Hindawi Publishing Corporation
Journal of Complex Analysis
Volume 2013, Article ID 253168, 5 pages
http://dx.doi.org/10.1155/2013/253168

Research Article

Some Properties of Solutions of Second-Order Linear Differential Equations

Zinelaâbidine Latreuch and Benharrat Belaïdi

Laboratory of Pure and Applied Mathematics, Department of Mathematics, University of Mostaganem (UMAB), BP 227, 27000 Mostaganem, Algeria

Correspondence should be addressed to Benharrat Belaïdi; belaidi@univ-mosta.dz

Received 13 August 2012; Accepted 25 September 2012

Academic Editor: Rabha W. Ibrahim

We study the growth and oscillation of $g_f = d_1 f_1 + d_2 f_2$, where d_1 and d_2 are entire functions of finite order not all vanishing identically and f_1 and f_2 are two linearly independent solutions of the linear differential equation $f'' + A(z) f = 0$.

1. Introduction and Main Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [1–4]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote, respectively, the exponents of convergence of the zero sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f.

Definition 1 (see [4, 5]). Let f be a meromorphic function. Then the hyperorder of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}. \tag{1}$$

Definition 2 (see [4, 5]). Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N(r,1/f)}{\log r}, \tag{2}$$

where $N(r,1/f)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$. Similarly, the hyperexponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r,1/f)}{\log r}, \tag{3}$$

where $\overline{N}(r,1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

Suppose that f_1 and f_2 are two linearly independent solutions of the complex linear differential equation

$$f'' + A(z) f = 0, \tag{4}$$

and the polynomial of solutions

$$g_f = d_1 f_1 + d_2 f_2, \tag{5}$$

where A and d_j $(j = 1, 2)$ are entire functions of finite order in the complex plane. It is clear that if d_j $(j = 1, 2)$ are complex numbers or $d_1 = c d_2$ where c is a complex number, then g_f is a solution of (4) or has the same properties of the solutions.

It is natural to ask what can be said about the properties of g_f in the case when $d_1 \neq c d_2$ where c is a complex number and under what conditions g_f keeps the same properties of the solutions of (4).

In [6], Chen studied the fixed points and hyper-order of solutions of second-order linear differential equations with entire coefficients and obtained the following results.

Theorem A (see [6]). *For all nontrivial solutions f of (4) the following hold.*

(i) *If A is a polynomial with* $\deg A = n \geq 1$, *then one has*

$$\lambda(f - z) = \rho(f) = \frac{n+2}{2}. \tag{6}$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then one has*

$$\begin{aligned} \lambda(f - z) &= \rho(f) = \infty, \\ \lambda_2(f - z) &= \rho_2(f) = \rho(A). \end{aligned} \tag{7}$$

Before we state our results we define h and ψ by

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix},$$

$$\psi(z) = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi, \tag{8}$$

where $\varphi \not\equiv 0$ is entire function of finite order and

$$\begin{aligned} \phi_2 &= \frac{3d_2^2 d_1'' - 3d_1 d_2 d_2''}{h}, \\ \phi_1 &= \frac{2d_1 d_2 d_2' A + 6d_2 d_1' d_2'' - 6d_2 d_2' d_1'' - 2d_2^2 d_1' A}{h}, \\ \phi_0 &= \frac{2d_2 d_1' d_2''' - 2d_1 d_2' d_2''' - 3d_1 d_2 d_2'' A - 3d_2 d_1'' d_2''}{h} \\ &\quad + \frac{2d_1 d_2 d_2' A' - 4d_2 d_1' d_2' A - 6d_1' d_2' d_2'' + 3d_1 \left(d_2''\right)^2}{h} \\ &\quad + \frac{4d_1 \left(d_2'\right)^2 A + 3d_2^2 d_1'' A + 6\left(d_2'\right)^2 d_1'' - 2d_2^2 d_1' A'}{h}. \end{aligned} \tag{9}$$

The subject of this paper is to study the controllability of solutions of the differential equation (4). In fact, we study the growth and oscillation of $g_f = d_1 f_1 + d_2 f_2$ where f_1 and f_2 are two linearly independent solutions of (4) and d_1 and d_2 are entire functions of finite order not all vanishing identically and satisfying $d_1 \neq c d_2$ where c is a complex number, and we obtain the following results.

Theorem 3. *Let $A(z)$ be a transcendental entire function of finite order. Let $d_j(z)$ $(j = 1, 2)$ be finite-order entire functions that are not all vanishing identically such that* $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. *If f_1 and f_2 are two linearly independent solutions of* (4), *then the polynomial of solutions* (5) *satisfies*

$$\begin{aligned} \rho\left(g_f\right) &= \rho\left(f_j\right) = \infty \quad (j = 1, 2), \\ \rho_2\left(g_f\right) &= \rho_2\left(f_j\right) = \rho(A) \quad (j = 1, 2). \end{aligned} \tag{10}$$

Theorem 4. *Under the hypotheses of Theorem 3, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two linearly independent solutions of* (4), *then the polynomial of solutions* (5) *satisfies*

$$\begin{aligned} \overline{\lambda}\left(g_f - \varphi\right) &= \lambda\left(g_f - \varphi\right) = \rho\left(f_j\right) = \infty \quad (j = 1, 2), \\ \overline{\lambda}_2\left(g_f - \varphi\right) &= \lambda_2\left(g_f - \varphi\right) = \rho_2\left(f_j\right) = \rho(A) \quad (j = 1, 2). \end{aligned} \tag{11}$$

Theorem 5. *Let $A(z)$ be a polynomial of* $\deg A = n$. *Let $d_j(z)$ $(j = 1, 2)$ be finite-order entire functions that are not all vanishing identically such that $h \not\equiv 0$ and* $\max\{\rho(d_1), \rho(d_2)\} < (n+2)/2$. *If f_1 f_2 are two linearly independent solutions of* (4), *then the polynomial of solutions* (5) *satisfies*

$$\rho\left(g_f\right) = \rho\left(f_j\right) = \frac{n+2}{2} \quad (j = 1, 2). \tag{12}$$

Theorem 6. *Under the hypotheses of Theorem 5, let $\varphi(z) \not\equiv 0$ be an entire function with $\rho(\varphi) < (n+2)/2$ such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two linearly independent solutions of* (4), *then the polynomial of solutions* (5) *satisfies*

$$\overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \frac{n+2}{2}. \tag{13}$$

2. Auxiliary Lemmas

Lemma 7 (see [7, 8]). *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite-order meromorphic functions. If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{14}$$

with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies

$$\begin{aligned} \overline{\lambda}(f) &= \lambda(f) = \rho(f) = +\infty, \\ \overline{\lambda}_2(f) &= \lambda_2(f) = \rho_2(f) = \rho. \end{aligned} \tag{15}$$

Here, we give a special case of the result due to Cao et al. in [9].

Lemma 8. *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite-order meromorphic functions. If f is a meromorphic solution of* (14) *with*

$$\max\left\{\rho\left(A_j\right) (j = 0, 1, \ldots, k-1), \rho(F)\right\} < \rho(f) < +\infty, \tag{16}$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f). \tag{17}$$

3. Proofs of the Theorems

Proof of Theorem 3. Suppose that f_1 and f_2 are two linearly independent solutions of (4). Then by Theorem A, we have

$$\begin{aligned} \rho(f_1) &= \rho(f_2) = \infty, \\ \rho_2(f_1) &= \rho_2(f_2) = \rho(A). \end{aligned} \tag{18}$$

Suppose that $d_1 = cd_2$, where c is a complex number. Then, by (5) we obtain

$$g_f = cd_2 f_1 + d_2 f_2 = (cf_1 + f_2) d_2. \tag{19}$$

Since $f = cf_1 + f_2$ is a solution of (4) and $\rho(d_2) < \rho(A)$, then we have

$$\begin{gathered} \rho(g_f) = \rho(cf_1 + f_2) = \infty, \\ \rho_2(g_f) = \rho_2(cf_1 + f_2) = \rho(A). \end{gathered} \tag{20}$$

Suppose now that $d_1 \not\equiv cd_2$ where c is a complex number. Differentiating both sides of (5), we obtain

$$g_f' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2'. \tag{21}$$

Differentiating both sides of (21), we obtain

$$g_f'' = d_1'' f_1 + 2d_1' f_1' + d_1 f_1'' + d_2'' f_2 + 2d_2' f_2' + d_2 f_2''. \tag{22}$$

Substituting $f_j'' = -Af_j$ $(j = 1, 2)$ into (22), we obtain

$$g_f'' = \left(d_1'' - d_1 A\right) f_1 + 2d_1' f_1' + \left(d_2'' - d_2 A\right) f_2 + 2d_2' f_2'. \tag{23}$$

Differentiating both sides of (23) and by substituting $f_j'' = -Af_j$ $(j = 1, 2)$, we obtain

$$\begin{aligned} g_f''' = {} & \left(d_1''' - 3d_1' A - d_1 A'\right) f_1 + \left(d_1'' - d_1 A + 2d_1''\right) f_1' \\ & + \left(d_2''' - 3d_2' A - d_2 A'\right) f_2 + \left(d_2'' - d_2 A + 2d_2''\right) f_2'. \end{aligned} \tag{24}$$

By (5), (21), (23), and (24) we have

$$\begin{aligned} g_f &= d_1 f_1 + d_2 f_2, \\ g_f' &= d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2', \\ g_f'' &= \left(d_1'' - d_1 A\right) f_1 + 2d_1' f_1' + \left(d_2'' - d_2 A\right) f_2 + 2d_2' f_2', \\ g_f''' &= \left(d_1''' - 3d_1' A - d_1 A'\right) f_1 + \left(d_1'' - d_1 A + 2d_1''\right) f_1' \\ & \quad + \left(d_2''' - 3d_2' A - d_2 A'\right) f_2 + \left(d_2'' - d_2 A + 2d_2''\right) f_2'. \end{aligned} \tag{25}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations we obtain

$$\begin{aligned} h &= \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix} \\ &= \left(4d_1^2\left(d_2'\right)^2 + 4d_2^2\left(d_1'\right)^2 - 8d_1 d_2 d_1' d_2'\right) A + 2d_1 d_2 d_1' d_2''' \\ & \quad + 2d_1 d_2 d_2' d_1''' - 6d_1 d_2 d_1'' d_2'' - 6d_1 d_1' d_2' d_2'' - 6d_2 d_1' d_2' d_1'' \\ & \quad + 6d_1\left(d_2'\right)^2 d_1'' + 6d_2\left(d_1'\right)^2 d_2'' - 2d_2^2 d_1' d_1''' - 2d_1^2 d_2' d_2''' \\ & \quad + 3d_1^2\left(d_2''\right)^2 + 3d_2^2\left(d_1''\right)^2. \end{aligned} \tag{26}$$

To show that $4d_1^2(d_2')^2 + 4d_2^2(d_1')^2 - 8d_1 d_2 d_1' d_2' \not\equiv 0$, we suppose that

$$d_1^2\left(d_2'\right)^2 + d_2^2\left(d_1'\right)^2 - 2d_1 d_2 d_1' d_2' = 0. \tag{27}$$

Dividing both sides of (27) by $(d_1 d_2)^2$, we obtain

$$\left(\frac{d_2'}{d_2}\right)^2 + \left(\frac{d_1'}{d_1}\right)^2 - 2\frac{d_1'}{d_1}\frac{d_2'}{d_2} = 0 \tag{28}$$

equivalent to

$$\left(\frac{d_1'}{d_1} - \frac{d_2'}{d_2}\right)^2 = 0, \tag{29}$$

which implies that $d_1 = cd_2$ where c is a complex number and this is a contradiction. Since $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$ and $4d_1^2(d_2')^2 + 4d_2^2(d_1')^2 - 8d_1 d_2 d_1' d_2' \not\equiv 0$, we can deduce from (26) that

$$\rho(h) = \rho(A) > 0. \tag{30}$$

Hence $h \not\equiv 0$. By Cramer's method we have

$$\begin{aligned} f_1 &= \frac{\begin{vmatrix} g_f & 0 & d_2 & 0 \\ g_f' & d_1 & d_2' & d_2 \\ g_f'' & 2d_1' & d_2'' - d_2 A & 2d_2' \\ g_f''' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}}{h} \\ &= \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g_f^{(3)} + \phi_2 g_f'' + \phi_1 g_f' + \phi_0 g_f, \end{aligned} \tag{31}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions of finite order defined in (9). Suppose now that $\rho(g_f) < \infty$, then by (31) we obtain $\rho(f_1) < \infty$, which is a contradiction, hence $\rho(g_f) = \infty$. By (5) we have $\rho_2(g_f) \leq \rho(A)$. Suppose that $\rho_2(g_f) < \rho(A)$, then by (31) we obtain $\rho_2(f_1) < \rho(A)$, which is a contradiction. Hence $\rho_2(g_f) = \rho(A)$. □

Proof of Theorem 4. By Theorem 3 we have $\rho(g_f) = \infty$ and $\rho_2(g_f) = \rho(A)$. Set $w(z) = d_1 f_1 + d_2 f_2 - \varphi$. Since $\rho(\varphi) < \infty$, then we have $\rho(w) = \rho(g_f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho(A)$. In order to prove that $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$, $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho(A)$ we need to prove only that $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho(A)$. By $g_f = w + \varphi$ we get from (31)

$$f_1 = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(3)} + \phi_2 w'' + \phi_1 w' + \phi_0 w + \psi, \tag{32}$$

where

$$\psi = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi. \tag{33}$$

Substituting (32) into (4), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(5)} + \sum_{j=0}^{4} \beta_j w^{(j)} = -\left(\psi'' + A\psi\right) = B, \tag{34}$$

where β_j $(j = 0, \dots, 4)$ are meromorphic functions of finite order. Since $\psi \not\equiv 0$ and $\rho(\psi) < \infty$, it follows that ψ is not a solution of (4), which implies that $B \not\equiv 0$. Then by applying Lemma 7 we obtain (11). □

Proof of Theorem 5. Suppose that f_1 and f_2 are two linearly independent solutions of (4). Then by Theorem A

$$\rho(f_1) = \rho(f_2) = \frac{n+2}{2}. \tag{35}$$

By the same reasoning as in Theorem 3, we have

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}. \tag{36}$$

Since $h \not\equiv 0$ and by Cramer's method we have

$$\begin{aligned} f_1 &= \frac{\begin{vmatrix} g_f & 0 & d_2 & 0 \\ g_f' & d_1 & d_2' & d_2 \\ g_f'' & 2d_1' & d_2'' - d_2 A & 2d_2' \\ g_f''' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}}{h} \\ &= \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g_f^{(3)} + \phi_2 g_f'' + \phi_1 g_f' + \phi_0 g_f, \end{aligned} \tag{37}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < (n+2)/2$ $(j = 0, 1, 2)$ defined in (9). By (5) we have $\rho(g_f) \le (n+2)/2$. Suppose that $\rho(g_f) < (n+2)/2$, then by (37) we obtain $\rho(f_1) < (n+2)/2$, which is a contradiction. Hence, $\rho(g_f) = (n+2)/2$. □

Proof of Theorem 6. By Theorem 5 we have $\rho(g_f) = (n+2)/2$. Set $w(z) = d_1 f_1 + d_2 f_2 - \varphi$. Since $\rho(\varphi) < (n+2)/2$, then we have $\rho(w) = \rho(g_f) = (n+2)/2$. In order to prove that $\overline{\lambda}(g_f - \varphi) = (n+2)/2$, we need to prove only that $\overline{\lambda}(w) = (n+2)/2$. By $g_f = w + \varphi$ we get from (37)

$$\begin{aligned} f_1 = {} & \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(3)} \\ & + \phi_2 w'' + \phi_1 w' + \phi_0 w + \psi, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \psi = {} & \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} \\ & + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi. \end{aligned} \tag{39}$$

Substituting (38) into (4), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(5)} + \sum_{j=0}^{4} \beta_j w^{(j)} = -\left(\psi'' + A\psi\right) = B, \tag{40}$$

where β_j $(j = 0, \dots, 4)$ are meromorphic functions with $\rho(\beta_j) < (n+2)/2$. Since $\psi \not\equiv 0$ and $\rho(\psi) < (n+2)/2$, it follows that ψ is not a solution of (4), which implies that $B \not\equiv 0$. Then by applying Lemma 8 we obtain (13). □

Acknowledgments

The authors would like to thank the referees for their helpful remarks and suggestions to improve the paper. This research is supported by Agence Nationale pour le Développement de la Recherche Universitaire (ANDRU) and University of Mostaganem (UMAB), (PNR Project Code 8/u27/3144).

References

[1] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, UK, 1964.

[2] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, vol. 15 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1993.

[3] R. Nevanlinna, *Eindeutige analytische Funktionen*, vol. 46 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, Berlin, Germany, 2nd edition, 1974.

[4] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[5] Z. X. Chen and C. C. Yang, "Some further results on the zeros and growths of entire solutions of second order linear differential equations," *Kodai Mathematical Journal*, vol. 22, no. 2, pp. 273–285, 1999.

[6] Z. X. Chen, "The fixed points and hyper-order of solutions of second order complex differential equations," *Acta Mathematica Scientia A*, vol. 20, no. 3, pp. 425–432, 2000 (Chinese).

[7] B. Belaïdi, "Growth and oscillation theory of solutions of some linear differential equations," *Matematicki Vesnik*, vol. 60, no. 4, pp. 233–246, 2008.

[8] Z. X. Chen, "Zeros of meromorphic solutions of higher order linear differential equations," *Analysis*, vol. 14, no. 4, pp. 425–438, 1994.

[9] T. B. Cao, Z. X. Chen, X. M. Zheng, and J. Tu, "On the iterated order of meromorphic solutions of higher order linear differential equations," *Annals of Differential Equations*, vol. 21, no. 2, pp. 111–122, 2005.