ \alpha \left(\log_{q-1} |z| \right)^\mu \right\}}{\exp_{p-1} \left\{ \gamma \left(\log_{q-1} |z| \right)^\mu \right\}} \rightarrow 0 \quad (6.65)$$

as $|z| \rightarrow +\infty$ for $z \in H$. It follows by (6.64) and (6.65) that

$$\lim_{|z| \rightarrow +\infty, z \in H} \frac{T(r, A_1)}{T(r, A_0)} \neq 1. \quad (6.66)$$

Obviously, (6.63) contradicts with (6.66). Hence, $-A_1 - zA_0 \not\equiv 0$. Clearly, $A_0 \not\equiv 0$. We can get the conclusion of Theorem 6.1.10, by reasoning in the same way as we did in the proof of Theorem 6.1.9.

Conclusion

In this thesis, we have focused on the study of several problems related to the growth order of solutions to complex functional equations. Throughout this work, we have made extensive use of Nevanlinna theory in the complex plane $\mathbb{C}$, particularly, the First Main Theorem of Nevanlinna, the lemma on the logarithmic derivative, and various properties of the Nevanlinna characteristic of $f(z + \eta)$ where $\eta \in \mathbb{C}$.

This study has concentrated on linear differential-difference equations with entire and meromorphic coefficients, as well as on linear differential equations whose coefficients are entire or meromorphic functions of $[p, q]$ -order in the whole complex plane. Although the thesis has addressed several questions concerning the growth and value distribution of solutions to such linear differential equations, it also opens the way for numerous further investigations, including the following :

Q1: Can results analogous to those in Chapters 2–4 be obtained for the linear differential equations

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

and

$$A_n(z)f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

when the coefficients are entire functions of the form

$$A_j(z) = P_j(z) \exp\{h_j(z)\}, \quad j = 0, \dots, n,$$

where $h_j(z)$ is a polynomial and $P_j(z)$ is an entire function?

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