

Analysis 1

Course

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Foreword

This manual brings together the analysis 1 course intended for an official program taught to first-year mathematics and computer science students. These are the courses that I presented at the Faculty of Exact Sciences and Computer Science of Abdelhamid Ibn Badis University of Mostaganem in the first semester during the academic years 2023 – 2025, for first-year students. After reviewing and reorganizing the content of these courses and adding some necessary concepts which, in my opinion, fit with the content of the program. This manual was divided into seven chapters.

The first chapter lays the foundation for understanding all the topics in the following chapters. Initially, we focused on basic logical concepts such as implication, converse of an implication, and the contrapositive of an implication. We also presented some patterns proof with simple examples, aiming to simplify them as much as possible so that students could become familiar with and understand these methods.

The second chapter deals with some concepts that we will use later in certain mathematical constructions. In the latter we have defined the concepts : the algebraic properties of the commutative field $(\mathbb{R}, +, \times)$ and that this field is totally ordered, intervals, absolute value with these properties, supremum and infimum of a subset of $\mathbb{R}$, characterization of the supremum and infimum, the function of the whole part of a real number and we give the Archimedean property finally the density of the set of rational numbers in $\mathbb{R}$.

The third chapter is devoted to the study of sequences including the definition of a finite limit of a sequence, convergent sequences, bounded sequences, operations on convergent sequences, monotone sequences and subsequences, recursive sequences and sequences defined by a function. Cauchy sequences and their properties are treated in detail. The chapter concludes with divergent sequences and their properties.

The fourth chapter is also important, in which we define a real function of a real variable, the domain of a function, the parity of a function, periodic functions, monotonic functions, and several other definitions such as local maximum and minimum values, as well as bounded functions. We concluded this chapter by defining a finite and infinite limit at a finite point and the limit at infinity, along with their properties.

In the fifth chapter, we dealt with the concept of the continuity of a real function and its properties. Several fundamental theorems were presented with their proofs. We concluded this chapter with the concepts of continuity by continuity, uniform continuity, and Lipschitzian functions.

The last two chapters deal with the differentiability of a real function at a point and operations on differentiable functions. The fundamental theorems on differentiable functions are given in detail. Elementary functions are treated in detail in the last chapter.

1

Preliminaries

1.1 Notations and Preliminaries

If an element x is in a set A , we write

$$x \in A,$$

and say that x is a member of A , or that x belongs to A . If x is not in A , we write

$$x \notin A.$$

If every element of a set A also belongs to a set B , we say that A is a subset of B and write

$$A \subseteq B \text{ or } B \supseteq A.$$

We say that a set A is a proper subset of a set B if $A \subseteq B$, but there is at least one element of B that is not in A . In this case we sometimes write

$$A \subset B.$$

Definition 1.1. *Two sets A and B are said to be equal, and we write $A = B$, if they contain the same elements.*

Remarks 1.1.

1. *To prove that the sets A and B are equal, we must show that*

$$A \subseteq B \text{ and } B \subseteq A.$$

2. *A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set. If P denotes a property that is meaningful and unambiguous for elements of a set A , then we write*

$$\{x \in A : P(x)\}.$$

3. *A empty set is a set that contains no elements. It is denoted as $\emptyset$ or $\{ \}$.*

1.2 Special sets

Special sets are used throughout this course and are denoted by standard symbols as follows :

1. The set of natural numbers, denoted $\mathbb{N}$, is given by :

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

2. The set of integer numbers, denoted $\mathbb{Z}$, is given by :

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

3. The set of rational numbers, denoted $\mathbb{Q}$, is given by :

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

4. The set of real numbers is $\mathbb{R}$.

Remark 1.1. *The set of real numbers $\mathbb{R}$ is of great importance in the analysis, moreover it is fundamental and it will be studied throughout chapter 2.*

Examples 1.1. 1. *The set*

$$A = \{n \in \mathbb{N} : n^2 - 5n + 6 = 0\}$$

is equal to the set

$$\{1, 2\},$$

and we can write

$$A = \{1, 2\}.$$

2. *A natural number n is even if it has the form $n = 2k$ for some $k \in \mathbb{N}$. The set of even natural numbers can be written as*

$$\{2k : k \in \mathbb{N}\}.$$

Similarly, the set of odd natural numbers can be written

$$\{2k + 1 : k \in \mathbb{N}\}.$$

Remark 1.2. *Every natural number is either even or odd, and no natural number is both even and odd.*

3. *We have for example*

$$\emptyset = \{n \in \mathbb{N} : n^2 = 5\}.$$

1.3 Patterns of proof

In this part, we will give the different demonstration rules to demonstrate insertions of the type : "if p then q ", which we call implication or conditional, noted :

$$p \Rightarrow q,$$

and it reads : p implies q .

The negation of the implication $p \Rightarrow q$ is the expression " p and not q ". "not q " is noted $\bar{q}$.

The expression " $(p \Rightarrow q)$ and $(q \Rightarrow p)$ " is called logical equivalence or biconditional , noted

$$p \Leftrightarrow q,$$

and it reads : p is equivalent to q or p if and only if q .

Examples 1.2. a. *let n be a natural number.*

$$n^2 \text{ is even} \Rightarrow n \text{ is even.}$$

The negation of this implication is :

$$n^2 \text{ is even and } n \text{ is odd}$$

b. *The two implications " $p \Rightarrow q$ and " $\bar{q} \Rightarrow \bar{p}$ are equivalent since they have the same negation.*

We write :

$$(p \Rightarrow q) \Leftrightarrow (\bar{q} \Rightarrow \bar{p}).$$

1.3.1 Direct reasoning

This method is based on the following rule :

Rule 1.1. *To establish $p \Rightarrow q$, we assume that p is true and we show that q is true.*

Example 1.1. *Let a, b, c and d be any elements of $\mathbb{R}^*$, such as*

$$5a - 7b \neq 0 \text{ and } 5c - 7d \neq 0.$$

We want to show that :

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{2a + 3b}{5a - 7b} = \frac{2c + 3d}{5c - 7d}. \quad (1.1)$$

We use direct reasoning, so we assume that,

$$\frac{a}{b} = \frac{c}{d}.$$

Then we can write :

$$\frac{2a + 3b}{5a - 7b} = \frac{2\frac{a}{b} + 3}{5\frac{a}{b} - 7} = \frac{\frac{2c+3d}{d}}{\frac{5c-7d}{d}} = \frac{2c + 3d}{5c - 7d}.$$

From where (1.1).

1.3.2 Proof by Contrapositive

The principle of demonstration by the contrapositive is based on the following rule :

Rule 1.2. *To show that $p \implies q$ it will suffice to prove $\bar{q} \implies \bar{p}$.*

Example 1.2. *Let $n \in \mathbb{N}$. We want to show that :*

$$n^2 \text{ is even} \implies n \text{ is even.} \quad (1.2)$$

It will suffice to show that :

$$n \text{ is an odd number} \implies n^2 \text{ is an odd number.}$$

Let n be an odd number. Then there exists $k \in \mathbb{N}$, such that $n = 2k + 1$. So we can write

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2\ell + 1 \text{ where } \ell = (2k^2 + 2k) \in \mathbb{N}.$$

Thus n^2 is an odd number. Hence (1.2).

1.3.3 Proof by absurd

Absurd reasoning depends on the following rule :

Rule 1.3. *To show that $p \implies q$, we assume that p is true and q is false, and we show that this leads to a contradiction.*

Example 1.3. *Let x and y two real positives. We want to show that :*

$$\frac{3x + 1}{2y + 5} = \frac{3y + 1}{2x + 5} \implies x = y. \quad (1.3)$$

We consider that

$$\frac{3x + 1}{2y + 5} = \frac{3y + 1}{2x + 5} \text{ and } x \neq y.$$

Let x and y two real positives. Firstly, we have

$$\begin{aligned} \frac{3x + 1}{2y + 5} = \frac{3y + 1}{2x + 5} &\Leftrightarrow (3x + 1)(2x + 5) = (2y + 5)(3y + 1) \\ &\Leftrightarrow 6x^2 + 17x + 5 = 6y^2 + 17y + 5 \\ &\Leftrightarrow 6(x^2 - y^2) + 17(x - y) = 0 \\ &\Leftrightarrow (x - y)(6x + 6y + 17) = 0 \\ &\Leftrightarrow x = y, \text{ (because } 6x + 6y + 17 > 0\text{)}. \end{aligned}$$

Which is absurd to the hypothesis ($x \neq y$). Hence (1.3).

1.3.4 Principle of Mathematical Induction

Principle 1.1. Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a statement for each natural number $n \geq n_0$. Suppose that :

- (1.) The statement $P(n_0)$ is true.
 - (2.) For all $k \geq n_0$, the truth of $P(k)$ implies the truth of $P(k+1)$.
- Then $P(n)$ is true for all $n \geq n_0$.

Example 1.4. [Bernoulli Inequality] Let $x \in \mathbb{R}$ with $x > -1$. We want to show that :

$$\text{for all } n \in \mathbb{N}, (1+x)^n \geq 1+nx.$$

We will prove this with induction.

When $n = 1$ the statement is $(1+x)^1 \geq 1+1.x$, and this is true because both sides equal $1+x$.

We assume that, for some $k \geq 1$, the statement $(1+x)^k \geq 1+kx$ is true. From this we need to prove

$$(1+x)^{k+1} \geq 1+(k+1)x.$$

Now, $(1+x)$ is positive because $x > -1$, so we can multiply both sides of $(1+x)^k \geq (1+kx)$ by $(1+x)$, and we obtain

$$\begin{aligned} (1+x)^{k+1} &\geq (1+kx)(1+x) \\ &\geq kx^2 + (k+1)x + 1 \\ &\geq (k+1)x + 1. \end{aligned}$$

Because the above term kx^2 is positive. Thus we get

$$(1+x)^{k+1} \geq 1+(k+1)x.$$

It follows by induction that

$$\text{for all } n \in \mathbb{N}, (1+x)^n \geq 1+nx.$$

1.3.5 Rational Numbers

On the set $\mathbb{Q}$ of rational numbers there are two binary operations, denoted by $+$ and $\cdot$ called addition and multiplication, respectively. These operations satisfy the following properties :

1. for all $a, b \in \mathbb{Q}$, $a+b \in \mathbb{Q}$, (addition is a law of composition on $\mathbb{Q}$);
2. for all $a, b \in \mathbb{Q}$, $a+b = b+a$, (commutative property of addition);
3. for all $a, b, c \in \mathbb{Q}$, $(a+b)+c = a+(b+c)$, (associative property of addition);
4. there exists an element $0 \in \mathbb{Q}$ such that : for all $a \in \mathbb{Q}$, $0+a = a$ and $a+0 = a$ (0 is an identity element or a zero element);

5. for each $a \in \mathbb{Q}$, there exists an element $-a \in \mathbb{Q}$, such that :

$$a + (-a) = 0 \text{ and } (-a) + a = 0,$$

($-a$ is called a inverse of a or the negative element of a);

6. for all $a, b \in \mathbb{Q}$, $a.b \in \mathbb{Q}$, (multiplication is a law of composition on $\mathbb{Q}$);
 7. for all $a, b \in \mathbb{Q}$, $a.b = b.a$, (commutative property of multiplication);
 8. for all $a, b, c \in \mathbb{Q}$, $(a.b).c = a.(b.c)$, (associative property of multiplication);
 9. there exists an element $1 \in \mathbb{Q}$ distinct from 0 such that :

$$\text{for all } a \in \mathbb{Q}, 1.a = a \text{ and } a.1 = a;$$

(1 is an identity element or a unit element),

10. for each $a \neq 0$ in $\mathbb{Q}$, there exists an element $\frac{1}{a} \in \mathbb{Q}$, such that :

$$a.\frac{1}{a} = 1 \text{ and } \frac{1}{a}.a = 1,$$

($\frac{1}{a}$ is called a inverse of a or the reciprocal element of a);

11. for all $a, b, c \in \mathbb{Q}$, $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$, (distributive property of multiplication over addition).

Definition 1.2. The set $\mathbb{Q}$ equipped by the two laws of composition $+$ and $.$, satisfying above axioms, is called a field, noted by $(\mathbb{Q}, +, .)$.

1.4 The Order Properties of $\mathbb{Q}$

Throughout this part, we designate by $\mathbb{Q}_+$ the set of the nonnegative rational numbers and by $\mathbb{Q}_-$ the set of the nonpositive rational numbers. The set of the positive or strictly positive rational numbers is denoted by $\mathbb{Q}_+^*$ and the set of the negative or strictly negative rational numbers is denoted by $\mathbb{Q}_-^*$.

We verify that on $\mathbb{Q}$ the relation $\leq$, reads from left to right "less than or equal to" and from right to left "greater than or equal to", defined by :

$$\text{for all } a, b \in \mathbb{Q}, a \leq b \Leftrightarrow (b - a) \in \mathbb{Q}_+,$$

is an order relation on $\mathbb{Q}$. That is means, this relation satisfy the following properties :

Proposition 1.1.

1. For all $a \in \mathbb{Q}$, $a \leq a$.
2. For all $a, b \in \mathbb{Q}$, $(a \leq b \text{ and } b \leq a) \Rightarrow a = b$.
3. For all $a, b \text{ and } c \in \mathbb{Q}$,

$$\begin{cases} a \leq b \\ b \leq c \end{cases} \Rightarrow a \leq c.$$

We easily verify by returning to the definition that the order relation $\leq$ on $\mathbb{Q}$ is a total order relation, moreover, it is compatible with the two laws $+$ and $\times$. More precisely, we have the following proposition.

Proposition 1.2. *The order relation $\leq$ on $\mathbb{Q}$ has the following properties :*

1. *it is a total order :*

$$\text{for all } x, y \in \mathbb{Q}, x \leq y \text{ or } y \leq x.$$

2. *it is compatible with the two operations $+$ and $\times$:*

$$\text{for all } x, y, z \in \mathbb{Q}, \begin{cases} x \leq y \Rightarrow x + z \leq y + z. \\ \text{and} \\ x \leq y \text{ and } 0 \leq z \Rightarrow x \times z \leq y \times z. \end{cases}$$

1.5 Supremum and the Infimum

Definition 1.3. *Let T be a nonempty subset of $\mathbb{Q}$.*

1. *The set T is said to be **bounded above** if there exists a number $\alpha \in \mathbb{Q}$ such that :*

$$\text{for all } x \in T, x \leq \alpha.$$

*Each such number α is called an **upper bound** of T .*

2. *The set T is said to be **bounded below** if there exists a number $\beta \in \mathbb{Q}$ such that :*

$$\text{for all } x \in T, \beta \leq x.$$

*Each such number β is called a **lower bound** of T .*

3. *A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.*

For example, the set $T = \{x \in \mathbb{Q} : x \leq 3\}$ is bounded above; the number 3 and any number larger than 3 is an upper bound of T . This set has no lower bounds, so that the set is not bounded below. Thus it is unbounded.

Definition 1.4. *Let T be a nonempty subset of $\mathbb{Q}$.*

1. If T is bounded above, then a number u is said to be a **supremum** or a **least upper bound** of T if it satisfies the conditions :
 - a. u is an upper bound of T ,
 - b. if v is any upper bound of T , then $u \leq v$.
2. If T is bounded below, then a number w is said to be an **infimum** or a **greatest lower bound** of T if it satisfies the conditions :
 - a. w is a lower bound of T ,
 - b. if t is any lower bound of T , then $t \leq w$.

Notation 1. If the supremum or the infimum of a set T exists, we will denote them by

$$\sup T \text{ and } \inf T.$$

For example the set $T = \{x \in \mathbb{Q} : 2 \leq x \leq 3\}$ is bounded and

$$\sup T = 3 \text{ and } \inf T = 2.$$

The field of rational numbers is not sufficient. The following result gives us the answer.

Lemma 1.1. *There does not exist a rational number r such that $r^2 = 2$.*

Proof :

Suppose, on the contrary, that p and q are integers such that $\ell = \frac{p}{q}$ is a solution of the equation $r^2 = 2$. We may assume that p and q are positive and have no common integer factors other than 1. Since $p^2 = 2q^2$, we see that p^2 is even. Example 1.2. This implies that p is also even so there exist $k \in \mathbb{N}$ such that $p = 2k$. Hence $4k^2 = 2q^2$, so that $2k^2 = q^2$. Therefore, q^2 is even, and it follows that q is an even natural number. Since the hypothesis that $\left(\frac{p}{q}\right)^2 = 2$ leads to the contradictory conclusion that q is both even and odd, it must be false.

2

Field of reals

2.1 Algebraic Properties of $\mathbb{R}$

On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$, called addition and multiplication, respectively. These operations satisfy the following properties :

1. for all $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$, (addition is a law of composition on $\mathbb{R}$) ;
2. for all $a, b \in \mathbb{R}$, $a + b = b + a$, (commutative property of addition) ;
3. for all $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$, (associative property of addition) ;
4. there exists an element $0 \in \mathbb{R}$ such that : for all $a \in \mathbb{R}$, $0 + a = a$ and $a + 0 = a$ (0 is an identity element or a zero element) ;
5. for each $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$, such that :

$$a + (-a) = 0 \text{ and } (-a) + a = 0,$$

($-a$ is called a inverse of a or the negative element of a) ;

6. for all $a, b \in \mathbb{R}$, $a \cdot b \in \mathbb{R}$, (multiplication is a law of composition on $\mathbb{R}$) ;
7. for all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$, (commutative property of multiplication) ;
8. for all $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, (associative property of multiplication) ;
9. there exists an element $1 \in \mathbb{R}$ distinct from 0 such that :

$$\text{for all } a \in \mathbb{R}, 1 \cdot a = a \text{ and } a \cdot 1 = a;$$

(1 is an identity element or a unit element),

10. for each $a \neq 0$ in $\mathbb{R}$, there exists an element $\frac{1}{a} \in \mathbb{R}$, such that :

$$a \cdot \frac{1}{a} = 1 \text{ and } \frac{1}{a} \cdot a = 1,$$

($\frac{1}{a}$ is called a inverse of a or the reciprocal element of a) ;

11. for all $a, b, c \in \mathbb{R}$, $a.(b+c) = a.b+a.c$ and $(b+c).a = b.a+c.a$, (distributive property of multiplication over addition).

Definition 2.1. The set $\mathbb{R}$ equipped by the two laws of composition $+$ and $.$, satisfying above axioms, is called a field, noted by $(\mathbb{R}, +, .)$.

Theorem 2.1. 1. If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$.

2. If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u.b = b$, then $u = 1$.

3. If $a \in \mathbb{R}$, then $a.0 = 0$.

Proof :

Using the properties above then about (1) we can get :

$$1. z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0.$$

For (2) we have

$$u = u.1 = u.(b.(\frac{1}{b})) = (u.b)\frac{1}{b} = (b.\frac{1}{b}) = 1.$$

For (3) we can write

$$a + a.0 = a.1 + a.0 = a.(1 + 0) = a.1 = a.$$

Therefore, we conclude from (1) that $a.0 = 0$. □

Theorem 2.2. 1. If $a \neq 0$ and b in $\mathbb{R}$ such that $a.b = 1$, then $b = \frac{1}{a}$.

2. If a and b are elements in $\mathbb{R}$ such that $a.b = 0$, then either $a = 0$ or $b = 0$.

Proof :

To prove (1), we using the properties above and we get

$$b = 1.b = (\frac{1}{a}.a).b = \frac{1}{a}.(a.b) = \frac{1}{a}.1 = \frac{1}{a}.$$

For (2), it suffices to assume $a \neq 0$ and prove that $b = 0$. Using the properties above , to obtain

$$b = 1.b = (\frac{1}{a}.a).b = \frac{1}{a}.(a.b) = \frac{1}{a}.0 = 0.$$

□

Remarks 2.1. We can define on $\mathbb{R}$ other operations for examples :

A. The operation of *subtraction* is defined by :

$$\text{for all } a, b \in \mathbb{R}, a - b = a + (-b)$$

B. The operation of *division* is defined by :

$$\text{for all } a, b \in \mathbb{R} \text{ with } b \neq 0, \frac{a}{b} = a \cdot \frac{1}{b}.$$

C. In practice, we will write ab instead of $a \cdot b$. Similarly, we will use the usual notation for exponents and write a^2 for aa , a^3 for a^2a ; and, in general, we define

$$a^{n+1} = a^n a \text{ for } n \in \mathbb{N}^*,$$

with the convention that $a^1 = a$. Further, if $a \neq 0$, we write $a^0 = 1$ and $a^{-1} = \frac{1}{a}$ and if $n \in \mathbb{N}$, we will write $a^{-n} = \left(\frac{1}{a}\right)^n$.

2.2 The Order Properties of $\mathbb{R}$

Throughout this part, we designate by $\mathbb{R}_+$ the set of the nonnegative real numbers and by $\mathbb{R}_-$ the set of the nonpositive real numbers. The set of the positive or strictly positive rational numbers is denoted by $\mathbb{R}_+^*$ and the set of the negative or strictly negative rational numbers is denoted by $\mathbb{R}_-^*$.

If $a \in \mathbb{R}_+^*$, we write $a > 0$ and say that a is a positive (or a strictly positive) real number.

If $a \in \mathbb{R}_+$, we write $a \geq 0$ and say that a is a nonnegative real number.

Similarly, if $a \in \mathbb{R}_-^*$, we write $a < 0$ and say that a is a negative (or a strictly negative) real number.

If $a \in \mathbb{R}_-$, we write $a \leq 0$ and say that a is a nonpositive real number.

Definition 2.2. Let a, b be elements of $\mathbb{R}$.

(a) If $a - b \in \mathbb{R}_+^*$, then we write $a > b$ or $b < a$.

(b) If $a - b \in \mathbb{R}_+$, then we write $a \geq b$ or $b \leq a$.

Remarks 2.2. Let a, b be elements of $\mathbb{R}$.

a. Exactly one of the following will hold :

$$a > b, \quad a = b, \quad a < b.$$

b. If both $a \leq b$ and $b \leq a$, then $a = b$.

c. We write $a < b < c$ to mean that both $a < b$ and $b < c$ are satisfied. The other "double" inequalities $a \leq b < c$, $a \leq b \leq c$, and $a < b \leq c$ are defined in a similar manner.

Exercise 2.1. Let a, b, c be any elements of $\mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.

2. If $a > b$, then $a + c > b + c$.

3. If $a > b$ and $c > 0$, then $ca > cb$.

4. If $a > b$ and $c < 0$, then $ca < cb$.

We verify that on $\mathbb{R}$ the relation $\leq$, defined by :

$$\text{for all } a, b \in \mathbb{R}, a \leq b \Leftrightarrow (b - a) \in \mathbb{R}_+,$$

is an order relation on $\mathbb{R}$. That is means, this relation satisfy the following properties :

Proposition 2.1.

1. For all $a \in \mathbb{R}$, $a \leq a$.

2. For all $a, b \in \mathbb{R}$, $(a \leq b \text{ and } b \leq a) \Rightarrow a = b$.

3. For all a, b and $c \in \mathbb{R}$,

$$\begin{cases} a \leq b \\ b \leq c \end{cases} \Rightarrow a \leq c.$$

We easily verify by returning to the definition that the order relation $\leq$ on $\mathbb{R}$ is a total order relation, moreover, it is compatible with the two laws $+$ and $\cdot$. More precisely, we have the following proposition.

Proposition 2.2. The order relation $\leq$ on $\mathbb{R}$ has the following properties :

1. it is a total order :

$$\text{for all } x, y \in \mathbb{R}, x \leq y \text{ or } y \leq x.$$

2. it is compatible with the two operations $+$ and $\times$:

$$\text{for all } x, y, z \in \mathbb{R}, \begin{cases} x \leq y \Rightarrow x + z \leq y + z. \\ \text{and} \\ x \leq y \text{ and } 0 \leq z \Rightarrow x \times z \leq y \times z. \end{cases}$$

2.3 Supremum and the Infimum

Definition 2.3. Let T be a nonempty subset of $\mathbb{R}$.

1. The set T is said to be **bounded above** if there exists a number $\alpha \in \mathbb{R}$ such that :

$$\text{for all } x \in T, x \leq \alpha.$$

Each such number α is called an **upper bound** of T .

2. The set T is said to be **bounded below** if there exists a number $\beta \in \mathbb{R}$ such that :

$$\text{for all } x \in T, \beta \leq x.$$

Each such number β is called a **lower bound** of T .

3. A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.

For example, the set $T = \{x \in \mathbb{R} : x \leq 3\}$ is bounded above; the number 3 and any number larger than 3 is an upper bound of T . This set has no lower bounds, so that the set is not bounded below. Thus it is unbounded.

Definition 2.4. Let T be a nonempty subset of $\mathbb{R}$.

1. If T is bounded above, then a number u is said to be a **supremum** or a **least upper bound** of T if it satisfies the conditions :
 - a. u is an upper bound of T ,
 - b. if v is any upper bound of T , then $u \leq v$.
2. If T is bounded below, then a number w is said to be an **infimum** or a **greatest lower bound** of T if it satisfies the conditions :
 - a. w is a lower bound of T ,
 - b. if t is any lower bound of T , then $t \leq w$.

Notation 2. If the supremum or the infimum of a set T exists, we will denote them by

$$\sup T \text{ and } \inf T.$$

For example the set $T = \{x \in \mathbb{R} : \sqrt{2} \leq x \leq 3\}$ is bounded and

$$\sup T = 3 \text{ and } \inf T = \sqrt{2}.$$

2.3.1 Characterization of the supremum and the infimum

Proposition 2.3. Let A a bounded subset of $\mathbb{R}$. We have this property of the least upper bound number :

$$M = \sup(A) \Leftrightarrow (M \text{ is an upper bound number of } A \text{ and for all } \varepsilon > 0, \text{ there exists } x_\varepsilon \in A; M - \varepsilon < x_\varepsilon).$$

The property of the greatest lower bound number is as follow :

$$m = \inf(A) \Leftrightarrow (m \text{ is a lower bound number of } A \text{ and for all } \varepsilon > 0, \text{ there exists } x_\varepsilon \in A; x_\varepsilon < m + \varepsilon).$$

2.3.2 Upper bound axiom

Axiom 2.1. Any upper part of $\mathbb{R}$ admits a least upper bound number.

As a consequence of this Axiom, we have :

Proposition 2.4. Any minor part of $\mathbb{R}$ admits a greatest lower bound number.

Proof : Exercise for students.

2.4 Absolute Value And Some Properties

Definition 2.5. The absolute value of a real number a , denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

Example 2.1. For example, $|5| = 5$, $|- \frac{1}{2}| = \frac{1}{2}$ and $|1 - \sqrt{3}| = \sqrt{3} - 1$.

Remarks 2.3. We see from the definition that :

1. For all $x \in \mathbb{R}$, $|x| \geq 0$.
2. For all $x \in \mathbb{R}$, $|x| = |-x|$.
3. For all $x \in \mathbb{R}$, $|x| = 0$ if and only if $x = 0$.

Some additional properties are as follows.

Theorem 2.3. (a) for all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

(b) for all $a \in \mathbb{R}$, $|a|^2 = a^2$.

(c) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

(d) for all $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Proof :

(a) If either a or b is 0, then both sides are equal to 0. There are four other cases to consider.

If $a > 0$; $b > 0$, then $ab > 0$, so that

$$|ab| = ab = |a||b|.$$

If $a > 0$; $b < 0$, then $ab < 0$, so that

$$|ab| = -ab = a(-b) = |a||b|.$$

The remaining cases are treated similarly.

(b) Since $a^2 \geq 0$, we have

$$a^2 = |a^2| = |aa| = |a||a| = |a|^2.$$

(c) If $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$, which is equivalent to $-c \leq a \leq c$. Conversely, if $-c \leq a \leq c$, then we have both $a \leq c$ and $-a \leq c$, so that $|a| \leq c$.

(d) Take $c = |a|$ in part (c). □

The following important inequality will be used frequently.

Lemma 2.1 (Triangle Inequality).

For all $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Proof :

From 2.3(d), we have

$$\left\{ \begin{array}{l} -|a| \leq a \leq |a| \\ \text{and} \\ -|b| \leq b \leq |b|. \end{array} \right.$$

On adding these inequalities, we obtain

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Hence, by 2.3(c) we have

$$|a + b| \leq |a| + |b|.$$

□

Corollary 2.1. *For all $a, b \in \mathbb{R}$, then*

(a)

$$||a| - |b|| \leq |a - b|.$$

(b)

$$|a - b| \leq |a| + |b|.$$

Proof :

(a) We write $a = a + b - b$ and then apply the Triangle Inequality to get

$$|a| = |a - b + b| \leq |a - b| + |b|.$$

Now subtract $|b|$ to get

$$|a| - |b| \leq |a - b|.$$

Similarly, we can write

$$|b| = |b - a + a| \leq |b - a| + |a|.$$

Now subtract $|a|$ to get

$$|b| - |a| \leq |b - a|.$$

So

$$-|a - b| = -|b - a| \leq |a| - |b|.$$

Combine these two inequalities, using 2.3, we get the inequality in (a).

(b) Replace b in the Triangle Inequality by $-b$ to get

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|.$$

we obtain the inequality in (b).

□

Corollary 2.2. *If $a_1, a_2, a_3; \dots, a_n$, are any real numbers, then*

$$|a_1 + a_2 + a_3 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|.$$

2.5 Intervals

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called intervals.

2.5.1 Open interval

Definition 2.6. Let $a, b \in \mathbb{R}$ satisfy $a < b$. The open interval determined by a and b , denoted (a, b) is the set

$$(a; b) = \{x \in \mathbb{R} : a < x < b\}.$$

Remark 2.1. The points a and b are called the endpoints of the interval; however, the endpoints are not included in an open interval.

2.5.2 Closed interval

Definition 2.7. Let $a, b \in \mathbb{R}$ satisfy $a \leq b$. The closed interval determined by a and b , denoted $[a, b]$ is the set

$$[a; b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Remark 2.2. The endpoints a and b are adjoined to this closed interval.

2.5.3 Half-Open and Half-Closed intervals

Definition 2.8. The two half-open (or half-closed) intervals determined by a and b are defined by

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

and

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

Remarks 2.4. 1.) The half-open interval $[a, b)$, includes the endpoint a , and the half-open interval $(a, b]$, includes the endpoint b .

2.) These four intervals is bounded and has length defined $b - a$. If $a = b$, the corresponding open interval is the empty set $(a; a) = \emptyset$, whereas the corresponding closed interval is the singleton set $[a; a] = \{a\}$.

2.5.4 Infinite open intervals

Definition 2.9. The infinite open intervals are the sets of the form :

$$(a; +\infty) = \{x \in \mathbb{R} : x > a\} \text{ and } (-\infty; b) = \{x \in \mathbb{R} : x < b\}.$$

2.5.5 Infinite closed intervals

Definition 2.10. *The infinite closed intervals are the sets of the form :*

$$[a; +\infty) = \{x \in \mathbb{R} : x \geq a\} \text{ and } (-\infty; b] = \{x \in \mathbb{R} : x \leq b\}.$$

Remarks 2.5. 1.) *It is often convenient to think of the entire set $\mathbb{R}$ as an infinite interval; in this case, we write*

$$(-\infty; +\infty) = \mathbb{R}.$$

Furthermore, no point is an endpoint of $(-\infty, +\infty)$.

2.) *It must be emphasized that $+\infty$ and $-\infty$ are not elements of $\mathbb{R}$, but only convenient symbols.*

2.5.6 Characterization of Intervals

Theorem 2.4 (Exercise :). *If T is a subset of $\mathbb{R}$ that contains at least two points and has the property*

$$\text{if } x, y \in T \text{ and } x < y, \text{ then } [x; y] \subseteq T;$$

then T is an interval.

2.5.7 Absolute values and intervals

Let $x_0 \in \mathbb{R}$. For $\alpha > 0$, we have the following equalities :

1. $\{x \in \mathbb{R} : |x - x_0| \leq \alpha\} = [x_0 - \alpha, x_0 + \alpha],$
2. $\{x \in \mathbb{R} : |x - x_0| < \alpha\} = (x_0 - \alpha, x_0 + \alpha),$
3. $\{x \in \mathbb{R} : |x - x_0| > \alpha\} = (-\infty, x_0 - \alpha) \cup (x_0 + \alpha, +\infty).$

2.6 Archimedean Property

Property 2.1 (Archimed's additive property).

For all real numbers a, b with $a > 0$, there exists an integer natural number n such that $na > b$.

Proof :

We are going to demonstrate through the absurd. Suppose that :

$$\text{For all } n \in \mathbb{N}, na \leq b.$$

Let the set

$$A = \{x = na \mid n \in \mathbb{N}\}.$$

The set A is a non-empty subset of $\mathbb{R}$ because it contains for example the element a , and it is bounded above by b . It therefore admits an upper bound denoted M . We have

$$na \leq M, \text{ for all } n \in \mathbb{N} \Rightarrow (n+1)a \leq M, \text{ for all } n \in \mathbb{N}.$$

So

$$na \leq M - a, \text{ for all } n \in \mathbb{N}.$$

Then $M - a$ is an upper bound of A and $M - a < M = \sup(A)$, which contradicts the definition of the upper bound (the smallest of the upper bounds.) $\square$

Theorem 2.5 (Archimed's Multiplicative property).

For all real numbers a, b with $a > 1$, there exists an integer natural number n such that $a^n > b$.

Proof :

(1) If $b \leq 1$ then the result is true for $n = 1$.

(2) We assume that $b > 1$. We have $a > 1$ then there exists two real numbers $\alpha > 0$ and $\beta > 0$ such as

$$a = 1 + \alpha \text{ and } b = 1 + \beta.$$

Using (1.4), so there exists a natural number n , for all $\alpha > 0$ such that

$$a^n = (1 + \alpha)^n \geq 1 + n\alpha.$$

It is therefore sufficient to show that

$$\text{For all } \alpha > 0 \text{ and } \beta > 0, 1 + n\alpha > 1 + \beta,$$

that's to say

$$\text{For all } \alpha > 0 \text{ and } \beta > 0, n\alpha > \beta.$$

This result is true according to the Archimed additive property. $\square$

2.7 Whole part of the real number

Proposition and Definition 2.1. *For every real number x , there exists one and only one integer p (p is in $\mathbb{Z}$) checking :*

$$p \leq x < p + 1. \quad (2.1)$$

The number p above is called the whole part of the number x and denoted $E(x)$ or $[x]$.

Proof :

(1). Let us first show uniqueness. Suppose that x has two integer parts p and p' . As p and p' satisfy the double inequality (2.1) we then deduce that :

$$\begin{aligned} \begin{cases} p < p' + 1 \\ p' < p + 1 \end{cases} &\Rightarrow (-1 < p - p' < 1 \text{ and } p - p' \in \mathbb{Z}) \\ &\Rightarrow p - p' = 0 \\ &\Rightarrow p' = p. \end{aligned}$$

(2). Let us show the existence of the whole part.

(a.) It is obvious that $E(0) = 0$.

(b.) Suppose that $x < 0$ and let :

$$C = \{n \in \mathbb{Z} : n \leq x\} \subset \mathbb{Z} \subset \mathbb{R}.$$

The set C is majored by x then, it admits an upper bound, which is in fact the least upper bound element of C . Let p be this element. We have :

(i) As $p \in C$ then $p \leq x$.

(ii) The fact that p is the least upper bound element of C then $p+1 \notin C$ therefore $x < p+1$. Which gives the double inequality (2.1). , Moreover $p \leq -1$.

(3) We assume that $x > 0$ and we set :

$$D = \{n \in \mathbb{Z} : x < n\} \subset \mathbb{Z} \subset \mathbb{R}.$$

The set D admits the greatest lower bound element denoted m (because D is minimized). Note that $m \geq 1$.

(i) As $m \in D$ then $x < m$.

(ii) The number m is the greatest lower bound element of D then $m-1 \notin D$, and therefore $m-1 \leq x$.

The number $p = m-1$ verifies (2.1). Furthermore we have $p \geq 0$. □

Remarks 2.6.

1. $\forall n \in \mathbb{Z}, E(n) = n$.

2. $\forall x \in \mathbb{R}, x > 0 \Rightarrow E(x) \geq 0$.

3. $\forall x \in \mathbb{R}, x < 0 \Rightarrow E(x) \leq -1$.

4. Any real number x is written in the form :

$$x = E(x) + \alpha, \text{ with } 0 \leq \alpha < 1.$$

2.8 Density of Rational Numbers in $\mathbb{R}$

Theorem 2.6. [The Density Theorem] *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof : Let x, y two real numbers with $x < y$. We know that :

$$nx < E(nx) + 1.$$

Since $y - x > 0$, it follows from the Archimed property, that there exists $n \in \mathbb{N}$ such that

$$n(y - x) > 1.$$

Therefore, we have

$$\begin{aligned} E(nx) + 1 &< E(nx) + n(y - x) \\ &< nx + n(y - x) = ny. \end{aligned}$$

So,

$$(nx < E(nx) + 1 < ny) \Rightarrow x < \frac{E(nx) + 1}{n} < y.$$

. Thus, the rational number $r = \frac{E(nx)+1}{n}$ satisfies $x < r < y$. □

2.9 The binomial formula

Proposition 2.5. [Newton's binomial formula] *For all $a, b \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have the formula :*

$$(x + y)^n = \sum_{k=0}^n C_n^k a^k b^{n-k},$$

with $C_n^k = \frac{n!}{k!(n-k)!}$; $n! = 1 \times 2 \times 3 \times \dots \times n$; and $0! = 1$.

Proof : Exercise for students. (You can use the demonstration by induction).

Exercise 2.2. *If $a \in \mathbb{R}$ is such that $|a| < \epsilon$ for every $\epsilon > 0$, then $a = 0$.*

Exercise 2.3. *Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that*

$$\sup(A \cup B) = \sup(\sup(A), \sup(B)).$$

Exercise 2.4. *Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let*

$$A + B = \{a + b : a \in A; b \in B\}.$$

Prove that

$$\sup(A + B) = \sup(A) + \sup(B) \text{ and } \inf(A + B) = \inf(A) + \inf(B).$$

Exercise 2.5. *Let A and B be two bounded parts of $\mathbb{R}$. Say whether the following insertions are True or False ?. Justify your answer.*

1. $A \subseteq B \Rightarrow \sup(A) \leq \sup(B)$.
2. $A \subseteq B \Rightarrow \inf(A) \leq \inf(B)$.
3. $\sup(-A) = -\inf(A)$.
4. $\sup(A) + \inf(B) \leq \sup(A + B)$.

3

Sequences

3.1 Sequences

Throughout the chapter, we designate by $\mathbb{N}_0$ the set of natural numbers n such that $n \geq n_0$, with n_0 is in $\mathbb{N}$.

Definition 3.1. A sequence of real numbers or a sequence in $\mathbb{R}$ is a function defined on the set $\mathbb{N}_0$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

If $u : \mathbb{N}_0 \rightarrow \mathbb{R}$ is a sequence, we will usually denote the value of u at n by the symbol u_n rather than using the function notation $u(n)$. The values u_n are also called the terms or the elements of the sequence. We will denote this sequence by the notations $(x_n)_{n \in \mathbb{N}_0}$. Sequences are often defined by giving a formula for the n^{th} term x_n and write for example

$$\forall n \in \mathbb{N}, x_n = \frac{1}{3n+1}.$$

Another way of defining a sequence is to specify the value of x_{n_0} or x_{n_0+1} and give a formula for x_{n+1} ($n \geq n_0$) in terms of x_n . More generally, we may specify x_{n_0} and give a formula for obtaining x_{n+1} from $x_{n_1+1}; x_{n_1+2}; \dots; x_n$. Sequences defined in this manner are said to be inductively (or recursively) defined.

Examples 3.1.

(1) If $\alpha \in \mathbb{R}$, the sequence

$$\forall n \in \mathbb{N}, u_n = \alpha.$$

all of whose terms equal α , is called the constant sequence.

(2) The sequence $(2n)_{n \in \mathbb{N}}$ of even natural numbers can be defined inductively by

$$\begin{cases} x_0 = 0 \\ \forall n \in \mathbb{N}, x_{n+1} = x_n + 2. \end{cases}$$

(3) The celebrated **Fibonacci sequence** $(f_n)_{n \in \mathbb{N}}$ is given by the inductive definition

$$\begin{cases} f_0 = 1, f_1 = 1; \\ \forall n \in \mathbb{N}, f_{n+2} = f_{n+1} + f_n. \end{cases}$$

Thus each term past the second is the sum of its two immediate predecessors. The first ten terms of $(f_n)_{n \in \mathbb{N}}$ are seen to be 1 ; 1 ; 2 ; 3 ; 5 ; 8 ; 13 ; 21 ; 34 ; 55.

3.2 The Limit and convergence of a sequence

Definition 3.2. A sequence $(x_n)_{n \in \mathbb{N}_0}$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$, or x is said to be a limit of $(x_n)_{n \in \mathbb{N}_0}$, if for every $\varepsilon > 0$ there exists a natural number K such that

$$\forall n \in \mathbb{N}_0, n \geq K \Rightarrow |x_n - x| < \varepsilon.$$

If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent.

When a sequence $(x_n)_{n \in \mathbb{N}_0}$ has limit x , we will use the notation

$$\lim_{n \rightarrow +\infty} x_n = x.$$

3.2.1 Uniqueness of Limits

Lemma 3.1. A sequence in $\mathbb{R}$ can have at most one limit.

Proof :

Suppose that x' and x'' are both limits of $(x_n)_{n \in \mathbb{N}_0}$. For each $\varepsilon > 0$ there exist K' such that

$$\forall n \in \mathbb{N}_0, n \geq K' \Rightarrow |x_n - x'| < \frac{\varepsilon}{2},$$

and there exists K'' such that

$$\forall n \in \mathbb{N}_0, n \geq K'' \Rightarrow |x_n - x''| < \frac{\varepsilon}{2}.$$

We let K be the larger of K' and K'' . Then for $n \geq K$ we apply the Triangle Inequality to get

$$|x' - x''| = |x' - x_n + x_n - x''| \leq |x' - x_n| + |x_n - x''| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $x' - x'' = 0$. □

Example 3.1. We show that the limit of the sequence $(u_n)_{n \in \mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^*, u_n = \frac{1}{n},$$

is 0.

If $\varepsilon > 0$ is given, then $\frac{1}{\varepsilon} > 0$. By the Archimedean Property, there is a natural number K such that $K\varepsilon > 1$. Then, if $n \geq K$, we have

$$\frac{1}{n} \leq \frac{1}{K} < \varepsilon.$$

Consequently, if $n \geq K$, then

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon.$$

Therefore, we can assert that the sequence $(u_n)_{n \in \mathbb{N}^*}$ converges to 0.

Theorem 3.1. Let $(x_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers and let $x \in \mathbb{R}$. If $(a_n)_{n \in \mathbb{N}_0}$ is a sequence of positive real numbers with $\lim_{n \rightarrow +\infty} a_n = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have

$$\forall n \geq m, |x_n - x| \leq Ca_n,$$

then it follows that $\lim_{n \rightarrow +\infty} x_n = x$.

Proof :

If $\varepsilon > 0$ is given, then since $\lim_{n \rightarrow +\infty} a_n = 0$, we know there exists $K = K\left(\frac{\varepsilon}{C}\right)$ such that

$$n \geq K \Rightarrow a_n = |a_n - 0| < \frac{\varepsilon}{C}.$$

Therefore it follows that if both $n \geq K$ and $n \geq m$, then

$$|x_n - x| \leq Ca_n < C \frac{\varepsilon}{C} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $x = \lim_{n \rightarrow +\infty} x_n$. □

Example 3.2.

(a) let $a > 0$. We show that the limit of the sequence $(v_n)_{n \in \mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^*, v_n = \frac{1}{1 + na},$$

is 0.

If $a > 0$, then $\lim_{n \rightarrow +\infty} \frac{1}{1+na} = 0$.

Since $a > 0$, then $0 < na < 1 + na$, and therefore $0 < \frac{1}{(1+na)} < \frac{1}{(na)}$. Thus we have

$$\forall n \in \mathbb{N}^*, \frac{1}{1 + na} \leq \frac{1}{a} \left(\frac{1}{n} \right).$$

Since $\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$, we may invoke Theorem above with $C = \frac{1}{a}$ and $m = 1$ to infer that $\lim_{n \rightarrow +\infty} v_n = 0$.

(b) let $b \in \mathbb{R}$ with $0 < b < 1$, we show that the limit of the sequence $(h_n)_{n \in \mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^*, h_n = b^n,$$

is 0.

Since $0 < b < 1$, we can write $b = \frac{1}{1+a}$, where $a = \frac{1}{b} - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1+a)^n \geq 1+na$. Hence

$$0 < b^n = \left(\frac{1}{1+a} \right)^n \leq \frac{1}{1+na} < \frac{1}{na}.$$

Thus from Theorem below we conclude that $\lim_{n \rightarrow +\infty} h_n = 0$.

Exercise 3.1. Use the definition of the limit of a sequence to establish the following limits.

$$\lim_{n \rightarrow +\infty} \left(\frac{3n+1}{2n+5} \right) = \frac{3}{2}.$$

3.3 Bounded Sequences

Definition 3.3. A sequence $(u_n)_{n \in \mathbb{N}_0}$ is said to be bounded from above if there exists a number $M \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}_0, u_n \leq M.$$

Any such M is called an upper bound. Likewise, $(u_n)_{n \in \mathbb{N}_0}$ is said to be bounded from below if there exists a number $m \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}_0, u_n \geq m.$$

Any such m is called a lower bound.

Definition 3.4. A sequence $(x_n)_{n \in \mathbb{N}_0}$ of real numbers is said to be bounded if there exists a real number $\alpha > 0$ such that

$$\forall n \in \mathbb{N}_0, |x_n| \leq \alpha.$$

Corollary 3.1. If a sequence is both bounded from above and bounded from below, then the sequence is bounded.

Proof :

The proof is trivial.

Theorem 3.2. A convergent sequence of real numbers is bounded.

Proof :

Suppose that $(x_n)_{n \in \mathbb{N}_0}$ is a convergent sequence such that $\lim_{n \rightarrow +\infty} x_n = x$ and let $\varepsilon = 1$. Then

there exists a natural number $K = K(1)$ such that $|x_n - x| < 1$ for all $n \geq K$. If we apply the Triangle Inequality with $n \geq K$ we obtain

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

If we set

$$M = \sup\{|x_{n_0+1}|, |x_{n_0+2}|, \dots, |x_{K-1}|, 1 + |x|\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

Exercise 3.2. Let $(u_n)_{n \in \mathbb{N}}$ be the sequence define by $u_0 \in]0, 1]$ and by the recurrence relation

$$\forall n \in \mathbb{N}, u_n = \frac{2u_n + u_n^2}{4}.$$

Prove that, for all $n \in \mathbb{N}$, we have $0 < u_n \leq 1$.

3.4 Operations on convergent sequences

Theorem 3.3.

- a. Let $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ be sequences of real numbers that converge to α and β , respectively, and let $c \in \mathbb{R}$. Then the sequences $(u_n + v_n)_{n \in \mathbb{N}_0}$, $(u_n - v_n)_{n \in \mathbb{N}_0}$, $(u_n \cdot v_n)_{n \in \mathbb{N}_0}$, and $(cu_n)_{n \in \mathbb{N}_0}$ converge to $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, and $c\alpha$, respectively.
- b. If $(x_n)_{n \in \mathbb{N}_0}$ converges to x and (z_n) is a sequence of nonzero real numbers that converges to z and if $z \neq 0$, then the quotient sequence $(\frac{x_n}{z_n})_{n \in \mathbb{N}_0}$ converges to $\frac{x}{z}$.

Proof :

(a) By hypothesis, if $\varepsilon > 0$ there exists two natural numbers K_1 and K_2 such that

$$\begin{cases} n \geq K_1 \Rightarrow |u_n - \alpha| < \frac{\varepsilon}{2}, \\ n \geq K_2 \Rightarrow |v_n - \beta| < \frac{\varepsilon}{2}. \end{cases}$$

Hence if $K = \sup\{K_1; K_2\}$, it follows that if $n \geq K$ and using the Triangle Inequality to obtain

$$|(u_n + v_n) - (\alpha + \beta)| = |(u_n - \alpha) + (v_n - \beta)| \leq |u_n - \alpha| + |v_n - \beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $(u_n + v_n)_{n \in \mathbb{N}_0}$ converges to $\alpha + \beta$.

Precisely the same argument can be used to show that $(u_n - v_n)$ converges to $\alpha - \beta$.

Note that

$$\forall n \in \mathbb{N}_0, u_n \cdot v_n - \alpha \cdot \beta = v_n(u_n - \alpha) + \alpha(v_n - \beta).$$

As the sequence $(v_n)_{n \in \mathbb{N}_0}$ is convergent therefore it is bounded so, there exists a real number $M_1 > 0$ such that

$$\forall n \in \mathbb{N}_0, |v_n| \leq M_1.$$

So we can write

$$\begin{aligned}
 |u_n \cdot v_n - \alpha\beta| &= |v_n(u_n - \alpha) + \alpha(v_n - \beta)| \\
 &\leq |v_n(u_n - \alpha)| + |\alpha(v_n - \beta)| \\
 &\leq |v_n||u_n - \alpha| + |\alpha||v_n - \beta| \\
 &\leq M_1|u_n - \alpha| + |\alpha||v_n - \beta| \\
 &\leq M|u_n - \alpha| + M|v_n - \beta|,
 \end{aligned}$$

where $M = \sup\{M_1, |\alpha|\}$.

From the convergence of $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ we conclude that if $\varepsilon > 0$ is given, then there exist natural numbers K_1 and K_2 such that

$$\begin{cases} n \geq K_1 \Rightarrow |u_n - \alpha| < \frac{\varepsilon}{2M}, \\ n \geq K_2 \Rightarrow |v_n - \beta| < \frac{\varepsilon}{2M}. \end{cases}$$

Now let $K = \sup\{K_1; K_2\}$; then if $n \geq K$ we infer that,

$$|u_n v_n - \alpha\beta| \leq M|u_n - \alpha| + M|v_n - \beta| < \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that the sequence $(u_n \cdot v_n)_{n \in \mathbb{N}_0}$ converges to $\alpha\beta$.

The fact that $(cu_n)_{n \in \mathbb{N}_0}$ converges to cx can be proved in the same way; it can also be deduced by taking (v_n) to be the constant sequence where for all n in $\mathbb{N}_0$ $v_n = c$.

(b) First, we prove that

$$\begin{cases} \lim_{n \rightarrow +\infty} z_n = z \neq 0 \\ \text{and} \\ \forall n \in \mathbb{N}_0, z_n \neq 0 \end{cases} \Rightarrow \lim_{n \rightarrow +\infty} \frac{1}{z_n} = \frac{1}{z}.$$

Let us first prove that the sequence (z_n) is bounded from below by a strictly positive number. Indeed, $z \neq 0$ implies that the interval $]0, |z|$ is non-empty. Let ε an element of this interval. So, since $\lim_{n \rightarrow +\infty} z_n = z$, there exists a natural number K such that

$$n \geq K \Rightarrow z - \varepsilon < z_n < z + \varepsilon \Rightarrow |z_n| > |z| - \varepsilon.$$

Let's now

$$a = \inf \{|z_{n_0}|, |z_{n_0+1}|, \dots, |z_K|, |z| - \varepsilon\}.$$

Hence we have

$$\forall n \in \mathbb{N}_0, |z_n| \geq a.$$

We have, according to the above,

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| = \frac{|z_n - z|}{|z||z_n|} \leq \frac{|z_n - z|}{a^2}.$$

As $\lim_{n \rightarrow +\infty} z_n = z$, so for all $\varepsilon > 0$ there exists p such that

$$n > p \Rightarrow |z_n - z| < a^2 \varepsilon \Rightarrow \left| \frac{1}{z_n} - \frac{1}{z} \right| < \varepsilon.$$

Whence

$$\lim_{n \rightarrow +\infty} \frac{1}{z_n} = \frac{1}{z}.$$

Since $\lim_{n \rightarrow +\infty} \frac{1}{z_n} = \frac{1}{z}$ and $\lim_{n \rightarrow +\infty} x_n = x$, so we can apply the statement (a) to conclude that

$$\lim_{n \rightarrow +\infty} \frac{x_n}{z_n} = \lim_{n \rightarrow +\infty} x_n \left(\frac{1}{z_n} \right) = x \left(\frac{1}{z} \right) = \frac{x}{z}.$$

Theorem 3.4. *If $(u_n)_{n \in \mathbb{N}_0}$ is a convergent sequence of real numbers and if $u_n \geq 0$ for all $n \in \mathbb{N}_0$, then $\alpha = \lim_{n \rightarrow +\infty} u_n \geq 0$.*

Proof :

Suppose the conclusion is not true and that $\alpha < 0$; then we can take $\varepsilon = -\alpha$ is positive. Since $(u_n)_{n \in \mathbb{N}_0}$ converges to α , there is a natural number K such that

$$\forall n \geq K, \alpha - \varepsilon < u_n < \alpha + \varepsilon.$$

In particular, we have $u_K < \alpha + \varepsilon = \alpha + (-\alpha) = 0$. But this contradicts the hypothesis that $u_n \geq 0$ for all $n \in \mathbb{N}_0$. Therefore, this contradiction implies that $\alpha \geq 0$. $\square$

Theorem 3.5. *If $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ are convergent sequences of real numbers and if $u_n \leq v_n$ for all $n \in \mathbb{N}_0$, then $\lim_{n \rightarrow +\infty} u_n \leq \lim_{n \rightarrow +\infty} v_n$.*

Proof :

Let $z_n = v_n - u_n$ so that $(z_n)_{n \in \mathbb{N}_0} = (v_n - u_n)_{n \in \mathbb{N}_0}$ and $z_n \geq 0$ for all $n \in \mathbb{N}_0$. It follows from Theorems (3.5) that

$$0 \leq \lim_{n \rightarrow +\infty} z_n = \lim_{n \rightarrow +\infty} (v_n - u_n);$$

so that $\lim_{n \rightarrow +\infty} u_n \leq \lim_{n \rightarrow +\infty} v_n$. $\square$

Theorem 3.6. *If $(u_n)_{n \in \mathbb{N}_0}$ is a convergent sequence and if $a \leq u_n \leq b$, for all $n \in \mathbb{N}_0$, then $a \leq \lim_{n \rightarrow +\infty} u_n \leq b$.*

Proof :

Let $(b_n)_{n \in \mathbb{N}_0}$ be the constant sequence given by

$$\forall n \in \mathbb{N}_0, b_n = b.$$

Theorem (3.5) implies that

$$\lim_{n \rightarrow +\infty} u_n \leq \lim_{n \rightarrow +\infty} b_n = b.$$

Similarly one shows that $a \leq \lim_{n \rightarrow +\infty} u_n$. $\square$

Theorem 3.7. *Let $a \in \mathbb{R}$. Suppose that $(u_n)_{n \in \mathbb{N}_0}$, $(v_n)_{n \in \mathbb{N}_0}$ and $(z_n)_{n \in \mathbb{N}_0}$ are sequences of real numbers such that*

$$\forall n \in \mathbb{N}_0, u_n \leq v_n \leq z_n \text{ and } \lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} z_n = a.$$

Then $(v_n)_{n \in \mathbb{N}_0}$ is convergent and $\lim_{n \rightarrow +\infty} v_n = a$.

Proof :

Let $a = \lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} z_n$. If $\varepsilon > 0$ is given, then it follows from the convergence of $(u_n)_{n \in \mathbb{N}_0}$ and $(z_n)_{n \in \mathbb{N}_0}$ to a that there exists a natural number K such that if $n \geq K$ then

$$|u_n - a| < \varepsilon \text{ and } |z_n - a| < \varepsilon.$$

Since the hypothesis implies that

$$\forall n \in \mathbb{N}_0, u_n - a \leq v_n - a \leq z_n - a,$$

it follows that for all $n \geq K$,

$$-\varepsilon < v_n - a < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that $\lim_{n \rightarrow +\infty} v_n = a$. □

Example 3.3. The sequence $(h_n)_{n \in \mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^*, h_n = \frac{\sin(n)}{n}.$$

We have

$$\forall n \in \mathbb{N}^* \quad -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

From the Theorem (3.7), we deduce that

$$0 = -\lim_{n \rightarrow +\infty} \frac{1}{n} = \lim_{n \rightarrow +\infty} \frac{1}{n}.$$

Hence, $\lim_{n \rightarrow +\infty} \frac{\sin(n)}{n} = 0$. So $(h_n)_{n \in \mathbb{N}^*}$ is convergent sequence.

Theorem 3.8. Let the sequence $(u_n)_{n \in \mathbb{N}_0}$ converge to u . Then the sequence $(|u_n|)_{n \in \mathbb{N}_0}$ of absolute values converges to $|u|$. That is, if $u = \lim_{n \rightarrow +\infty} u_n$, then $|u| = \lim_{n \rightarrow +\infty} |u_n|$.

Proof :

It follows from the Triangle Inequality that

$$\forall n \in \mathbb{N}_0, ||u_n - |u|| \leq |u_n - u|.$$

The convergence of $(|u_n|)_{n \in \mathbb{N}_0}$ to $|u|$ is then an immediate consequence of the convergence of $(u_n)_{n \in \mathbb{N}_0}$. □

3.5 Monotone Sequences

Definition 3.5. Let $(x_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers. We say that $(x_n)_{n \in \mathbb{N}_0}$ is increasing if it satisfies the inequalities

$$\forall n \in \mathbb{N}_0, x_{n+1} - x_n \geq 0,$$

and it is said to be strictly increasing if it satisfies the inequalities

$$\forall n \in \mathbb{N}_0, x_{n+1} - x_n > 0.$$

We say that $(x_n)_{n \in \mathbb{N}_0}$ is decreasing if it satisfies the inequalities

$$\forall n \in \mathbb{N}_0, x_{n+1} - x_n \leq 0,$$

and it is said to be strictly decreasing if it satisfies the inequalities

$$\forall n \in \mathbb{N}_0, x_{n+1} - x_n < 0.$$

We say that $(x_n)_{n \in \mathbb{N}_0}$ is monotone (strictly monotone) if it is either increasing (strictly increasing) or decreasing (strictly decreasing).

Theorem 3.9. [Monotone Convergence Theorem]

A monotone sequence of real numbers is convergent if and only if it is bounded. Further :

(a) If $(x_n)_{n \in \mathbb{N}_0}$ is a bounded increasing sequence, then

$$\lim_{n \rightarrow +\infty} x_n = \sup \{x_n : n \in \mathbb{N}_0\}.$$

(b) If $(x_n)_{n \in \mathbb{N}_0}$ is a bounded decreasing sequence, then

$$\lim_{n \rightarrow +\infty} x_n = \inf \{x_n : n \in \mathbb{N}_0\}.$$

Proof :

It was seen that a convergent sequence must be bounded.

Conversely, let $(x_n)_{n \in \mathbb{N}_0}$ be a bounded monotone sequence. Then $(x_n)_{n \in \mathbb{N}_0}$ is either increasing or decreasing.

(a) We first treat the case where $(x_n)_{n \in \mathbb{N}_0}$ is a bounded, increasing sequence. Since $(x_n)_{n \in \mathbb{N}_0}$ is bounded, there exists a real number M such that

$$\forall n \in \mathbb{N}_0, x_n < M.$$

So $\ell = \sup \{x_n : n \in \mathbb{N}_0\}$ exists in $\mathbb{R}$. We will show that $\ell = \lim_{n \rightarrow +\infty} x_n$.

If $\varepsilon > 0$ is given, then $\ell - \varepsilon$ is not an upper bound of the set $\{x_n : n \in \mathbb{N}_0\}$, and hence there exists x_K such that $\ell - \varepsilon < x_K$. The fact that $(x_n)_{n \in \mathbb{N}_0}$ is an increasing sequence implies that $x_K \leq x_n$ whenever $n \geq K$, so that

$$\text{for all } n \geq K, \ell - \varepsilon < x_K \leq x_n \leq \ell < \ell + \varepsilon.$$

Therefore we have

$$\text{for all } n \geq K, |x_n - \ell| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(x_n)_{n \in \mathbb{N}_0}$ converges to ℓ .

(b) If $(x_n)_{n \in \mathbb{N}_0}$ is a bounded decreasing sequence, then it is clear that $(-x_n)_{n \in \mathbb{N}_0}$ is a bounded increasing sequence. It was shown in part (a) that

$$\lim_{n \rightarrow +\infty} (-x_n) = \sup \{-x_n : n \in \mathbb{N}_0\}.$$

And also, by Exercise (2.5).Statement (3), we have

$$\sup \{-x_n : n \in \mathbb{N}_0\} = -\inf \{x_n : n \in \mathbb{N}_0\}.$$

Therefore

$$\lim_{n \rightarrow +\infty} (-x_n) = -\lim_{n \rightarrow +\infty} (x_n) = -\inf \{x_n : n \in \mathbb{N}_0\}.$$

Hence

$$\lim_{n \rightarrow +\infty} (x_n) = \inf \{x_n : n \in \mathbb{N}_0\}.$$

□

Examples 3.2.

(1) Let $(u_n)_{n \in \mathbb{N}}$ sequence defined by

$$\begin{cases} u_0 = \frac{1}{2} \\ \forall n \in \mathbb{N}, u_{n+1} = \frac{1}{\sqrt{2}}u_n + 2. \end{cases}$$

1. Show that $(u_n)_{n \in \mathbb{N}}$ is an increasing sequence.
2. Deduce that it is convergent and determine its limit.

(2) Let $(v_n)_{n \in \mathbb{N}^*}$ sequence defined by

$$\begin{cases} v_1 \geq 2 \\ \forall n \in \mathbb{N}^*, v_{n+1} = 1 + \sqrt{v_n - 1}. \end{cases}$$

1. Show that $(v_n)_{n \in \mathbb{N}^*}$ is decreasing sequence.
2. Deduce that it is convergent and determine its limit.

(3) Let $(h_n)_{n \in \mathbb{N}}$ sequence defined by

$$\begin{cases} |h_0| \leq 1 \\ \forall n \in \mathbb{N}^*, h_{n+1} = \frac{1+h_n^2}{2}. \end{cases}$$

1. Show that $(h_n)_{n \in \mathbb{N}}$ is an increasing sequence.
2. Deduce that it is convergent and determine its limit.

Exercise 3.3. Let $a > 0$. $(s_n)_{n \in \mathbb{N}}$ is the sequence defined by

$$\begin{cases} |s_0| > 0 \text{ be arbitrary} \\ \forall n \in \mathbb{N}, s_{n+1} = \frac{1}{2} \left(s_n + \frac{a}{s_n} \right). \end{cases}$$

Show that $(s_n)_{n \in \mathbb{N}}$ is convergent and determine its limit.

3.5.1 Adjacent sequences

Definition 3.6. Two real sequences $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ are called adjacent if

1. $(u_n)_{n \in \mathbb{N}_0}$ is increasing.
2. $(v_n)_{n \in \mathbb{N}_0}$ is decreasing.
3. $\lim_{n \rightarrow +\infty} (u_n - v_n) = 0$.

Example 3.4. We defined a sequence $(u_n)_{n \in \mathbb{N}}$ by :

$$\begin{cases} u_0 = \frac{1}{2} \\ u_{n+1} = \frac{2}{1+u_n}. \end{cases}$$

And we define two other sequences $(k_n)_{n \in \mathbb{N}}$, $(h_n)_{n \in \mathbb{N}}$ by

$$\forall n \in \mathbb{N}, k_n = u_{2n} \text{ and } h_n = u_{2n+1}.$$

We want to answer the following questions.

1. Show that, $\forall n \in \mathbb{N}, \frac{1}{4} < u_n < 2$.
2. Prove that, $\forall n \in \mathbb{N}, k_n < 1 < h_n$.
3. Show that the sequence $(k_n)_{n \in \mathbb{N}}$ is increasing and the sequence $(h_n)_{n \in \mathbb{N}}$ is decreasing.
4. Deduce the limit of the sequence $(k_n)_{n \in \mathbb{N}}$ thus the limit $(h_n)_{n \in \mathbb{N}}$.
5. Deduct.
6. Establish the following inequality

$$\forall n \in \mathbb{N}, |u_{n+1} - 1| \leq \frac{4}{5} |u_n - 1|.$$

7. Prove that the sequence $(u_n)_{n \in \mathbb{N}}$ is convergent.

Theorem 3.10. Two adjacent sequences converge and their limits are equal.

Proof :

Let $(u_n)_{n \in \mathbb{N}}$, $(v_n)_{n \in \mathbb{N}}$ be two adjacent sequences and we assume that $(u_n)_{n \in \mathbb{N}}$ is increasing and $(v_n)_{n \in \mathbb{N}}$ is decreasing.

Note that, for all n in $\mathbb{N}$,

$$u_{n+1} - u_n \geq 0 \text{ and } v_{n+1} - v_n \leq 0.$$

Therefore, the sequence $(u_n - v_n)_{n \in \mathbb{N}}$ is increasing :

$$\forall n \in \mathbb{N}, (u_{n+1} - v_{n+1}) - (u_n - v_n) = (u_{n+1} - u_n) - (v_{n+1} - v_n) \geq 0.$$

It is therefore bounded from above by its limit 0, so

$$\forall n \in \mathbb{N}, u_n \leq v_n.$$

Then we can write

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \quad (3.1)$$

From (3.1) we notice that the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing and bounded from below by v_0 so, it converges.

We assume that

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

Likewise, the sequence $(v_n)_{n \in \mathbb{N}}$ is decreasing and bounded from above by u_0 so, it converges.

We assume that

$$\lim_{n \rightarrow +\infty} v_n = \ell'.$$

For all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$|u_k - v_k| < \frac{\varepsilon}{3}, \quad |u_k - \ell| < \frac{\varepsilon}{3}, \quad |v_k - \ell'| < \frac{\varepsilon}{3}.$$

The triangular inequality gives

$$|\ell - \ell'| \leq |\ell - u_k| + |u_k - v_k| + |v_k - \ell'| < \varepsilon,$$

and this for all $\varepsilon > 0$, therefore $\ell - \ell' = 0$.

3.5.2 Nested intervals

To two adjacent sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ we associate the family of intervals

$$\forall n \in \mathbb{N}, I_n = [u_n, v_n].$$

So,

$$I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$$

We say that family $(I_n)_{n \in \mathbb{N}}$ is made up of nested intervals whose length approaches to 0. As both sequences have the same limit ℓ , then

$$\bigcap_{n \in \mathbb{N}} I_n = \{\ell\}.$$

Reciprocally, For any sequence $(I_n)_{n \in \mathbb{N}}$ of nested intervals of $\mathbb{R}$ whose length approaches to 0, the sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ of the lower and upper extremities of the intervals are two adjacent sequences which therefore have a common limit ℓ , intersection of all the intervals I_n .

Proposition 3.1. [Nested Intervals Property] *If $I_n = [a_n; b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\ell \in \mathbb{R}$ such that $\ell \in I_n$ for all $n \in \mathbb{N}$. That's to say*

$$\bigcap_{n \in \mathbb{N}} I_n = \{\ell\}.$$

3.6 Subsequences

Definition 3.7. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and let

$$\begin{aligned} \varphi : \mathbb{N} &\rightarrow \mathbb{N} \\ n &\mapsto \varphi(n), \end{aligned}$$

be a strictly increasing function. Then the sequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ is called a subsequence of $(x_n)_{n \in \mathbb{N}}$.

Remark 3.1. The fact that the function φ is strictly increasing then we have

$$\forall n \in \mathbb{N}, \varphi(n) \geq n.$$

Example 3.5. If $(x_n)_{n \in \mathbb{N}}$ is the sequence given by

$$\forall n \in \mathbb{N}^*, x_n = \frac{1}{n},$$

then these sequences $(x_{2n})_{n \in \mathbb{N}}$, $(x_{2n+1})_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^*, v_n = \frac{1}{(2n)!},$$

are subsequences of $(x_n)_{n \in \mathbb{N}}$.

Theorem 3.11. If a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers converges to a real number x , then any subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ also converges to x .

Proof :

Let $\varepsilon > 0$ be given and let $K = K(\varepsilon)$ be such that if $n \geq K$, then $|x_n - x| < \varepsilon$.

As $\forall n \in \mathbb{N}, \varphi(n) \geq n$ so we have

$$\forall n \in \mathbb{N}, n \geq K \Rightarrow \varphi(n) \geq K.$$

Then we deduce that :

$$\forall n \in \mathbb{N}, n \geq K \Rightarrow |x_{\varphi(n)} - x| < \varepsilon.$$

Therefore the subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ also converges to x . □

3.7 Recursive Sequences

Let f be a function defined on an interval $I \subset \mathbb{R}$ such that $f(I) \subset I$.

Definition 3.8. The sequence $(u_n)_{n \in \mathbb{N}}$ defined by :

$$\begin{cases} u_0 \in I \\ \forall n \in \mathbb{N}, u_{n+1} = f(u_n), \end{cases}$$

is called a recursive sequence (or sequence defined recursively).

Example 3.6. Let $(v_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$\begin{cases} v_0 = 3 \\ \forall n \in \mathbb{N}, v_{n+1} = \frac{4v_n+2}{v_n+1}. \end{cases}$$

The sequence $(v_n)_{n \in \mathbb{N}}$ is defined by the function :

$$f(x) = \frac{4x+2}{x+1}, \quad I = [0, +\infty[.$$

We have $f(I) \subset I$ and therefore the sequence $(v_n)_{n \in \mathbb{N}}$ is well defined.

3.7.1 Monotonicity transfers from f to $(u_n)_{n \in \mathbb{N}}$

Case 1 : f is an increasing function

Proposition 3.2. If the function f is increasing on I then the sequence $(u_n)_{n \in \mathbb{N}}$ defined by f is monotone.

- a. $(u_n)_{n \in \mathbb{N}}$ is increasing if $u_1 - u_0 \geq 0$.
- b. $(u_n)_{n \in \mathbb{N}}$ is decreasing if $u_1 - u_0 \leq 0$.

Proof :

We assume that $u_1 - u_0 \geq 0$ and f is an increasing function. we prove by induction that the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing

We denoted by $P(n)$ the statement,

$$\forall n \in \mathbb{N}, u_{n+1} - u_n \geq 0.$$

1. $P(0)$ is true by hypothesis.
2. We assume that $P(n)$ is true for some natural numbers n and we need to prove that $P(n+1)$ is also true. We therefore have $u_{n+1} \geq u_n$, and as f is increasing. We can write

$$\begin{aligned} u_{n+1} \geq u_n &\Leftrightarrow f(u_{n+1}) \geq f(u_n) \\ &\Leftrightarrow u_{n+2} \geq u_{n+1}. \end{aligned}$$

Hence $P(n+1)$ is true. According to the principle of induction, we have

$$\forall n \in \mathbb{N}, u_{n+1} \leq u_n.$$

So the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing.

We proceed in the same way if $u_1 - u_0 \leq 0$.

□

Case 2 : f is decreasing function

Let $I \subset \mathbb{R}$, the function $f : I \rightarrow I$ is decreasing on I and the sequence $(u_n)_{n \in \mathbb{N}}$ is defined by f .

The sequence of even terms $(u_{2n})_{n \in \mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N}, u_{2n+2} = f(u_{2n+1}) = f(f(u_{2n})) = f \circ f(u_{2n}).$$

Similarly, the sequence of odd terms $(u_{2n+1})_{n \in \mathbb{N}}$ verifies

$$\forall n \in \mathbb{N}, u_{2n+3} = f(u_{2n+2}) = f(f(u_{2n+1})) = f \circ f(u_{2n+1}).$$

Furthermore, we have this result

$$f \text{ is decreasing} \Rightarrow f \circ f \text{ is increasing.}$$

Proposition 3.3. *If the function f is decreasing on I then the sequence $(u_n)_{n \in \mathbb{N}}$ defined by f is not monotone. Furthermore we have*

$$\begin{aligned} u_2 \geq u_0 &\Rightarrow \begin{cases} (u_{2n})_{n \in \mathbb{N}} \text{ is increasing} \\ (u_{2n+1})_{n \in \mathbb{N}} \text{ is decreasing.} \end{cases} \\ u_2 \leq u_0 &\Rightarrow \begin{cases} (u_{2n})_{n \in \mathbb{N}} \text{ is decreasing} \\ (u_{2n+1})_{n \in \mathbb{N}} \text{ is increasing.} \end{cases} \end{aligned}$$

Proof :

We assume that $u_2 \geq u_0$ and we prove by induction that the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing. $P(n)$ is the stament

$$\forall n \in \mathbb{N}, u_{2n+2} \geq u_{2n}.$$

$P(0)$ is true by hypothesis.

We assume that $P(n)$ is true for some natural numbers n and we need to prove that $P(n+1)$ is also true. We have

$$\begin{aligned} u_{2n+2} \geq u_{2n} &\Leftrightarrow f \circ f(u_{2n+2}) \geq f \circ f(u_{2n}) \\ &\Leftrightarrow u_{2n+4} \geq u_{2n+2}. \end{aligned}$$

We deduce that $P(n+1)$ is true. According to the principle of induction, we have

$$\forall n \in \mathbb{N}, u_{2n+2} \geq u_{2n}.$$

So the sequence $(u_{2n})_{n \in \mathbb{N}}$ is increasing.

Let us now show that $(u_{2n+1})_{n \in \mathbb{N}}$ is decreasing.

As we have $u_2 \geq u_0$ and f decreasing, we have $f(u_2) \leq f(u_0)$ ie $u_3 \geq u_1$.

We would then prove, by induction, that

$$\forall n \in \mathbb{N}, u_{2n+3} \leq u_{2n+1}.$$

We proceed in the same way if $u_2 \leq u_0$.

□

Definition 3.9. Let $\ell \in I$ where $I \subset \mathbb{R}$ is an interval. We say that ℓ is a fixed point of f if and only if $f(\ell) = \ell$.

Theorem 3.12. Let $\ell \in I$ where $I \subset \mathbb{R}$ is an interval and $(u_n)_{n \in \mathbb{N}}$ be a recursive sequence defined by a function $f : I \rightarrow \mathbb{R}$. We assume that f is continuous on I . So, we have

$$u_n \longrightarrow \ell \implies f(\ell) = \ell.$$

Proof :

From the equation $u_{n+1} = f(u_n)$, we obtain by passing to the limit :

$$\ell = \lim_{n \rightarrow +\infty} u_{n+1} = \lim_{n \rightarrow +\infty} f(u_n).$$

The continuity of the function f at point ℓ gives the equation $\ell = f(\ell)$. □

Example 3.7. Let the sequence $(u_n)_{n \in \mathbb{N}}$ be defined by :

$$\begin{cases} u_0 = 2 \\ \forall n \in \mathbb{N}, u_{n+1} = \frac{5u_n + 4}{u_n + 2}. \end{cases}$$

We can prove by induction, that

$$\forall n \in \mathbb{N}, 0 < u_n < 4. \quad (3.2)$$

We have

$$f(x) = \frac{5x + 4}{x + 2} \text{ and } \forall x > 0, f'(x) > 0.$$

Therefore f is increasing on $I =]0, +\infty[$ then $(u_n)_{n \in \mathbb{N}}$ is monotone sequence.

As

$$u_1 = \frac{5u_0 + 4}{u_0 + 2} = \frac{14}{4} = \frac{7}{2} > u_0.$$

So, the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing and bounded from above by 4 then it's convergent.

Let $\ell = \lim_{n \rightarrow +\infty} u_n$. Since the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing then verify (3.2), therefore ℓ belongs to $[0; 4]$.

As the function f defined by $f(x) = \frac{5x+4}{x+2}$ is continuous on the interval $[0; 4]$, then ℓ is solution of the equation

$$f(x) = x, x \in [0; 4].$$

We have, for all $x \in [0; 4]$

$$f(x) = x \Leftrightarrow \frac{5x + 4}{x + 2} = x \Leftrightarrow x^2 - 3x - 4 = 0 \Leftrightarrow x = -1 \text{ or } x = 4.$$

As $-1 \notin [0; 4]$ so, $\ell = 4$.

3.8 The Cauchy Criterion

Definition 3.10. A sequence $(x_n)_{n \in \mathbb{N}_0}$ of real numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a natural number $K = K(\varepsilon)$ such that for all natural numbers $n, m \geq K$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

Examples 3.3.

1. The sequence $(\frac{1}{n})_{n \in \mathbb{N}^*}$ is a Cauchy sequence. If $\varepsilon > 0$ is given, we choose a natural number $K = K(\varepsilon)$ such that $K > \frac{2}{\varepsilon}$. Then if $m, n \geq K$, we have

$$\frac{1}{n} \leq \frac{1}{K} < \frac{\varepsilon}{2}, \text{ and } \frac{1}{m} < \frac{\varepsilon}{2}.$$

Therefore, it follows that if $m, n \geq K$, then

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(\frac{1}{n})_{n \in \mathbb{N}^*}$ is a Cauchy sequence.

2. The sequence $(x_n = 2 + (-1)^n)_{n \in \mathbb{N}}$ is not a Cauchy sequence. The negation of the definition of Cauchy sequence is : There exists $\varepsilon > 0$ such that for every K there exist at least one $n > K$ and at least one $m > K$ such that $|x_n - x_m| \geq \varepsilon$. For the terms $x_n = 2 + (-1)^n$, we observe that if n is even, then $x_n = 3$ and $x_{n+1} = 1$. If we take $\varepsilon = 2$, then for any K we can choose an even number $n > K$ and let $m = n + 1$ to get

$$|x_n - x_{n+1}| = 2 = \varepsilon.$$

We conclude that $(x_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence.

Lemma 3.2. Lemma If $(x_n)_{n \in \mathbb{N}_0}$ is a convergent sequence of real numbers, then it is a Cauchy sequence.

Proof :

If $x = \lim_{n \rightarrow +\infty} x_n$, then given $\varepsilon > 0$ there is a natural number $K = K(\frac{\varepsilon}{2})$ such that if $n \geq K$ then $|x_n - x| < \frac{\varepsilon}{2}$.

Thus, if $K' = K'(\varepsilon) = K$ and if $n, m \geq K'$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $(x_n)_{n \in \mathbb{N}_0}$ is a Cauchy sequence. $\square$

In order to establish that a Cauchy sequence is convergent, we will need the following result.

Lemma 3.3. A Cauchy sequence of real numbers is bounded.

Proof :

Let $(x_n)_{n \in \mathbb{N}_0}$ be a Cauchy sequence and let $\varepsilon = 1$.

If $K = K(1)$ and $n \geq K$, then $|x_n - x_K| < 1$.

Hence, by the Triangle Inequality, we have

$$\forall n \geq K, |x_n| \leq |x_K| + 1.$$

If we set $M = \sup\{|x_1|, |x_2|, \dots, |x_{K-1}|, |x_K| + 1\}$, then it follows that

$$\forall n \in \mathbb{N}_0, |x_n| < M.$$

□

Proposition 3.4. *If a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ contains a convergent subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} x_{\varphi(n)} = \ell$, then the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to ℓ .*

Proof :

Let $(x_{\varphi(n)})_{n \in \mathbb{N}}$ be a subsequence of $(x_n)_{n \in \mathbb{N}}$ and which converges to ℓ . We have

$$|x_n - \ell| = |x_n - x_{\varphi(n)} + x_{\varphi(n)} - \ell| \leq |x_n - x_{\varphi(n)}| + |x_{\varphi(n)} - \ell|.$$

The sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and from the fact that *forall* $n \in \mathbb{N}$, $\varphi(n) \geq n$, then

$$\forall \varepsilon > 0, \exists K \in \mathbb{N} : \forall n, n \geq K \Rightarrow |x_{\varphi(n)} - x_n| < \frac{\varepsilon}{2},$$

and since the subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ is convergent to ℓ , so we can write

$$\forall \varepsilon > 0, \exists K' \in \mathbb{N} : \forall n, n \geq K' \Rightarrow |x_{\varphi(n)} - \ell| < \frac{\varepsilon}{2}.$$

For $k'' = \sup\{K, K'\}$, we get

$$\forall \varepsilon > 0, \forall n, n \geq K'' \Rightarrow |x_n - \ell| < \varepsilon.$$

Then the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and converges to ℓ .

□

Theorem 3.13. [Monotone Subsequence Theorem] *If $(x_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, then there is a subsequence of $(x_n)_{n \in \mathbb{N}}$ that is monotone.*

Proof :

The proof is for the students.

□

Theorem 3.14. [The Bolzano-Weierstrass Theorem] *A bounded sequence of real numbers has a convergent subsequence.*

Proof :

It follows from the Monotone Subsequence Theorem that if $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence, then it has a subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem that the subsequence is convergent. $\square$

We now present the important Cauchy Convergence Criterion.

Theorem 3.15. [Cauchy Convergence Criterion] *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof :

We have seen, in Lemma 3.2, that a convergent sequence is a Cauchy sequence.

Conversely, let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence, we will show that $(x_n)_{n \in \mathbb{N}}$ is convergent to some real number.

First we observe from Lemma 3.3 that the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded.

Therefore, by the Bolzano-Weierstrass Theorem, there is a subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ that converges to some real number ℓ . We shall complete the proof by showing that $(x_n)_{n \in \mathbb{N}}$ converges to ℓ .

Since $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $K = K(\frac{\varepsilon}{2})$ such that if $n, m \geq K$ then

$$|x_n - x_m| < \frac{\varepsilon}{2}. \quad (3.3)$$

Since the subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ converges to ℓ , there is a natural number $K' = K'(\frac{\varepsilon}{2}) \geq K$ belongs to $\{\varphi(n) : n \in \mathbb{N}\}$ such that

$$|x'_{K'} - \ell| < \frac{\varepsilon}{2}. \quad (3.4)$$

Since $K' \geq K$, it follows from (3.3) with $m = K'$ that

$$\forall n \geq K, |x_n - x_{K'}| < \frac{\varepsilon}{2}$$

Therefore, if $n \geq K$, we have

$$\begin{aligned} |x_n - \ell| &= |x_n - x_{K'} + x_{K'} - \ell| \\ &\leq |x_n - x_{K'}| + |x_{K'} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\lim_{n \rightarrow +\infty} x_n = \ell$. Therefore the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent. $\square$

Examples :

$\square$ Let $(u_n)_{n \in \mathbb{N}^*}$ be defined by

$$u_1 = 1, u_2 = 2, \text{ and } \forall n > 2, u_n = \frac{1}{2}(u_{n-2} + u_{n-1}).$$

It can be shown by Induction that

$$\forall n \in \mathbb{N}^*, 1 \leq u_n \leq 2.$$

We can prove that the sequence $(u_n)_{n \in \mathbb{N}^*}$ is not monotone. However, it is readily seen that

$$\forall n \in \mathbb{N}^*, |u_{n+1} - u_n| = \frac{1}{2^{n-1}}.$$

Thus, if $m > n$, we may employ the Triangle Inequality to obtain

$$\begin{aligned} |u_n - u_m| &\leq |u_n - u_{n+1}| + |u_{n+1} - u_{n+2}| + \cdots + |u_{m-1} - u_m| \\ &= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \\ &= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}} \right) < \frac{1}{2^{n-2}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $\frac{1}{2^n} < \frac{\text{varepsilon}}{4}$ and if $m \geq n$, then it follows that

$$|u_n - u_m| < \varepsilon.$$

Therefore, $(u_n)_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{R}$.

By the Cauchy Criterion we infer that the sequence $(u_n)_{n \in \mathbb{N}^*}$ converges to a number ℓ .

The subsequence $(u_{2n+1})_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}^*}$ satisfies

$$\begin{aligned} u_{2n+1} &= 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} \\ &= 1 + \frac{2}{3} \left(1 - \frac{1}{4^n} \right). \end{aligned}$$

From (3.4), we can right

$$\ell = \lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} u_{2n+1} = 1 + \frac{2}{3} = \frac{5}{3}.$$

Definition 3.11. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers is contractive if there exists a constant α , $0 < \alpha < 1$, such that

$$\forall n \in \mathbb{N}, |x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|.$$

The number α is called the constant of the contractive sequence.

Theorem 3.16. Every contractive sequence is a Cauchy sequence, and therefore is convergent.

Proof :

Let $(x_n)_{n \in \mathbb{N}}$ be contractive sequence. So there exists real number $0 < \alpha < 1$ such that

$$\forall n \in \mathbb{N}, |x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|.$$

By induction, we can write

$$\begin{aligned} |x_{n+2} - x_{n+1}| &\leq \alpha |x_{n+1} - x_n| \\ &\leq \alpha^2 |x_n - x_{n-1}| \\ &\leq \vdots \vdots \vdots \vdots \\ &\leq \alpha^n |x_2 - x_1|. \end{aligned}$$

For $m > n$, by the Triangle Inequality and then using the formula for the sum of a geometric progression, we obtain

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq (\alpha^{m-2} + \alpha^{m-3} + \cdots + \alpha^{n-1}) |x_2 - x_1| \\ &\leq \alpha^{n-1} \left(\frac{1 - \alpha^{m-n}}{1 - \alpha} \right) |x_2 - x_1| \\ &\leq \alpha^{n-1} \left(\frac{1}{1 - \alpha} \right) |x_2 - x_1| \end{aligned}$$

Since $0 < \alpha < 1$, we know $\lim_{n \rightarrow +\infty} \alpha^n = 0$. Therefore, we infer that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. It now follows from the Cauchy Convergence Criterion that $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence.

Exercise 3.4. We consider the sequence of Fibonacci fractions $x_n = \frac{f_n}{f_{n+1}}$ where

$$f_1 = f_2 = 1 \text{ and } \forall n \in \mathbb{N}^*, f_{n+1} = f_n + f_{n-1}.$$

1. Show that

$$\forall n \in \mathbb{N}^*, \frac{1}{2} \leq x_n \leq 1.$$

2. Prove that

$$\forall n \in \mathbb{N}^*, |x_{n+1} - x_n| \leq \frac{4}{9} |x_n - x_{n-1}|.$$

3. Deduce that the sequence $(x_n)_{n \in \mathbb{N}^*}$ is convergent and determine its limit.

3.9 Properly Divergent Sequences

Definition 3.12. A sequence which is not convergent is called a divergent sequence.

Divergence Criteria 3.1. If a sequence $(x_n)_{n \in \mathbb{N}_0}$ of real numbers has either of the following properties, then $(x_n)_{n \in \mathbb{N}_0}$ is divergent.

1. $(x_n)_{n \in \mathbb{N}_0}$ has two convergent subsequences $(x_{\varphi(n)})_{n \in \mathbb{N}_0}$ and $(x_{\psi(n)})_{n \in \mathbb{N}_0}$ whose limits are not equal.
2. $(x_n)_{n \in \mathbb{N}_0}$ is unbounded.

Example :

▼ Let $(u_n)_{n \in \mathbb{N}}$ be a sequence defined by

$$\forall n \in \mathbb{N}, u_n = (-1)^n,$$

We have $(u_{2n})_{n \in \mathbb{N}}$ and $(u_{2n+1})_{n \in \mathbb{N}}$ are two subsequences of $(u_n)_{n \in \mathbb{N}}$ whose limits are 1 and (-1) respectively. So the sequence $(u_n)_{n \in \mathbb{N}}$ is divergent sequence.

▼▼ Let $(h_n)_{n \in \mathbb{N}}$ be a sequence defined by

$$\forall n \in \mathbb{N}, h_n = (n)^2 + n.$$

This sequence is divergent because it is unbounded. □

Definition 3.13. Let $(x_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers.

(1) We say that $(x_n)_{n \in \mathbb{N}_0}$ tends to $+\infty$, and write $\lim_{n \rightarrow +\infty} x_n = +\infty$, if

$$\forall A \in \mathbb{R} \exists K = K(A), n \geq K \Rightarrow x_n > A.$$

(2) We say that $(x_n)_{n \in \mathbb{N}_0}$ tends to $-\infty$ and write $\lim_{n \rightarrow +\infty} x_n = -\infty$, if

$$\forall B \in \mathbb{R} \exists K = K(B), n \geq K \Rightarrow x_n < B.$$

We say that is properly divergent in case we have either $\lim_{n \rightarrow +\infty} x_n = +\infty$ or $\lim_{n \rightarrow +\infty} x_n = -\infty$

Theorem 3.17. A monotone sequence of real numbers is properly divergent if and only if it is unbounded.

(a) If $(x_n)_{n \in \mathbb{N}_0}$ is an unbounded increasing sequence, then $\lim_{n \rightarrow +\infty} x_n = +\infty$.

(b) If $(x_n)_{n \in \mathbb{N}_0}$ is an unbounded decreasing sequence, then $\lim_{n \rightarrow +\infty} x_n = -\infty$.

Proof :

(a) Suppose that $(x_n)_{n \in \mathbb{N}_0}$ is an increasing sequence. We know that if $(x_n)_{n \in \mathbb{N}_0}$ is bounded, then it is convergent.

If $(x_n)_{n \in \mathbb{N}_0}$ is unbounded, then for any $a \in \mathbb{R}$ there exists $n = n(a) \in \mathbb{N}$ such that $a < x_{n(a)}$. But since $(x_n)_{n \in \mathbb{N}_0}$ is increasing, we have for all $n \geq n(a)$ $x_n \geq x_{n(a)} > a$.

Since a is arbitrary, it follows that $\lim_{n \rightarrow +\infty} x_n = +\infty$.

(b) is proved in a similar way. □

Theorem 3.18. Let $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ be two sequences of real numbers and suppose that

$$\forall n \in \mathbb{N}_0, u_n \leq v_n. \quad (3.5)$$

(1) If $\lim_{n \rightarrow +\infty} u_n = +\infty$, then $\lim_{n \rightarrow +\infty} v_n = +\infty$

(2) If $\lim_{n \rightarrow +\infty} v_n = -\infty$, then $\lim_{n \rightarrow +\infty} u_n = -\infty$

Proof :

(1) If $\lim_{n \rightarrow +\infty} u_n = +\infty$, and if $a \in \mathbb{R}$ is given, then there exists a natural number $K(a)$ such that if $n \geq K(a)$, then $a < u_n$. In view of (3.5), it follows that $a < v_n$ for all $n \geq K(a)$. Since a is arbitrary, it follows that $\lim_{n \rightarrow +\infty} v_n = +\infty$.

The proof of (2) is similar of (1). □

3.10 Limit Superior and Limit Inferior

Definition 3.14. Let $(x_n)_{n \in \mathbb{N}_0}$ be a bounded sequence of real numbers.

(1) The limit superior of $(x_n)_{n \in \mathbb{N}_0}$ is the infimum of the set

$$X = \{\alpha \in \mathbb{R} : x_n > \alpha \text{ for at most a finite number of } n \in \mathbb{N}_0\}.$$

It is denoted by $\limsup(x_n)$ or $\overline{\lim}(x_n)$.

(2) The limit inferior of $(x_n)_{n \in \mathbb{N}_0}$ is the supremum of the set

$$S = \{w \in \mathbb{R} : x_m < w \text{ for at most a finite number of } m \in \mathbb{N}_0\}.$$

It is denoted by $\liminf(x_n)$ or $\underline{\lim}(x_n)$.

Theorem 3.19. If $(x_n)_{n \in \mathbb{N}_0}$ is a bounded sequence of real numbers, then the following statements for a real number δ are equivalent.

1. $\delta = \limsup(x_n)$.
2. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}_0$ such that $\delta + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}_0$ such that $\delta - \varepsilon < x_n$.
3. If $u_m = \sup\{x_n : n \geq m\}$; then $\delta = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.
4. If S is the set of subsequential limits of x_n , then $\delta = \sup S$.

Proof :

The proof is for the students.

4

Real Functions Of A Real Variable

4.1 Generalities

Definition 4.1. Let X and Y are nonempty sets. Then $f : X \longrightarrow Y$ is called a function from the set X to the set Y if for every $x \in X$ there is at most only one element $y \in Y$ such that $y = f(x)$.

► If $E = F = \mathbb{R}$, f is said a real function of a real variable and is denoted by

$$\begin{array}{ccc} f : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x). \end{array}$$

► To simplify, we will say a function instead of real function of a real variable.

Definition 4.2. Let f be a function. The set, denoted by $\mathcal{D}_f$, of values of x for which $f(x)$ is defined is called domain of definition of f . In another way

$$\mathcal{D}_f = \{x \in \mathbb{R} : f(x) \text{ exists}\}.$$

Examples :

1. We want to determine the domain of definition set of the following function

$$\begin{array}{ccc} f : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) = \frac{x^2+1}{|x|-x}. \end{array}$$

We have

$$\mathcal{D}_f = \{x \in \mathbb{R} : |x| - x \neq 0\} =]-\infty, 0[$$

2. The domain of the function g defined by

$$\begin{array}{ccc} g : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & g(x) = \sqrt{\frac{x+1}{x-1}}. \end{array}$$

is

$$\begin{aligned}\mathcal{D}_g &= \left\{ x \in \mathbb{R} : \frac{x+1}{x-1} \geq 0 \text{ and } x-1 \neq 0 \right\} \\ &= \{ x \in \mathbb{R} : (x+1)(x-1) \geq 0 \text{ and } x \neq 1 \} \\ &=]-\infty, -1] \cup]1, +\infty[.\end{aligned}$$

4.1.1 Graph of a function

Definition 4.3. The graph of a function f is the set, denoted $\Gamma(f)$, defined by

$$\Gamma(f) = \{ (x, y) \in \mathbb{R}^2 : x \in \mathcal{D}_f \text{ and } y = f(x) \}.$$

4.2 Operations on functions

Definition 4.4. Let f, g are two functions, $\mathcal{D}_f$ and $\mathcal{D}_g$ are their domains of definition respectively and λ is a real number.

1. The addition of these two function is the function, denoted by $f + g$ given as

$$\mathcal{D}_{f+g} = \mathcal{D}_f \cap \mathcal{D}_g \text{ and } [\forall x \in \mathcal{D}_{f+g}, (f+g)(x) = f(x) + g(x)].$$

2. The product of λ by the function f is the function, denoted by $\lambda.f$ given as

$$\mathcal{D}_{\lambda.f} = \mathcal{D}_f \text{ and } [\forall x \in \mathcal{D}_f, (\lambda.f)(x) = \lambda.f(x)].$$

3. The product of these two function is the function, denoted by $f.g$ given as

$$\mathcal{D}_{f.g} = \mathcal{D}_f \cap \mathcal{D}_g \text{ and } [\forall x \in \mathcal{D}_{f.g}, (f.g)(x) = f(x).g(x)].$$

4. The quotient of these two function is the function, denoted by $\frac{f}{g}$ given as

$$\forall x \in \mathcal{D}_{\frac{f}{g}}, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

where

$$\mathcal{D}_{\frac{f}{g}} = \{ x \in \mathcal{D}_f \cap \mathcal{D}_g : g(x) \neq 0 \}.$$

5. The composite function $g \circ f$ is the function defined by

$$\forall x \in \mathcal{D}_{f \circ g}, (g \circ f)(x) = g(f(x)),$$

where

$$\mathcal{D}_{g \circ f} = \{ x \in \mathbb{R} : x \in \mathcal{D}_f \text{ and } f(x) \in \mathcal{D}_g \}.$$

4.2.1 Even and odd functions

Definition 4.5. Let f be a function and $\mathcal{D}_f$ its domain of definition.

► f is called an **even function** if it satisfies

$$\forall x \in \mathcal{D}_f, -x \in \mathcal{D}_f \text{ and } f(-x) = f(x).$$

► f is called an **odd function** if it satisfies

$$\forall x \in \mathcal{D}_f, -x \in \mathcal{D}_f \text{ and } f(-x) = -f(x).$$

Examples :

[a.] Let f the function defined by

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{x^2 + 2|x| + 1}{x^2 + |x|}. \end{aligned}$$

The domain of definition of f is $\mathcal{D}_f = \mathbb{R}^*$.

Let $x \in \mathbb{R}^*$. We have $-x \in \mathbb{R}^*$. Furthermore

$$f(-x) = \frac{(-x)^2 + 2|-x| + 1}{(-x)^2 + |-x|} = \frac{x^2 + 2|x| + 1}{x^2 + |x|} = f(x).$$

Since $x \in \mathbb{R}^*$ is arbitrary, we conclude that f is an even function.

[b.] Let g the function defined by

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) = x\sqrt{\frac{x^2 + 1}{x^2 - 1}}. \end{aligned}$$

The function g is an odd function. Indeed,

$$\mathcal{D}_g = \left\{ x \in \mathbb{R} : \frac{x^2 + 1}{x^2 - 1} \geq 0 \text{ and } x^2 - 1 \neq 0 \right\} =]-\infty, -1[\cup]1, +\infty[,$$

moreover, if x is in $\mathcal{D}_g$, $-x$ belongs to $\mathcal{D}_g$. Furthermore

$$g(-x) = -x\sqrt{\frac{(-x)^2 + 1}{(-x)^2 - 1}} = -x\sqrt{\frac{x^2 + 1}{x^2 - 1}} = -g(x).$$

□

Properties 4.2.1. Some basic properties of odd and even functions are :

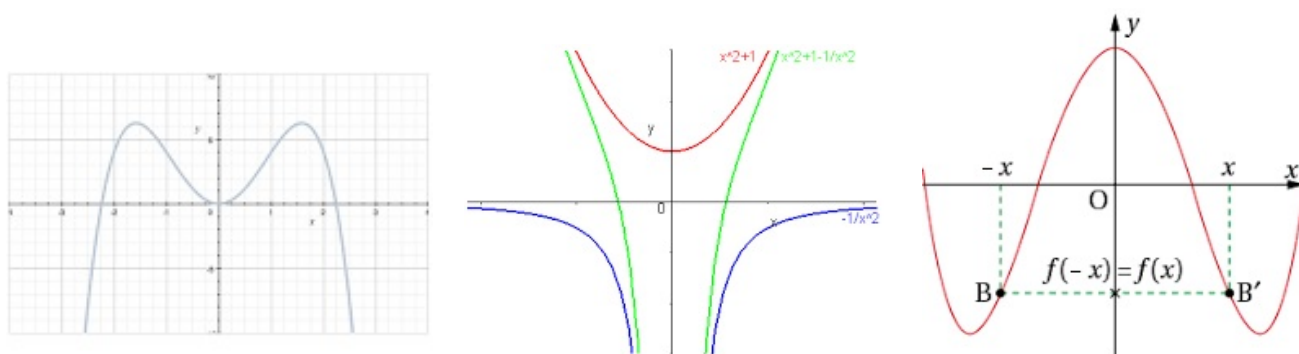
1. The only function whose domain is all real numbers which is both odd and even, is the constant function which is identically zero,

$$\forall x \in \mathbb{R}, f(x) = 0.$$

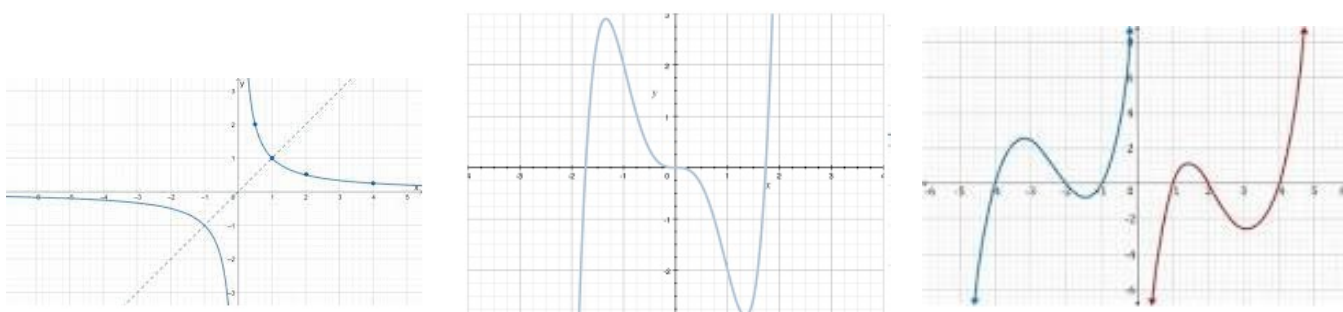
2. The sum of two even functions is even, and the sum of two odd functions is odd.
3. The product of two even functions is even, and the product of two odd functions is even.
4. The product of an even function and an odd function is an odd function.
5. The quotient of two even functions is even, and the quotient of two odd functions is even.
6. The quotient of an even function and an odd function is an odd function.
7. If f is an even function then the function $g \circ f$ is an even function.
8. If f is an odd function and g is an even function then $g \circ f$ is an even function and if g is odd then $g \circ f$ is an odd function.

4.2.2 Even and Odd Functions Graph

The graph of an even function is symmetric with respect to the y -axis. In other words, the graph of an even function remains the same after reflection about the y -axis. For any two opposite input values of x , the function value will remain the same all along the curve. Whereas the graph of an odd function is symmetric with respect to the origin. In other words, the graph of an odd function is at the same distance from the origin but in opposite directions. For any two opposite input values of x , the function has opposite y values.



Graphs of even functions



Graphs of odd functions

4.2.3 Periodic Functions

Definition 4.6. Let f be a function and $\mathcal{D}_f$ its domain of definition. f is said to be periodic if there exists a strictly positive real number P such that

$$\forall x \in \mathcal{D}_f, (x - P) \in \mathcal{D}_f, (x + P) \in \mathcal{D}_f, \text{ and } f(x + P) = f(x).$$

Remark 4.1. If f admits P for period then f also admits nP with $n \in \mathbb{Z}$ as period. This is why we will always determine the smallest period when looking for the period of a function.

Definition 4.7. We say that f is P -periodic (or periodic with period P) if P is the smallest period for f . Furthermore We then define the pulsation from f by :

$$\omega = \frac{2\pi}{P},$$

and the amplitude A of f by :

$$A = \frac{\max(f) - \min(f)}{2}.$$

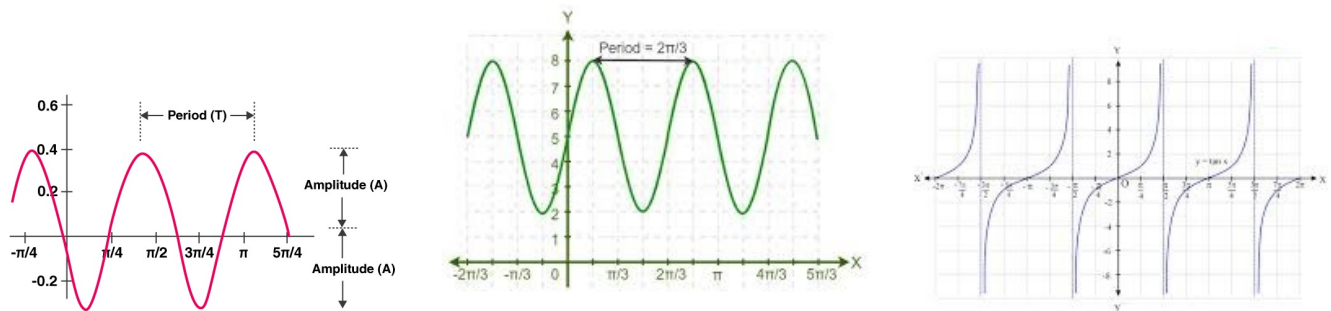
Examples :

1. The function $x \mapsto \cos(x)$ is 2π -periodic.
2. The function $x \mapsto \tan(x)$ is π -periodic.

Remark 4.2.

1. To define a P -periodic function, it is enough to know it over an interval of length P .
2. The graph of P -periodic function is symmetrical and repeats along the horizontal axis over intervals of length P .

Examples : Graphs of P -periodic functions :



Graphs of P -periodic functions

4.3 Restrictions of Functions

Definition 4.8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, $\mathcal{D}_f$ its domain of definition and $A \subset \mathbb{R}$, we can define a function $f_1 : A \rightarrow \mathbb{R}$ by

$$\forall x \in A \cap \mathcal{D}_f, f_1(x) = f(x).$$

The function f_1 is called the restriction of f to A . Sometimes it is denoted by $f_1 = f \setminus A$.

Example :

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ given by :

$$f(x) = \ln\left(\frac{x+1}{x-1}\right).$$

We know that $\mathcal{D}_f =]-\infty, -1[\cup]1, +\infty[$.

Let $A = [0, +\infty[$ and we define the function $g : A \rightarrow \mathbb{R}$ by

$$\forall x \in A \cap \mathcal{D}_f, g(x) = f(x).$$

The function g is the restriction of f to A . □

4.4 Bounded Functions

Definition 4.9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, $\mathcal{D}_f$ its domain of definition, and A is a subset of $\mathcal{D}_f$. We say that f is bounded above on A if the set

$$f(A) = \{f(x) : x \in A\}$$

is bounded above in $\mathbb{R}$, that is, there exists $B \in \mathbb{R}$ such that

$$\forall x \in A, f(x) \leq B.$$

Similarly, the function f is bounded below on A if the set $f(A)$ is bounded below in $\mathbb{R}$, that is, there exists $m \in \mathbb{R}$ such that

$$\forall x \in A, f(x) \geq m.$$

We say that f is bounded on A if it is bounded above and below on A ; this is equivalent to saying that there exists $M \in \mathbb{R}$ such that

$$\forall x \in A, |f(x)| \leq M.$$

Examples :

1. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$x \mapsto f(x) = \frac{3}{2+x^2}$$

is bounded. In fact, we have

$$\mathcal{D}_f = \mathbb{R}, \forall x \in \mathcal{D}_f, 0 \leq f(x) \leq \frac{3}{2}.$$

2. We show that the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$x \mapsto g(x) = \frac{4x}{4+x^2}$$

is bounded.

We have $\mathcal{D}_g = \mathbb{R}$.

We notice that g is an odd function and for all $x \in \mathbb{R}_+$, $g(x) \geq 0$.

On the other hand, we have

$$\forall x \in \mathbb{R}_+, g(x) = \frac{4x}{4+x^2} \leq \frac{4+x^2}{4+x^2} = 1.$$

since g is odd then we obtain

$$\forall x \in \mathbb{R}, -1 \leq g(x) \leq 1.$$

Therefore g is bounded.

3. The function $f : [2; +\infty[\rightarrow \mathbb{R}; f(x) = \frac{x^2+2x+2}{x+1}$ it is bounded above . Since,

$$\mathcal{D}_f = [2; +\infty[\text{ and } \forall x \in \mathcal{D}_f, f(x) \geq 2.$$

4.5 Monotone functions

Definition 4.10. Let f be a function $\mathcal{D}_f$ its domain of definition and I be an interval of $\mathcal{D}_f$.

1. f is said to be strictly increasing on I if

$$\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

2. f is said to be increasing on I if

$$\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2).$$

3. f is said to be strictly decreasing on I if

$$\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

4. f is said to be decreasing on I if

$$\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2).$$

5. f is said to be constant on I if

$$\forall x_1, x_2 \in I, f(x_1) = f(x_2).$$

6. f is said to be strictly monotone on I if f is strictly increasing or strictly decreasing on I .

7. f is said to be monotone on I if f is increasing or decreasing on I .

Remark 4.3. We can use the following fraction called growth rate or the decay rate of f

$$x_1, x_2 \in I, \frac{f(x_1) - f(x_2)}{x_1 - x_2} \text{ and } x_1 \neq x_2.$$

We can say that

1. f is strictly increasing on I if

$$\forall x_1, x_2 \in I, x_1 \neq x_2 \text{ and } \frac{f(x_1) - f(x_2)}{x_1 - x_2} > 0.$$

2. f is increasing on I if

$$\forall x_1, x_2 \in I, x_1 \neq x_2 \text{ and } \frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq 0.$$

3. f is strictly decreasing on I if

$$\forall x_1, x_2 \in I, x_1 \neq x_2 \text{ and } \frac{f(x_1) - f(x_2)}{x_1 - x_2} < 0.$$

4. f is decreasing on I if

$$\forall x_1, x_2 \in I, x_1 \neq x_2 \text{ and } \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq 0.$$

5. f is constant on I if

$$\forall x_1, x_2 \in I, x_1 \neq x_2 \text{ and } \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0.$$

Examples :

(1) We show that the function f defined on $\mathbb{R}$ by $f(x) = x^2 + x + 1$ is strictly increasing on $[-\frac{1}{2}, +\infty[$ and strictly decreasing on $] -\infty, -\frac{1}{2}]$. We can write f in the form :

$$\forall x \in \mathbb{R}, f(x) = (x + \frac{1}{2})^2 + \frac{3}{4}.$$

Firstly, let x_1 and x_2 be in $I_1 = [-\frac{1}{2}, +\infty[$. So

$$\begin{aligned}
 x_1 < x_2 &\Rightarrow \underbrace{x_1 + \frac{1}{2}}_{\geq 0 \text{ on } I_1} < \underbrace{x_2 + \frac{1}{2}}_{\geq 0 \text{ on } I_1} \\
 &\Rightarrow (x_1 + \frac{1}{2})^2 < (x_2 + \frac{1}{2})^2 \\
 &\Rightarrow \left(x_1 + \frac{1}{2}\right)^2 + \frac{3}{4} < \left(x_2 + \frac{1}{2}\right)^2 + \frac{3}{4} \\
 &\Rightarrow f(x_1) < f(x_2).
 \end{aligned}$$

Then f is increasing in $[-\frac{1}{2}, +\infty[$.

In the same way we can prove that f is strictly decreasing on $] -\infty, -\frac{1}{2}]$.

4.6 Local maximum, Local minimum of function

Let f be a function defined on the set $\mathcal{D}_f$.

4.6.1 Local maximum

Definition 4.11. *The function f has a local maximum value at x_0 if and only if there exists an open interval I such that*

$$\begin{cases} x_0 \in I \text{ and } I \subset \mathcal{D}_f, \\ \forall x \in I, f(x_0) \geq f(x). \end{cases}$$

4.6.2 Local minimum

Definition 4.12. *The function f has a local minimum value at x_0 if and only if there exists an open interval I such that*

$$\begin{cases} x_0 \in I \text{ and } I \subset \mathcal{D}_f, \\ \forall x \in I, f(x) \geq f(x_0). \end{cases}$$

4.6.3 Boundary value of a function

Definition 4.13. *If $f(x_0)$ is a local minimum value of f or a local maximum value then $f(x_0)$ is said a boundary value of a function f .*

4.7 Limits of Functions

4.7.1 Finite Limit at a point

Definition 4.14. Let A is non-empty set of $\mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if for every $\varepsilon > 0$ there exists at least one point $d \in A$; $d \neq c$ such that $d \in]c - \varepsilon, c + \varepsilon[$.
In other words

$$c \text{ is a cluster point of } A \Leftrightarrow \forall \varepsilon > 0, \exists d \neq c : d \in]c - \varepsilon, c + \varepsilon[\cap A.$$

We have the following easy-to-establish result.

Theorem 4.1. A number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence $(a_n)_n$ in A such that

$$\lim_{n \rightarrow +\infty} a_n = c \text{ and } \forall n \in \mathbb{N}, a_n \neq c.$$

Remarks 4.1.

1. It is not required in the definition that $c \in A$.
2. If A is an interval with endpoints a and b then the set of cluster points of A is $[a, b]$.
3. A finite set has no cluster points.

Definition 4.15. [Cauchy definition of the limit at a point] Let $A \subset \mathbb{R}$, and let c be a cluster point of A . For a function $f : A \rightarrow \mathbb{R}$, a real number ℓ is said to be a limit of f at c if

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, [x \in A \text{ and } 0 < |x - c| < \delta] \Rightarrow |f(x) - \ell| < \varepsilon.$$

We often write $\ell = \lim_{x \rightarrow c} f(x)$.

We also say that " $f(x)$ approaches ℓ as x approaches c ".

Exercise 4.1. Show using the definition that :

$$\begin{aligned} (a) \quad \lim_{x \rightarrow 0} \frac{x^2+4x}{x} &= 4, & (b) \quad \lim_{x \rightarrow 1} \frac{x^2+4x-5}{x^2-1} &= 3, & (c) \quad \lim_{x \rightarrow 0} \frac{x^2+4x+5}{x+2} &= \frac{5}{2}, \\ (d) \quad \lim_{x \rightarrow 2} \frac{x^3-4}{x^2+1} &= \frac{4}{5}. \end{aligned}$$

Solution .1.

(a) We pose that $f(x) = \frac{x^2+4x}{x}$. Let $\varepsilon > 0$. So, we have

$$|f(x) - 4| = \left| \frac{x^2+4x}{x} - 4 \right| = \left| \frac{x^2+4x-4x}{x} \right| = \left| \frac{x^2}{x} \right| = |x|.$$

Whence,

$$|x| < \varepsilon \Rightarrow |f(x) - 4| < \varepsilon.$$

Thus, we can take $\delta = \varepsilon$.

(b) We pose that $f(x) = \frac{x^2+4x-5}{x^2-1}$ and let $\varepsilon > 0$. We can write

$$\begin{aligned} |f(x) - 3| &= \left| \frac{x^2+4x-5}{x^2-1} - 3 \right| = \left| \frac{x^2+4x-5-3x^2+3}{x^2-1} \right| = \left| \frac{-2x^2+4x-2}{x^2-1} \right| = \left| \frac{-2(x^2-2x+1)}{x^2-1} \right| \\ &= \left| \frac{-2(x-1)^2}{(x-1)(x+1)} \right| = \left| \frac{-2(x-1)}{x+1} \right| = \frac{|2(x-1)|}{|x+1|}. \end{aligned}$$

As we look for the limit when x tends to 1, we can assume that $x \in]\frac{1}{3}, \frac{5}{3}[$. On this interval we have

$$\frac{4}{3} \leq |x+1| = x+1 \leq \frac{8}{3}.$$

Therefore,

$$x \in]\frac{1}{3}, \frac{5}{3}[\Rightarrow |f(x) - 3| = \frac{|2(x-1)|}{|x+1|} \leq \frac{3}{2} |x-1|.$$

So,

$$\left(x \in]\frac{1}{3}, \frac{5}{3}[\text{ and } \frac{3}{2} |x-1| < \varepsilon \right) \Rightarrow |f(x) - 3| < \varepsilon.$$

We can take $\delta = \inf \left\{ \frac{3}{2}, \frac{2\varepsilon}{3} \right\}$.

(c) We pose that $f(x) = \frac{x^2+4x+5}{x+2}$. Let $\varepsilon > 0$. So, we have

$$\left| f(x) - \frac{5}{2} \right| = \left| \frac{x^2+4x+5}{x+2} - \frac{5}{2} \right| = \left| \frac{2x^2+8x+10-5x-10}{2(x+2)} \right| = \left| \frac{2x^2+3x}{2(x+2)} \right| = \left| \frac{2x+3}{x+2} \right| \frac{|x|}{2}.$$

As we look for the limit when x tends to 0, we can assume that $x \in]-\frac{1}{4}, \frac{1}{4}[$. On this interval we have

$$\frac{10}{7} \leq \left| \frac{2x+3}{x+2} \right| = \left(2 - \frac{1}{x+2} \right) \leq \frac{14}{9}.$$

Hence,

$$x \in]-\frac{1}{4}, \frac{1}{4}[\Rightarrow \left| f(x) - \frac{5}{2} \right| = \left| \frac{2x+3}{x+2} \right| \frac{|x|}{2} \leq \frac{7}{9} |x|.$$

Therefore,

$$\left(x \in]-\frac{1}{4}, \frac{1}{4}[\text{ and } \frac{7}{9} |x| < \varepsilon \right) \Rightarrow \left| f(x) - \frac{5}{2} \right| < \varepsilon.$$

So, we can take $\delta = \inf \left\{ \frac{1}{4}, \frac{9\varepsilon}{7} \right\}$.

(d) Let $f(x) = \frac{x^3-4}{x^2+1}$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. So, we get

$$\begin{aligned} \left| f(x) - \frac{4}{5} \right| &= \left| \frac{x^3-4}{x^2+1} - \frac{4}{5} \right| = \left| \frac{5x^3-20-(4x^2+4)}{5(x^2+1)} \right| = \left| \frac{5x^3-20-4x^2-4}{5(x^2+1)} \right| = \left| \frac{5x^3-4x^2-24}{5(x^2+1)} \right| \\ &= \left| \frac{(x-2)(5x^2+6x+12)}{5(x^2+1)} \right| = \left| \frac{5x^2+6x+12}{5(x^2+1)} \right| |x-2|. \end{aligned}$$

As x tends to 2, so we can assume that $x \in]1, 3[$ and we obtain :

$$\begin{aligned} \begin{cases} 23 \leq 5x^2 + 6x^2 + 12 \leq 75 \\ 10 \leq 5(x^2 + 1) \leq 50 \end{cases} &\Rightarrow \begin{cases} 23 \leq 5x^2 + 6x^2 + 12 \leq 75 \\ \frac{1}{50} \leq \frac{1}{5(x^2+1)} \leq \frac{1}{10} \end{cases} \\ &\Rightarrow \frac{23}{50} \leq \frac{5x^2+6x^2+12}{5(x^2+1)} \leq \frac{75}{10} = \frac{15}{2}. \end{aligned}$$

Hence,

$$x \in]1, 3[\Rightarrow \left| f(x) - \frac{4}{5} \right| = \left| \frac{5x^2 + 6x + 12}{5(x^2 + 1)} \right| |x - 2| \leq \frac{15}{2} |x - 2|.$$

Therefore,

$$\left(x \in]1, 3[\text{ and } \frac{15}{2} |x - 2| < \varepsilon \right) \Rightarrow \left| f(x) - \frac{4}{5} \right| < \varepsilon.$$

So, we can take $\delta = \inf \left\{ 1, \frac{2\varepsilon}{15} \right\}$.

Proposition 4.1. *The limit of a function at a point, when it exists, is unique.*

Proof :

We assume that a function $f : I \rightarrow \mathbb{R}$ admits two limits ℓ_1 and ℓ_2 at a point c and let $\varepsilon > 0$. Of the definition of the limit, we can write

$$\exists \delta_1 = \delta_1(\varepsilon) > 0 \text{ and } \exists \delta_2 = \delta_2(\varepsilon) > 0 : \begin{cases} (x \in I \text{ and } |x - c| < \delta_1) \Rightarrow |f(x) - \ell_1| < \frac{\varepsilon}{2} \\ (x \in I \text{ and } |x - c| < \delta_2) \Rightarrow |f(x) - \ell_2| < \frac{\varepsilon}{2} \end{cases}$$

Let's put $\delta = \inf \{ \delta_1, \delta_2 \}$. So for $0 < |x - c| < \delta$ we get

$$|\ell_1 - \ell_2| = |\ell_1 - f(x) + f(x) - \ell_2| \leq |\ell_1 - f(x)| + |f(x) - \ell_2| < \varepsilon.$$

Thus, $\forall \varepsilon > 0$, $|\ell_1 - \ell_2| < \varepsilon$. From where $\ell_1 = \ell_2$. □

Definition 4.16. [Heine definition of the limit at a point] Let $f : A \rightarrow \mathbb{R}$ and let c be a cluster point of A . We say that $\lim_{x \rightarrow c} f(x) = \ell$ if for every sequence $(x_n)_n$ in A such that $\lim_{n \rightarrow +\infty} x_n = c$ then $\lim_{n \rightarrow +\infty} (f(x_n)) = \ell$.

Theorem 4.2. *Cauchy's definition of the limit at a point is equivalent to Heine's definition of the limit at a point.*

Proof :

Assume $f : A \rightarrow \mathbb{R}$ has limit ℓ at c . So by Cauchy definition we can write

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 : [x \in A \text{ and } |x - c| < \delta] \Rightarrow |f(x) - \ell| < \varepsilon. \quad (4.1)$$

Suppose $(x_n)_n$ is any sequence in A with $\lim_{n \rightarrow +\infty} x_n = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$. So we can write

$$\delta_\varepsilon > 0 \text{ and } \lim_{n \rightarrow +\infty} x_n = c \Rightarrow \exists n_{\delta_\varepsilon} \in \mathbb{N} : \forall n > n_{\delta_\varepsilon}, |x_n - c| < \delta_\varepsilon. \quad (4.2)$$

By (4.1) and (4.2) we obtain

$$\forall \varepsilon > 0, \exists n_\varepsilon = n_{\delta_\varepsilon} \in \mathbb{N} : \forall n > n_\varepsilon, |f(x_n) - \ell| < \varepsilon.$$

Hence $\lim_{n \rightarrow +\infty} f(x_n) = \ell$.

Conversely, by contrapositive argument, we assume that $\lim_{x \rightarrow c} f(x) = \ell$ in the Heine sense but not in the Cauchy sense. In this case we have

$$\exists \varepsilon > 0 : \forall \delta > 0, x \in A \text{ and } |x - c| < \delta \text{ and } |f(x) - \ell| \geq \varepsilon.$$

So

$$\forall n \in \mathbb{N}^*, \exists x_n \in A : |x_n - c| < \frac{1}{n} \text{ and } |f(x_n) - \ell| \geq \varepsilon.$$

We conclude that $\lim_{n \rightarrow +\infty} f(x_n) \neq \ell$, which contradicts the hypothesis. So we have

$$\left[\lim_{x \rightarrow c} f(x) = \ell \text{ in the Heine sense} \right] \Rightarrow \left[\lim_{x \rightarrow c} f(x) = \ell \text{ in the Cauchy sense} \right].$$

□

Remark 4.4. Heine's definition of the limit is mainly used to show that a function has no limit at a point.

Exercise 4.2. Show that the two functions $f, g: \mathbb{R}^* \rightarrow \mathbb{R}$ given by

$$f(x) = \sin\left(\frac{1}{x}\right) \text{ and } g(x) = \cos\left(\frac{1}{x}\right)$$

have no limits at point $a = 0$.

Solution .2. The two sequences $(x_n = \frac{1}{n\pi})_{n \in \mathbb{N}^*}$ and $(y_n = \frac{2}{(4n+1)\pi})_{n \in \mathbb{N}}$ are the sequences of $\mathbb{R}^*$ which converge to 0. So $a = 0$ is cluster point of $\mathbb{R}^*$. Moreover

$$\lim_{n \rightarrow +\infty} f(x_n) = \lim_{n \rightarrow +\infty} \sin\left(\frac{1}{x_n}\right) = \lim_{n \rightarrow +\infty} \sin(n\pi) = 0,$$

and

$$\lim_{n \rightarrow +\infty} f(y_n) = \lim_{n \rightarrow +\infty} \sin\left(\frac{1}{y_n}\right) = \lim_{n \rightarrow +\infty} \sin\left(\frac{(4n+1)\pi}{2}\right) = 1.$$

So f has no limit at $a = 0$. In the same way, we get

$$\lim_{n \rightarrow +\infty} g(x_n) = \lim_{n \rightarrow +\infty} \cos\left(\frac{1}{x_n}\right) = \lim_{n \rightarrow +\infty} \cos(n\pi) = (-1)^n.$$

So g has no limit at $a = 0$.

4.7.2 Finite Right-hand limit and Finite Left-hand limit of function at a point

Definition 4.17. [Cauchy definition of Finite Right-hand limit at a point]

Let $A \in \mathbb{R}$, $f: A \rightarrow \mathbb{R}$ be a function and a is cluster point of A . We say that $\ell \in \mathbb{R}$ is a right-hand limit of f at a and we write $\lim_{x \rightarrow a^+} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : (x \in A \text{ and } a < x < a + \delta) \Rightarrow |f(x) - \ell| < \varepsilon.$$

Definition 4.18. [Cauchy definition of Finite Left-hand limit at a point]

Let $A \in \mathbb{R}$, $f: A \rightarrow \mathbb{R}$ be a function and a is cluster point of A . We say that $\ell \in \mathbb{R}$ is a left-hand limit of f at a and we write $\lim_{x \rightarrow a^-} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : (x \in A \text{ and } a - \delta < x < a) \Rightarrow |f(x) - \ell| < \varepsilon.$$

Remark 4.5. Using the sequences, the finite right-hand limit and finite left-hand limit of function f at a point a can be write

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) = \ell &\Leftrightarrow \left(\forall (x_n)_n \subset A, x_n > a \forall n \text{ and } \lim_{n \rightarrow +\infty} x_n = a \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = \ell \right). \\ \lim_{x \rightarrow a^-} f(x) = \ell &\Leftrightarrow \left(\forall (x_n)_n \subset A, x_n < a \forall n \text{ and } \lim_{n \rightarrow +\infty} x_n = a \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = \ell \right). \end{aligned}$$

Theorem 4.3. Let A is non-empty set of $\mathbb{R}$ $f : A \rightarrow \mathbb{R}$ be a function and a is cluster point of A . Then

$$\left(\lim_{x \rightarrow a} f(x) = \ell \text{ and } b \in \mathbb{R} \right) \Leftrightarrow \left(\lim_{x \rightarrow a^+} f(x) = \ell = \lim_{x \rightarrow a^-} f(x) \right).$$

Proof :

The direct implication is trivial. Conversely, we assume that

$$\left(\lim_{x \rightarrow a^+} f(x) = \ell = \lim_{x \rightarrow a^-} f(x) \right).$$

So we can write

$$\forall \varepsilon > 0, \begin{cases} \exists \delta_1 = \delta_1(\varepsilon) (x \in A \text{ and } a < x < a + \delta_1) \Rightarrow |f(x) - \ell| < \varepsilon \\ \exists \delta_2 = \delta_2(\varepsilon) (x \in A \text{ and } a - \delta_2 < x < a) \Rightarrow |f(x) - \ell| < \varepsilon. \end{cases}$$

We can take $\delta = \delta(\varepsilon) = \inf\{\delta_1, \delta_2\}$ and we get

$$(x \in A \text{ and } |x - a| < \delta) \Rightarrow \begin{cases} (x \in A \text{ and } a < x < a + \delta \leq a + \delta_1) \\ \text{or} \\ (x \in A \text{ and } a - \delta \leq a - \delta < x < a) \end{cases} \Rightarrow \begin{cases} |f(x) - \ell| < \varepsilon \\ \text{or} \\ |f(x) - \ell| < \varepsilon. \end{cases}$$

Which completes the proof. $\square$

4.7.3 Infinite Limits at a point

Definition 4.19. Let I is non-empty set of $\mathbb{R}$ $f : I \rightarrow \mathbb{R}$ be a function and a is cluster point of I .

1. We say that f tends to $+\infty$ as $x \rightarrow a$, and write $\lim_{x \rightarrow a} f(x) = +\infty$, if

$$\forall A > 0, \exists \delta = \delta(A) > 0 : (x \in I \text{ and } |x - a| < \delta) \Rightarrow f(x) > A.$$

2. We say that f tends to $-\infty$ as $x \rightarrow a$, and write $\lim_{x \rightarrow a} f(x) = -\infty$, if

$$\forall B > 0, \exists \delta = \delta(B) > 0 : (x \in I \text{ and } |x - a| < \delta) \Rightarrow f(x) < B.$$

Exercise 4.3. Using the definition, show that

$$(a) \lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = +\infty, (b) \lim_{x \rightarrow 2} \frac{-3}{(x-2)^2} = -\infty$$

Solution .3. (a) Let $A > 0$. We have

$$\frac{1}{(x-2)^2} > A \Leftrightarrow \frac{1}{|x-2|} > \sqrt{A} \Rightarrow |x-2| < \frac{1}{\sqrt{A}} \Rightarrow \delta = \delta(A) = \frac{1}{\sqrt{A}}.$$

(b) Let $B < 0$. We have

$$\frac{-3}{(x-2)^2} < B \Leftrightarrow \frac{1}{|x-2|^2} > \frac{-B}{3} \Rightarrow |x-2| < \sqrt{\frac{-3}{B}} \Rightarrow \delta = \delta(B) = \sqrt{\frac{-3}{B}}.$$

4.7.4 Limits at Infinity

Definition 4.20. Let $I \subseteq \mathbb{R}$, let $f : I \rightarrow \mathbb{R}$ be a function, and suppose that $]a, +\infty[\subset I$ for some $a \in \mathbb{R}$.

1 . A real ℓ is said to be a limit of f when x tends to $+\infty$ and we write $\lim_{x \rightarrow +\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A = A(\varepsilon) > 0 : (x \in I \text{ and } x > A) \Rightarrow |f(x) - \ell| < \varepsilon.$$

2 . $+\infty$ is said to be a limit of f when x tends to $+\infty$ and we write $\lim_{x \rightarrow +\infty} f(x) = +\infty$ if

$$\forall M > 0, \exists A = A(M) > 0 : (x \in I \text{ and } x > A) \Rightarrow f(x) > M.$$

3 . $-\infty$ is said to be a limit of f when x tends to $+\infty$ and we write $\lim_{x \rightarrow +\infty} f(x) = -\infty$ if

$$\forall M < 0, \exists A = A(M) > 0 : (x \in I \text{ and } x > A) \Rightarrow f(x) < M.$$

Remark 4.6. Using the sequences, we can write

1.

$$\left(\lim_{x \rightarrow +\infty} f(x) = \ell \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = \ell \right).$$

2.

$$\left(\lim_{x \rightarrow +\infty} f(x) = +\infty \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = +\infty \right).$$

3.

$$\left(\lim_{x \rightarrow +\infty} f(x) = -\infty \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = -\infty \right).$$

Definition 4.21. Let $I \subseteq \mathbb{R}$, let $f : I \rightarrow \mathbb{R}$ be a function, and suppose that $] -\infty, a[\subset I$ for some $a \in \mathbb{R}$.

1 . A real ℓ is said to be a limit of f when x tends to $-\infty$ and we write $\lim_{x \rightarrow -\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A = A(\varepsilon) < 0 : (x \in I \text{ and } x < A) \Rightarrow |f(x) - \ell| < \varepsilon.$$

2 . $+\infty$ is said to be a limit of f when x tends to $+\infty$ and we write $\lim_{x \rightarrow +\infty} f(x) = +\infty$ if

$$\forall M > 0, \exists A = A(M) < +\infty : (x \in I \text{ hbox and } x > A) \Rightarrow f(x) > M.$$

3 . $-\infty$ is said to be a limit of f when x tends to $+\infty$ and we write $\lim_{x \rightarrow +\infty} f(x) = -\infty$ if

$$\forall M < 0, \exists A = A(M) < +\infty : (x \in I \text{ hbox and } x > A) \Rightarrow f(x) < M.$$

Remark 4.7. Using the sequences, we can write

1.

$$\left(\lim_{x \rightarrow +\infty} f(x) = \ell \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = \ell \right).$$

2.

$$\left(\lim_{x \rightarrow +\infty} f(x) = +\infty \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = +\infty \right).$$

3.

$$\left(\lim_{x \rightarrow +\infty} f(x) = -\infty \right) \Leftrightarrow \left(\forall (x_n)_n \subset I \text{ and } \lim_{n \rightarrow +\infty} x_n = +\infty \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) = -\infty \right).$$

4.7.5 Calculation rules

Let f and g be two functions admitting limits (finite or infinite) at a point $a \in \overline{\mathbb{R}} =]-\infty, +\infty[\cup \{-\infty, +\infty\}$. We have

1. $\left(\lim_{x \rightarrow a} f(x) \in \mathbb{R} \text{ and } \lim_{x \rightarrow a} g(x) \in \mathbb{R} \right) \Rightarrow \lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$
2. $\left(\lim_{x \rightarrow a} f(x) = +\infty \text{ and } g(x) \geq M \right) \Rightarrow \lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$
3. $\left(\lim_{x \rightarrow a} f(x) = -\infty \text{ and } g(x) \leq M \right) \Rightarrow \lim_{x \rightarrow a} (f(x) + g(x)) = -\infty.$
4. $\left(\lim_{x \rightarrow a} f(x) \in \mathbb{R} \text{ and } \lim_{x \rightarrow a} g(x) \in \mathbb{R} \right) \Rightarrow \lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$
5. $\left(\lim_{x \rightarrow a} f(x) \in \mathbb{R} \text{ and } \alpha \in \mathbb{R} \right) \Rightarrow \lim_{x \rightarrow a} (\alpha \cdot f)(x) = \alpha \cdot \lim_{x \rightarrow a} f(x).$
6. $\left(\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \alpha \in \mathbb{R}^* \right) \Rightarrow \lim_{x \rightarrow a} (\alpha \cdot f)(x) = \alpha \cdot \lim_{x \rightarrow a} f(x).$
7. $\left(\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty \right) \Rightarrow \lim_{x \rightarrow a} (f \cdot g)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$
8. $\left(\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) \neq 0 \right) \Rightarrow \lim_{x \rightarrow a} (f \cdot g)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$
9. $\left(\lim_{x \rightarrow a} f(x) = 0 \text{ and } g(x) \text{ is bounded} \right) \Rightarrow \lim_{x \rightarrow a} (f \cdot g)(x) = 0.$
10. $\left(\lim_{x \rightarrow a} f(x) \in \mathbb{R} \text{ and } \lim_{x \rightarrow a} g(x) \neq 0 \right) \Rightarrow \lim_{x \rightarrow a} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}.$

4.8 Limits Properties

Theorem 4.4. Let A nonempty set of $\mathbb{R}$, $f : A \rightarrow \mathbb{R}$ be a function defined on A , and let $c \in \mathbb{R}$ be a cluster point of A . If

$$\forall x \in A, x \neq c, a \leq f(x) \leq b,$$

and if $\lim_{x \rightarrow c} f(x)$ exists, then

$$a \leq \lim_{x \rightarrow c} f(x) \leq b.$$

Proof :

Indeed, if $\ell = \lim_{x \rightarrow c} f(x)$, then if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$ and if the sequence (x_n) converges to c , then the sequence $(f(x_n))$ converges to ℓ . Since, for all $n \in \mathbb{N}$, $a \leq f(x_n) \leq b$, it follows from sequences theorem that $a \leq \ell \leq b$. $\square$

Theorem 4.5. [Squeeze theorem] Let A an nonempty set of $\mathbb{R}$, $f, g, h : A \rightarrow \mathbb{R}$ are three functions that are defined on A and let $c \in \mathbb{R}$ be a cluster point of A . If

$$\forall x \in A, x \neq c, f(x) \leq g(x) \leq h(x),$$

and if $\lim_{x \rightarrow c} f(x) = \ell = \lim_{x \rightarrow c} h(x)$ then $\lim_{x \rightarrow c} g(x) = \ell$.

Proof :

Let (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$ witch converges to c , so there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow \lim_{n \rightarrow +\infty} x_n = c \Rightarrow \lim_{n \rightarrow +\infty} f(x_n) \leq \lim_{n \rightarrow +\infty} g(x_n) \leq \lim_{n \rightarrow +\infty} h(x_n).$$

Using sequences theorem, we obtain

$$\lim_{n \rightarrow +\infty} f(x_n) = \lim_{n \rightarrow +\infty} g(x_n) = \lim_{n \rightarrow +\infty} h(x_n).$$

We conclude that

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x).$$

$\square$

Theorem 4.6. Let A an nonempty set of $\mathbb{R}$, let $f, g : A \rightarrow \mathbb{R}$ are two functions that are defined on A and let $c \in \mathbb{R}$ be a cluster point of A . Suppose that $\forall x \in A, x \neq c, f(x) \leq g(x)$.

(1) If $\lim_{x \rightarrow c} f(x) = +\infty$, then $\lim_{x \rightarrow c} g(x) = +\infty$.

(2) If $\lim_{x \rightarrow c} g(x) = -\infty$, then $\lim_{x \rightarrow c} f(x) = -\infty$.

Proof :

We leave the proof to the reader. $\square$

Remark 4.8. The previous results remain true if the functions are defined on

1. $]c, +\infty[$ and $x \rightarrow +\infty$.
2. $] -\infty, c[$ and $x \rightarrow -\infty$.
3. $]c - \alpha, c[$, $\alpha > 0$ and $x \rightarrow c^-$.
4. $]c, c + \alpha[$, $\alpha > 0$ and $x \rightarrow c^+$.

Example :

On $]0, \frac{\pi}{2}[$, we have

$$\begin{aligned} \sin(x) \leq x \leq \tan(x) &\Rightarrow 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \\ &\Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq 1. \end{aligned}$$

Thus

$$\forall x \in]0, \frac{\pi}{2}[, \cos(x) \leq \frac{\sin(x)}{x} \leq 1. \quad (4.3)$$

On other hand, if $x \in] -\frac{\pi}{2}, 0[$ then $-x \in]0, \frac{\pi}{2}[$ so $(-x)$ checks (4.3) so,

$$\cos(-x) \leq \frac{\sin(-x)}{-x} \leq 1 \Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

Finally

$$\forall x \in] -\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}[, \cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

As $\lim_{x \rightarrow 0} \cos(x) = 1$, so $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$.

Exercise 4.4. Determine the following limits.

1. $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^{\frac{1}{x}}$, 2. $\lim_{x \rightarrow 0} \left(\frac{x \sin(x)}{1 - \cos(x)}\right)$, 3. $\lim_{x \rightarrow 0} \left(\frac{\tan(2x)}{\sqrt{1 - \cos(x)}}\right)$, 4. $\lim_{x \rightarrow 0} \left(\frac{\sin(2x)}{\sin(4x)}\right)$
5. $\lim_{x \rightarrow +\infty} \left(\frac{x}{\sqrt{x+1}} - \frac{x}{\sqrt{x+2}}\right)$, 6. $\lim_{x \rightarrow 0} \left(\frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2}\right)$, 7. $\lim_{x \rightarrow 1} \left(\frac{\sqrt[3]{x}-1}{x-1}\right)$, 8. $\lim_{x \rightarrow 1} \left(\frac{\sqrt[5]{x}-1}{\sqrt[3]{x}-1}\right)$
9. $\lim_{x \rightarrow +\frac{\pi}{6}} \left(\frac{2\cos(2x)-1}{\cos(3x)}\right)$, 10. $\lim_{x \rightarrow \frac{\pi}{3}} \left(\frac{\sin(3x)}{1-2\cos(x)}\right)$, 11. $\lim_{x \rightarrow +\infty} \left(x - \sqrt{x^2 + 2x + 1}\right)$,
12. $\lim_{x \rightarrow 1} \left(\frac{\sqrt[5]{x}-1}{\sqrt[3]{x}-1}\right)$.

Exercise 4.5. Let a an any real number and f, g are two functions that are defined on $]a, +\infty[$. We suppose that

$$\begin{cases} \lim_{x \rightarrow +\infty} f(x) = 0 \\ \lim_{x \rightarrow +\infty} g(x) = \pm\infty \\ \lim_{x \rightarrow +\infty} f(x).g(x) = \ell. \end{cases}$$

Determine the flowing limit

$$\lim_{x \rightarrow +\infty} (1 + f(x))^{g(x)}.$$

Use the previous result to determine the following limits

1. $\lim_{x \rightarrow +\infty} \left(\frac{x+2}{x-3}\right)^x$, 2. $\lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2+2}\right)^{\frac{x^3+1}{x+1}}$.

5

Continuous Functions

5.1 Continuous Function at a Point

Definition 5.1. Let A be non-empty set of $\mathbb{R}$, let $f : A \rightarrow \mathbb{R}$ a function and $a \in A$. We say that f is continuous at a if, $\lim_{x \rightarrow a} f(x) = f(a)$.

Remarks 5.1.

1. From the definition of the continuity of the function f at a , the point a must be in the domain of definition $\mathcal{D}_f$ of f .
2. Of the definition of the limit, we can write

$$[f \text{ is continuous at } a] \Leftrightarrow [\forall \varepsilon > 0, \exists \delta_\varepsilon = \delta_\varepsilon(a) > 0 : |x - a| < \delta_\varepsilon \Rightarrow |f(x) - f(a)| < \varepsilon].$$

3.

$$(f \text{ is continuous at } a) \Leftrightarrow \left(\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a) \right).$$

4. **Sequential Criterion for Continuity :** f is continuous at the point $a \in A$ if and only if for every sequence (x_n) in A that converges to a , the sequence $(f(x_n))$ converges to $f(a)$.
5. If the function f is not continuous at a , we say that it is discontinuous function at this point or that a is a point of discontinuity of f .

Definition 5.2. The function $f : A \rightarrow \mathbb{R}$ is said

1. continuous at the right of a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.
2. continuous at the left of a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.

Theorem 5.1. The function f is continuous at point a if and only if it is continuous at the right and at the left of a .

Examples :

1. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{\sin(x)}{x(x-1)} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0, \end{cases}$$

is continuous at the point 0, indeed

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 1 = f(0).$$

2. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g(x) = \begin{cases} \frac{1-\cos(x)}{x} & \text{if } x \geq 0 \\ 2-4x & \text{if } x < 0, \end{cases}$$

is discontinuous at the point 0, indeed

$$\lim_{x \rightarrow 0^+} g(x) = 0 \neq 2 = \lim_{x \rightarrow 0^-} g(x).$$

5.2 Combinations of Continuous Functions

Theorem 5.2. Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $\lambda \in \mathbb{R}$. Suppose that $a \in A$ and that f and g are continuous at a .

- (1) Then $f + g$, $f - g$, fg and $\lambda.f$ are continuous at a .
- (2) If $g(a) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at a .
- (3) Let $B \subseteq \mathbb{R}$ and let $h : B \rightarrow \mathbb{R}$ be a function such that $f(A \cap \mathcal{D}_f) \subseteq B$. If h is continuous at $b = f(a)$, then the composition $g \circ f : A \rightarrow \mathbb{R}$ is continuous at a .

Definition 5.3. Let $A \subseteq \mathbb{R}$ and let $f : A \rightarrow \mathbb{R}$. If B is a subset of A , we say that f is continuous on the set B if f is continuous at every point of B .

Examples :

- (a) The polynomial functions, $\cos$ and $\sin$ are continuous on $\mathbb{R}$.
- (b) The function $\ln$ is continuous on $]0, +\infty[$.
- (c) The function $\exp$ is continuous on $\mathbb{R}$.
- (d) The function $\tan$ is continuous on $\mathbb{R} \setminus \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$.
- (e) The function $\cot$ is continuous on $\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$.

5.3 Extension by continuity

Definition 5.4. If a function $f : A \rightarrow \mathbb{R}$ is discontinuous at a point a and if f has a limit ℓ at the point a then the function $F : A \cup \{a\} \rightarrow \mathbb{R}$ given by

$$F(x) = \begin{cases} f(x) & \text{for } x \neq a, \\ \ell & \text{for } x = a, \end{cases}$$

is continuous at a and it is called extension by continuity of the function f at point a .

Example :

(a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{x^3 + 2x - 3}{x^2 - 1}.$$

f is continuous function on $\mathcal{D}_f = \mathbb{R} - \{1, -1\}$.

Furthermore, for all $x \in \mathcal{D}_f$, we can write

$$f(x) = \frac{x^3 + 2x - 3}{x^2 - 1} = \frac{(x-1)(x^2 + x + 3)}{(x-1)(x+1)} = \frac{x^2 + x + 3}{x+1},$$

so

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \left(\frac{x^2 + x + 3}{x+1} \right) = \frac{5}{2}.$$

Therefore, the function f can be extended by continuity at 1 and its extension F is given by :

$$F(x) = \begin{cases} f(x) & \text{for } x \neq 1, \\ \frac{5}{2} & \text{for } x = 1. \end{cases}$$

The function f cannot be extended by continuity at -1 since

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1^+} \left(\frac{x^2 + x + 3}{x+1} \right) = +\infty.$$

Exercise 5.1.

a. Determine, as a function of α , the limit of f given by

$$f(x) = \frac{\alpha x^2 + (\alpha^2 + 1)x + \alpha}{x - 5}$$

at point $x = 5$.

2. Does the function f accept an extension by continuity at the point $x = 5$.

Exercise 5.2. Let $\alpha \in \mathbb{R}$. f_α is the function defined by

$$f_\alpha(x) = \frac{2x^2 + x - 3}{x^2 + (1 - \alpha)x - \alpha}.$$

- a. Determine the domain of definition of f_α .
- b. Determine the real numbers α for which the function f_α admits an extension by continuity at α . Then determine, for each of these numbers, the extension of the function f_α by continuity at the point α .

5.4 Continuous Functions on Intervals

Theorem 5.3. [Principle of conservation of the sign] *If the function f is continuous at a point a and if $f(a) \neq 0$ then, there exists $\delta > 0$ such that*

$$\forall x \in]a - \delta, a + \delta[, f(a) \cdot f(x) > 0.$$

Proof : Suppose that $f(a) > 0$. We show that there exists $\delta > 0$ such that for all $x \in]a - \delta, a + \delta[, f(x) > 0$. By continuity of f at a we have,

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 : x \in]a - \delta_\varepsilon, a + \delta_\varepsilon[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[.$$

We take $\varepsilon = \frac{f(a)}{2} > 0$. So there exists $\delta = \delta_\varepsilon$ such that

$$x \in]a - \delta, a + \delta[\Rightarrow 0 < f(a) - \varepsilon = \frac{f(a)}{2} \leq f(x) \leq \frac{3}{2}f(a).$$

□

Theorem 5.4. [Boundedness Theorem] *Let I be a closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I . Then f is bounded on I . In other way, there exists two real numbers m and M such that*

$$\forall x \in I, m \leq f(x) \leq M.$$

Proof : Suppose that f is not bounded on I . In this case we can, write

$$\forall \alpha \in \mathbb{R}, \exists x \in I : f(x) > \alpha.$$

Then, for any $n \in \mathbb{N}$ there is a number $x_n \in I$ such that $|f(x_n)| > n$.

Whence the sequence $(f(x_n))_n$ is not bounded then $\lim_{n \rightarrow +\infty} f(x_n) = +\infty$.

On the other hand, the sequence (x_n) is bounded. Therefore, by the Bolzano-Weierstrass theorem there exists a subsequence $(x_{\varphi(n)})$ of (x_n) that converges to a number $x \in [a, b]$, such as the $\lim_{n \rightarrow +\infty} f(x_{\varphi(n)}) = +\infty$. But f is continuous at x , so that $(f(x_{\varphi(n)}))_n$ converges to $f(x)$. Therefore the supposition that the continuous function f is not bounded on the closed bounded interval I leads to a contradiction. So the function f is bounded above.

By analogous way, we can also show that f is bounded below on I . □

Corollary 5.1. *If the function f is not bounded on a closed bounded interval then it cannot be continuous on this interval.*

Definition 5.5. Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$ and let $f : A \rightarrow \mathbb{R}$ be a function defined on A . We say that f has an absolute maximum on A if there is a point $x^* \in A$ such that

$$\forall x \in A, f(x^*) \geq f(x).$$

We say that f has an absolute minimum on A if there is a point $x_* \in A$ such that

$$\forall x \in A, f(x_*) \leq f(x).$$

We say that x^* is an absolute maximum point for f on A , and that x_* is an absolute minimum point for f on A , if they exist.

Theorem 5.5. Let I be a closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I . Then f has an absolute maximum and an absolute minimum on I .

Proof : Consider the nonempty set

$$f(I) = \{f(x) : x \in I\}.$$

of values of f on I . According to the previous theorem, the set $f(I)$ is a bounded subset of $\mathbb{R}$.

Let

$$c^* = \sup_{x \in I} f(x) \text{ and } c_* = \inf_{x \in I} f(x).$$

We show that there exist points x^* and x_* in I such that

$$c^* = f(x^*) \text{ and } c_* = f(x_*).$$

We will establish the existence of the point x_* , leaving the proof of the existence of x^* to the reader.

Since $c_* = \inf_{x \in I} f(x)$, then

$$\forall \varepsilon > 0, \exists x_\varepsilon \in I : c_* \leq f(x_\varepsilon) < c_* + \varepsilon.$$

From where,

$$\forall n \in \mathbb{N}^*, \exists x_n \in I : c_* \leq f(x_n) < c_* + \frac{1}{n}. \quad (5.1)$$

We thus obtain a sequence $(x_n)_n$ of elements of I , verifying (5.1). Therefore, by the Bolzano-Weierstrass Theorem, there is a subsequence (x_{n_k}) of $(x_n)_n$ that converges to some number x_* . Moreover, by continuity of f , we have

$$\left\{ \begin{array}{l} \lim_{n_k \rightarrow +\infty} x_{n_k} = x_* \\ c_* \leq f(x_n) < c_* + \frac{1}{n} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lim_{n_k \rightarrow +\infty} x_{n_k} = x_* \\ c_* \leq f(x_{n_k}) < c_* + \frac{1}{n_k} \end{array} \right\} \Rightarrow f(x_*) = \lim_{n_k \rightarrow +\infty} f(x_{n_k}) = c_*.$$

We conclude that x_* is an absolute minimum point of f on I . □

From the proof of the previous theorem, it appears that

Corollary 5.2. *If the function f is continuous on $[a, b]$ then*

$$f([a, b]) = [f(x_*), f(x^*)]$$

where x_* and x^* are the absolute minimum and the absolute maximum points for f on I respectively.

Theorem 5.6. Location of Roots Theorem :

Let $I = [a, b]$ and let $f : I \rightarrow \mathbb{R}$ be continuous on I . If $f(a) \cdot f(b) < 0$, then there exists a number $c \in]a, b[$ such that $f(c) = 0$.

Proof :

We assume that $f(a) < 0 < f(b)$. Let

$$Y^- = \{x \in [a, b] : f(x) < 0\} \text{ and } Y^+ = \{x \in [a, b] : f(x) > 0\}.$$

The set Y^- is nonempty bounded above (even bounded) and consequently admits a least upper bound $\alpha = \sup Y^- \in [a, b]$. Likewise, the set Y^+ is nonempty bounded below (even bounded) and consequently admits a greatest lower bound $\beta = \inf Y^+ \in [a, b]$.

Furthermore, there exists a sequence $(\alpha_n)_n$ of elements of Y^- which converges to α , just as there exists a sequence $(\beta_n)_n$ of elements of Y^+ which converges to β .

By continuity we have

$$f(\alpha) \leq 0 \text{ and } f(\beta) \geq 0.$$

We will show that

$$f(\alpha) = f(\beta) = 0.$$

We assume for example that $f(\alpha) < 0$. We will have, according to theorem 5.3) of conservation of the sign :

$$\exists \delta > 0 : \text{ such that } \forall x \in]\alpha - \delta, \alpha + \delta[, f(x) < 0.$$

In this case we have

$$\alpha + \frac{\delta}{2} > \alpha \text{ and } \alpha + \frac{\delta}{2} \in Y^-,$$

which contradicts the definition of α . So we have $f(\alpha) = 0$.

By analogous way, we also show that $f(\beta) = 0$. So we can take $c = \alpha$ or $c = \beta$. $\square$

Theorem 5.7. [Bolzano's Intermediate Value Theorem]

Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be continuous function on I . If $a, b \in I$, $a < b$ and if $\lambda \in \mathbb{R}$ satisfies $f(a) \leq \lambda \leq f(b)$ or $f(b) \leq \lambda \leq f(a)$, then there exists a point $c \in [a, b]$ such that $f(c) = \lambda$.

Proof :

It is enough to apply the previous theorem to the continuous function $g : [a, b] \rightarrow \mathbb{R}$, defined by : $g(x) = f(x) - \lambda$. $\square$

Exercise 5.3. Show that the equation $x^3 + x^2 - 1 = 0$ admits at least one solution α in the interval $[0, 1]$.

Exercise 5.4. Let

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{x+1}{2x-1}. \end{aligned}$$

Show that the equation $f(x) = 3$ admits at least one solution α in the interval $[\frac{7}{10}, \frac{9}{10}]$.

Exercise 5.5. Let f be a continuous function on the interval $[a, b]$ and α, β are two real numbers. Show that, there is at least c in $5a, b]$ such that

$$\alpha f(a) + \beta f(b) = (\alpha + \beta)f(c).$$

Theorem 5.8. Let $f : I = [a, b] \rightarrow \mathbb{R}$ be continuous function on I . If f is a strictly monotone function on I then f is a bijective function from I to $f(I)$.

In this case, we can say that f admits an inverse function, denoted by f^{-1} , defined from $f(I)$ to I such that

1. f^{-1} is continuous function on the interval $f(I)$.
2. f^{-1} is strictly monotone function on $f(I)$. In another way, if f is strictly increasing function on I then f^{-1} is strictly increasing function on $f(I) = [f(a), f(b)]$ and if f is strictly decreasing function on $[a, b]$ then f^{-1} is strictly decreasing function on $f(I) = [f(b), f(a)]$.

Proof :

Leaving the proof to the reader. □

Example : We define the function f on $I = [0, +\infty[$ by $f(x) = x^n$ where n is a non-zero natural number. f is continuous and strictly increasing function on I . So f admits an inverse function f^{-1} defined by

$$\begin{aligned} f^{-1} : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\ x &\longmapsto f^{-1}(x) = x^{\frac{1}{n}} = \sqrt[n]{x}. \end{aligned}$$

f^{-1} is continuous and strictly increasing function on $\mathbb{R}_+$. □

5.5 Uniform Continuity of functions

Definition 5.6. Let $f : A \rightarrow \mathbb{R}$ be a function defined on A . We say that f is uniformly continuous on A if

$$\forall \varepsilon > 0, \exists \delta_\varepsilon : \forall x_1, x_2 \in A, |x_1 - x_2| < \delta_\varepsilon \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Examples :

(1) We consider the continuous function $f(x) = x^3$ on $\mathbb{R}$. Let $\varepsilon = 1$. For all $\delta > 0$ we take $x_1 > \frac{1}{\sqrt{\delta}}$ and we put $x_2 = x_1 + \frac{\delta}{3}$. So, we have

$$|f(x_1) - f(x_2)| = |(x_1 + \frac{\delta}{3})^3 - x_1^3| = x_1^2\delta + x_1\frac{\delta^2}{3} > x_1^2\delta > 1 = \varepsilon.$$

We are proved that

$$\exists \varepsilon = 1 : \forall \delta > 0, \exists x_1, x_2 \in \mathbb{R} : |x_1 - x_2| < \delta \text{ and } |f(x_2) - f(x_1)| > \varepsilon.$$

Consequently, f is continuous function on $\mathbb{R}$ but not uniformly continuous function on $\mathbb{R}$.

(2) In the similar way, we can prove that the function $f(x) = \frac{1}{x}$ it's continuous function on $]0, 1]$ but not uniformly continuous function on this interval. $\square$

Theorem 5.9. [Uniform Continuity Theorem(Heine theorem)]

Let I be a closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I . Then f is uniformly continuous on I .

Proof :

If f is not uniformly continuous on I then,

$$\exists \varepsilon > 0 : \forall \delta > 0, \exists x_1, x_2 \in I : \text{ such that } |x_1 - x_2| < \delta \text{ and } |f(x_1) - f(x_2)| > \varepsilon.$$

Therefore,

$$\exists \varepsilon > 0 : \forall n \in \mathbb{N}^*, \exists x_n, y_n \in I : \text{ such that } |x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| > \varepsilon. \quad (5.2)$$

So, we obtain two sequences $(x_n)_{n \in \mathbb{N}^*}$ and $(y_n)_{n \in \mathbb{N}^*}$ in I verifies (5.2).

By the Bolzano-Weierstrass Theorem there is a subsequence $(x_{n_k})_{n_k \in \mathbb{N}^*}$ of $(x_n)_{n \in \mathbb{N}^*}$ that converges to an element $x \in I$. It is clear that the corresponding subsequence $(y_{n_k})_{n_k \in \mathbb{N}^*}$ also converges to x , since

$$|y_{n_k} - x| = |y_{n_k} - x_{n_k} + x_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x|.$$

Now if f is continuous we can write

$$\lim_{n_k \rightarrow +\infty} x_{n_k} = x = \lim_{n_k \rightarrow +\infty} y_{n_k} \Rightarrow \lim_{n_k \rightarrow +\infty} f(x_{n_k}) = \lim_{n_k \rightarrow +\infty} f(y_{n_k}) = f(x).$$

However, this is in contradiction with (5.2). So, f is uniformly continuous function. $\square$

5.6 Lipschitz Functions

Definition 5.7. Let $A \subseteq \mathbb{R}$ and let $f : A \rightarrow \mathbb{R}$ be a function defined on A . f is said to be a K -Lipschitz function (or to satisfy a Lipschitz condition) on A if there exists a constant $K > 0$ such that

$$\forall x, y \in A, |f(x) - f(y)| \leq K|x - y|. \quad (5.3)$$

Theorem 5.10. If $f : A \rightarrow \mathbb{R}$ is a K -Lipschitz function, then f is uniformly continuous on A (also is a continuous function on A).

Proof :

If condition (5.3) is satisfied, then given $\varepsilon > 0$, we can take $\delta = \frac{\varepsilon}{K}$. If $x_1, x_2 \in A$ satisfy $|x_1 - x_2| < \delta$ then

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2| < \varepsilon.$$

Therefore f is uniformly continuous on A so f is continuous on A . $\square$

6

Differentiability

6.1 Derivative at a point

Definition 6.1. Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \rightarrow \mathbb{R}$ be a function defined on I , and $a \in I$. We say that f is differentiable at a if the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a},$$

exists in $\mathbb{R}$. In this case $f'(a)$ is called the derivative of f at a .

Examples :

(1) Let f be a function given by : $f(x) = \sin(x)$. We show that f is differentiable at any point $a \in \mathbb{R}$. We have

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \rightarrow a} \frac{\sin(x) - \sin(a)}{x - a} = \lim_{x \rightarrow a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{x - a} \\ &= \lim_{x \rightarrow a} \left(\frac{2 \sin\left(\frac{x-a}{2}\right)}{x - a} \right) \cos\left(\frac{x+a}{2}\right) \\ &= \lim_{x \rightarrow a} \left(\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)} \right) \lim_{x \rightarrow a} \cos\left(\frac{x+a}{2}\right) \\ &= \cos(a). \end{aligned}$$

So, f is differentiable at any point $a \in \mathbb{R}$ and $f'(a) = \cos(a)$.

(2) In the same way, we can show that the function $x \mapsto f(x) = \cos(x)$ is differentiable at any point a of $\mathbb{R}$ and $f'(a) = -\sin(a)$.

(3) We can show that the function $x \mapsto g(x) = x^n$, $n \in \mathbb{N}$, is differentiable at any point a of $\mathbb{R}$ and $g'(a) = na^{n-1}$.

6.2 Right-hand derivative and left-hand derivative at a point

Definition 6.2. Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \rightarrow \mathbb{R}$ be a function defined on I , and $a \in I$.

1. **[The right-hand derivative at a]** We say that f admits a right-hand derivative at a if the limit

$$f'_r(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a},$$

exists in $\mathbb{R}$. In this case, $f'_r(a)$ is called the right-hand derivative of f at a .

2. **[The left-hand derivative at a]** We say that f admits a left-hand derivative at a if the limit

$$f'_l(a) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a},$$

exists in $\mathbb{R}$. In this case, $f'_l(a)$ is called the left-hand derivative of f at a .

Theorem 6.1. The function f is differentiable at point a of $\mathbb{R}$ if and only if it admits a right-hand derivative and a left-hand derivative at a and $f'_r(a) = f'_l(a)$.

Examples 6.1. (1) Let f the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \sqrt{x^2 + 2x} & \text{for } x \in [1, +\infty[\\ \sqrt{3}(2x^2 - 1) & \text{for } x \in]-\infty, 1]. \end{cases}$$

We study the differentiability of f at 1.

We have

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{x^2 + 2x} - \sqrt{3}}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(\sqrt{x^2 + 2x} - \sqrt{3})(\sqrt{x^2 + 2x} + \sqrt{3})}{(\sqrt{x^2 + 2x} + \sqrt{3})(x - 1)} \\ &= \lim_{x \rightarrow 1^+} \frac{x^2 + 2x - 3}{(\sqrt{x^2 + 2x} + \sqrt{3})(x - 1)} = \lim_{x \rightarrow 1^+} \frac{(x - 1)(x + 3)}{(\sqrt{x^2 + 2x} + \sqrt{3})(x - 1)} \\ &= \lim_{x \rightarrow 1^+} \frac{(x + 3)}{\sqrt{x^2 + 2x} + \sqrt{3}} = \frac{2}{\sqrt{3}}. \end{aligned}$$

Thus f admits a right-hand derivative at 1 and $f'_r(1) = \frac{2}{\sqrt{3}}$.

On the other hand,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{\sqrt{3}(2x^2 - 1) - \sqrt{3}}{x - 1} = \sqrt{3} \lim_{x \rightarrow 1^-} \frac{2x^2 - 2}{x - 1} = 2\sqrt{3} \lim_{x \rightarrow 1^-} (x + 1) \\ &= 4\sqrt{3}. \end{aligned}$$

Thus f admits a left-hand derivative at 1 and $f'_l(1) = 4\sqrt{3}$. It follows that f is not differentiable at 1 because $f'_l(1) = 4\sqrt{3} \neq \frac{2}{\sqrt{3}} = f'_r(1)$.

(2) We study the differentiability of the function f defined on $\mathbb{R}$ by

$$f(x) = |x|,$$

at the point 0. We have

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{|x| - |0|}{x} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1.$$

Thus f admits a left-hand derivative at 0 and $f'_l(0) = -1$.

On the other hand,

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{|x| - |0|}{x} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1.$$

Thus f admits a right-hand derivative at 0 and $f'_r(0) = 1$. But is not differentiable at 0.

6.3 Geometric Interpretation of Differentiation

Definition 6.3. If the function f is differentiable at the point a then the derivative $f'(a)$ of f at a is the slope of the tangent line to $y = f(x)$ at the point $P = (a, f(a))$. The tangent equation at the point P is given by

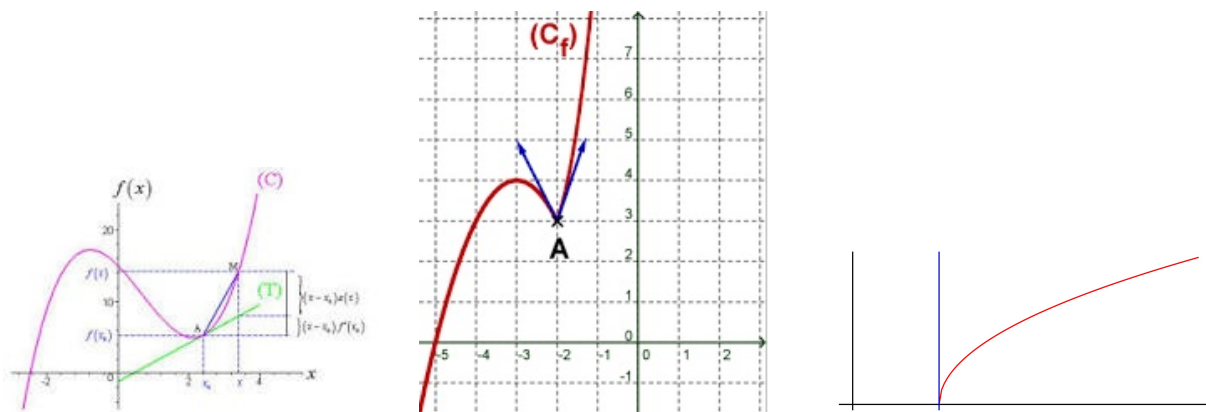
$$y = f'(a)(x - a) + f(a).$$

Remarks 6.1. 1. If the function f is not differentiable at the point a but admits a left-hand derivative and a right-hand derivative at this point then the graph of f admits two half tangents of equations :

$$y = f'_r(a)(x - a) + f(a) \quad x \geq a \quad \text{and} \quad y = f'_l(a)(x - a) + f(a) \quad x \leq a.$$

2. If the function $f'(a) = \pm\infty$ we say that f is not differentiable at the point a but the graph of f admits a tangent parallel to the ordinate axis at the point $(a, f(a))$. The equation of this tangent is given by $x = a$.

Examples :



6.4 Derivability and continuity

Theorem 6.2. *If a function f is differentiable at a point then, it is continuous at this point.*

Proof :

Let I be an open interval, $f : I \rightarrow \mathbb{R}$ be a function differentiable at a point $a \in I$. For all $x \in I$ and $x \neq a$, we have

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a} \right) (x - a).$$

Since $f'(a)$ exists, so we get

$$\begin{aligned} \lim_{x \rightarrow a} (f(x) - f(a)) &= \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) (x - a) \\ &= \left(\lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) \right) \times \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \times 0 = 0. \end{aligned}$$

Therefore, $\lim_{x \rightarrow a} f(x) = f(a)$, so that f is continuous at a . □

Remarks 6.2. 1. *If a function f is discontinuous at a point a then f is not differentiable at this point.*

2. *The continuity of a function f at a point does not assure the existence of the derivative at that point. See the Example (6.1).2.*

6.5 Derivative function

Definition 6.4. *Let f be a function defined on an open interval I of $\mathbb{R}$. We say that f is differentiable on I if it is differentiable at any point of I . We can associate to any element x of I , the derivative $f'(x)$ of f at x and we thus define a new function that is denoted by*

$$\begin{aligned} f' : I &\rightarrow \mathbb{R} \\ x &\mapsto f'(x). \end{aligned}$$

This function is called a derivative function of f on I .

Examples 6.2. *The functions $\sin$ and $\cos$ are differentiable on $\mathbb{R}$ and these derivative functions are $\cos$ and $-\sin$ respectively.*

6.6 Operations on differentiable functions at a point

Theorem 6.3. *Let $I \subseteq \mathbb{R}$ be an open interval, let $a \in I$, and let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be functions that are differentiable at a . Then :*

(1) The constant function is differentiable and its derivative is the zero function.

(2) If $\lambda \in \mathbb{R}$, then the function $\lambda \cdot f$ is differentiable at a , and

$$(\lambda \cdot f)'(a) = \lambda \cdot f'(a).$$

(3) The function $f + g$ is differentiable at a , and

$$(f + g)'(a) = f'(a) + g'(a).$$

(4) **[Product Rule]** The function fg is differentiable at a , and

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

(5) **[Quotient Rule]** If $g(a) \neq 0$, then the function $\frac{f}{g}$ is differentiable at a , and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

(6) **[Chain Rule]** Let J be interval in $\mathbb{R}$, let $\psi : J \rightarrow \mathbb{R}$ be function such that $\psi(J) \subseteq I$. If ψ is differentiable at $f(a)$ then the composite function $\psi \circ f$ is differentiable at a and

$$(\psi \circ f)'(a) = f'(a)\psi'(f(a)).$$

(7) If f is strictly monotone on I and $f'(a) \neq 0$, then the function f^{-1} is differentiable at $f(a)$, and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof :

The proof of (1), (2) and (3) is trivial. We shall prove (4) and (5). (4) Let $h = f \cdot g$, then for $x \in I$, $x \neq a$, we have

$$\begin{aligned} \frac{h(x) - h(a)}{x - a} &= \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x - a} = \frac{f(x)g(x) - f(a)g(a)}{x - a} \\ &= \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \\ &= \frac{(f(x) - f(a))g(x) + f(a)(g(x) - g(a))}{x - a} \\ &= \left(\frac{f(x) - f(a)}{x - a}\right)g(x) + f(a)\left(\frac{g(x) - g(a)}{x - a}\right). \end{aligned}$$

Since g is continuous at a , then $\lim_{x \rightarrow a} g(x) = g(a)$. Moreover f and g are differentiable at a , we deduce from the properties of limits that

$$\begin{aligned} h'(a) &= \lim_{x \rightarrow a} \left(\frac{h(x) - h(a)}{x - a}\right) = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a}\right) \times \lim_{x \rightarrow a} g(x) + f(a) \lim_{x \rightarrow a} \left(\frac{g(x) - g(a)}{x - a}\right) \\ &= f'(a)g(a) + f(a)g'(a). \end{aligned}$$

Finally

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

(5) Let $\psi = \frac{f}{g}$. Since g is differentiable at a , it is continuous at that point. Therefore, since $g(a) \neq 0$, that there exists an interval $J \subseteq I$ with $a \in J$ such that

$$\forall x \in J, g(x) \neq 0.$$

For all $x \in J, x \neq a$, we have

$$\begin{aligned} \frac{\psi(x) - \psi(a)}{x - a} &= \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(a)}{x - a} = \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} = \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)(x - a)} \\ &= \frac{\left(\frac{f(x)g(a) - f(a)g(x)}{x - a}\right)}{g(x)g(a)} = \frac{\left(\frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{x - a}\right)}{g(x)g(a)} \\ &= \frac{\left(\frac{(f(x) - f(a))g(a) - f(a)(g(x) - g(a))}{x - a}\right)}{g(x)g(a)} = \frac{\left(\frac{f(x) - f(a)}{x - a}\right)g(a) - f(a)\left(\frac{g(x) - g(a)}{x - a}\right)}{g(x)g(a)} \end{aligned}$$

Using the continuity of g at a and the differentiability of f and g at a , we get

$$\psi'(a) = \lim_{x \rightarrow a} \frac{\psi(x) - \psi(a)}{x - a} = \lim_{x \rightarrow a} \frac{\left(\frac{f(x) - f(a)}{x - a}\right)g(a) - f(a)\left(\frac{g(x) - g(a)}{x - a}\right)}{g(x)g(a)} = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Thus, $\psi = \frac{f}{g}$ is differentiable at a and equation (5) holds.

(6) For all $x \in I$ and $x \neq a$, we can write

$$\frac{\psi \circ f(x) - \psi \circ f(a)}{x - a} = \frac{\psi(f(x)) - \psi(f(a))}{f(x) - f(a)} = \left(\frac{\psi(f(x)) - \psi(f(a))}{f(x) - f(a)}\right) \left(\frac{f(x) - f(a)}{x - a}\right).$$

Then

$$\begin{aligned} \lim_{x \rightarrow a} \frac{\psi \circ f(x) - \psi \circ f(a)}{x - a} &= \lim_{x \rightarrow a} \left(\frac{\psi(f(x)) - \psi(f(a))}{f(x) - f(a)}\right) \left(\frac{f(x) - f(a)}{x - a}\right) \\ &= \lim_{y \rightarrow f(a)} \left(\frac{\psi(y) - \psi(f(a))}{y - f(a)}\right) \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a}\right) \\ &= \psi'(f(a)) \cdot f'(a). \end{aligned}$$

(7) As f is strictly monotone then we can write

$$\forall x \in I, f^{-1} \circ f(x) = x \Rightarrow (f^{-1} \circ f)'(x) = 1.$$

So

$$(f^{-1} \circ f)'(a) = (f^{-1})'(f(a))f'(a) = 1.$$

□

Examples 6.3. 1. Let $n \in \mathbb{N}^*$. The function $\sqrt[n]{\cdot}$ is the inverse of the function $x \mapsto x^n$ on the $[0, +\infty[$. We apply the formula of the derivative of inverse function and we can write, for all $y \in]0, +\infty[$,

$$(\sqrt[n]{y})' = \frac{1}{nx^{n-1}} = \frac{1}{(n\sqrt[n]{y})^{n-1}} = \frac{1}{n}y^{\frac{1}{n}-1}.$$

2. The function $\sin$ from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$ is strictly monotone so admits an inverse denoted $\sin^{-1}$ or $\arcsin$ defined from $[-1, 1]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where is strictly monotone on $[-1, 1]$. Moreover, applying the formula of the derivative of inverse function and we can write, for all $y \in]-1, 1[$,

$$\arcsin'(y) = \frac{1}{\sin'(x)} = \frac{1}{\sin'(\arcsin(y))} = \frac{1}{\cos(\arcsin(y))}.$$

However,

$$\cos^2(\arcsin(y)) = 1 - \sin^2(\arcsin(y)) = 1 - y^2,$$

and as y belongs to $] -1, 1[$, $\arcsin(y) \in] -\frac{\pi}{2}, \frac{\pi}{2}[$ then $\cos(\arcsin(y)) > 0$. Hence $\cos(\arcsin(y)) = \sqrt{1 - y^2}$. So we have

$$\arcsin'(y) = \frac{1}{\sqrt{1 - y^2}}.$$

3. In the same way, we can show that the function $\tan$ is strictly monotone from $] -\frac{\pi}{2}, \frac{\pi}{2}[$ to $\mathbb{R}$ so admits an inverse function denoted by $\tan^{-1}$ or $\arctan$ defined from $\mathbb{R}$ to $] -\frac{\pi}{2}, \frac{\pi}{2}[$ where is strictly monotone on $\mathbb{R}$ and with the formula for the inverse derivative we can write :

$$\forall x \in \mathbb{R}, \arctan'(x) = \frac{1}{1 + x^2}.$$

6.7 Fundamental theorems

Theorem 6.4. [Fermat theorem] Let I be an open interval of $\mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be a function. If f admits a boundary value at a point $a \in I$ and f is differentiable at a then $f'(a) = 0$.

Proof :

For example, suppose that f has a local maximum. The differentiability in a gives us that the right-hand and left-hand derivatives exist and are equal

$$f'_r(a) = f'_l(a) = f'(a).$$

However, the local maximum property gives

$$x \geq a \Rightarrow \frac{f(x) - f(a)}{x - a} \geq 0 \Rightarrow f'_r(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \geq 0,$$

and

$$x \leq a \Rightarrow \frac{f(x) - f(a)}{x - a} \leq 0 \Rightarrow f'_l(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

We conclude that $f'(a) = 0$. □

Remarks 6.3. 1. *If a function admits a boundary value at a point, then the tangent at this point to the curve of this function is parallel to the (ox) axis.*

2. *A function can have a zero derivative at a point, without having a value bounded at this point for example you can take the function $x \mapsto x^5$.*

3. *The theorem does not apply to right-hand or left-hand derivatives. In fact, the function $f(x) = |x|$ has a local minimum at $x = 0$, it is differentiable to the right of 0 but $f'_r(0) = 1$. Likewise, it is differentiable to the left of 0 but $f'_l(0) = -1$.*

Theorem 6.5. [Rolle's Theorem] *Suppose that f is continuous on a closed interval $I = [a, b]$, that the derivative f' exists at every point of the open interval $]a, b[$, and that $f(a) = f(b)$. Then there exists at least one point c in $]a, b[$ such that $f'(c) = 0$.*

Proof :

f is continuous function on $I = [a, b]$, so there exists two real numbers α and β in I such that

$$\forall x \in I, f(\alpha) \leq f(x) \leq f(\beta).$$

However,

$$f(\alpha) = \inf_{x \in I} f(x) \quad \text{and} \quad f(\beta) = \sup_{x \in I} f(x).$$

If $f(\alpha) = f(\beta)$ then the function f is constant, so for all $c \in]a, b[$, $f'(c) = 0$.

If $f(\alpha) < f(\beta)$ then $(\alpha, \beta) \neq (a, b)$. Hence

$$(\alpha, \beta) \neq (a, b) \Rightarrow \alpha \neq a \quad \text{or} \quad \beta \neq b \Rightarrow \alpha \in]a, b[\quad \text{or} \quad \beta \in]a, b[.$$

Just apply the Fermat Theorem to complete the proof. □

Geometric Interpretation of Rolle's Theorem : The geometric view of the Rolle's Theorem is that there is some point $A(c, f(c))$ on the curve $y = f(x)$ at which the tangent line is parallel to the axis (ox) .

Theorem 6.6. [Mean Value Theorem] *Suppose that f is continuous on a closed interval $I = [a, b]$, and that f has a derivative in the open interval $]a, b[$. Then there exists at least one point c in $]a, b[$ such that,*

$$f(b) - f(a) = (b - a)f'(c).$$

Proof :

Consider the function ψ defined on I by

$$\psi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Rolle's Theorem are satisfied by ψ since ψ is continuous on $[a, b]$, differentiable on (a, b) , and $\psi(a) = \psi(b) = 0$. Therefore, there exists a point c in (a, b) such that

$$0 = \psi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

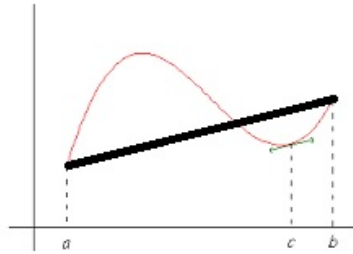
Hence,

$$f(b) - f(a) = (b - a)f'(c).$$

□

Geometric Interpretation of Mean Value Theorem :

The geometric view of the Mean Value Theorem is that there is some point on the curve $y = f(x)$ at which the tangent line is parallel to the line segment through the points $A(a, f(a))$ and $B(b, f(b))$.



Corollary 6.1. *If the function f verified on $[a, b]$ all the conditions of mean value theorem then,*

$$m \leq \frac{f(b) - f(a)}{b - a} \leq M \quad \text{where} \quad m = \inf_{x \in]a, b[} f'(x) \quad \text{and} \quad M = \sup_{x \in]a, b[} f'(x).$$

Corollary 6.2. *If the function f is continuous on $[a, b]$ and admits on $]a, b[$ an identically zero derivative then, the function is constant on $[a, b]$.*

Corollary 6.3. *If the function f verified on $[a, b]$ all the conditions of mean value theorem then,*

1.

$$\forall x \in]a, b[, \quad f'(x) > 0 \Rightarrow f \text{ is strictly increasing function.}$$

2.

$$\forall x \in]a, b[, \quad f'(x) < 0 \Rightarrow f \text{ is strictly decreasing function.}$$

Theorem 6.7. [Lagrange Theorem] *Let f and g be two definite functions on an interval $[a, b]$. We suppose that*

1. f and g are continuous functions on $[a, b]$.
2. f and g are differentiable functions on $]a, b[$.

In this case, there exists $c \in]a, b[$ such that

$$(f(b) - f(a)) g'(c) = (g(b) - g(a)) f'(c).$$

Proof :

Just take the function ψ defined on the interval $[a, b]$ by

$$\psi(x) = (f(b) - f(a)) g(x) - (g(b) - g(a)) f(x),$$

then apply mean value theorem on $[a, b]$. □

Exercise 6.1. 1. Using the finite mean value theorem, show that

$$\forall x \in]0, +\infty[, \quad \frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}.$$

2. Determine the limits

$$\lim_{x \rightarrow +\infty} \sqrt{x} (\ln(x+1) - \ln x) \quad \text{and} \quad \lim_{x \rightarrow +\infty} x (\ln(x+1) - \ln x).$$

3. Deduce that $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$.

4. Determine this limit $\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^x$.

Exercise 6.2. Prove the following double inequalities :

1.

$$\forall x \in [0, +\infty[, \quad x \leq e^x - 1 \leq xe^x.$$

2.

$$\forall x \in [0, +\infty[, \quad \frac{x}{x+1} \leq \ln(x+1) \leq x.$$

6.8 Hospital Rules

Theorem 6.8. [Hospital's Rule I] Let $a < b$ and let f, g be differentiable on $I =]a, b[$ such that $g'(x) \neq 0$ for all $x \in I$. Suppose that

$$\lim_{x \rightarrow a^+} f(x) = 0 = \lim_{x \rightarrow a^+} g(x).$$

If

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty, +\infty\},$$

then

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

Proof :

(★) : We consider the case that

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}.$$

F and G are the extensions by continuity of f and g respectively at a point a given by

$$F(x) = \begin{cases} f(x) & \text{for } x \in]a, b[\\ 0 & \text{for } x = a \end{cases} ; \quad G(x) = \begin{cases} g(x) & \text{for } x \in]a, b[\\ 0 & \text{for } x = a. \end{cases}$$

The functions F and G are defined and continuous on $]a, b[$ and these functions are differentiable on $]a, b[$. Moreover

$$\forall x \in]a, b[, \quad F'(x) = f'(x) \quad \text{and} \quad G'(x) = g'(x) \neq 0.$$

Let $\varepsilon > 0$. In this case we can write

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \Rightarrow \exists \delta_\varepsilon > 0 : \forall x \in]a, a + \delta_\varepsilon[, \quad \left| \frac{f'(x)}{g'(x)} - \ell \right| < \varepsilon.$$

By applying Lagrange's theorem (6.7) to any segment $[a, x] \subset [a, a + \delta_\varepsilon]$, so there exists $c \in]a, x[$ such that

$$\left| \frac{f(x)}{g(x)} - \ell \right| = \left| \frac{F(x) - F(a)}{G(x) - G(a)} - \ell \right| = \left| \frac{f'(c)}{g'(c)} - \ell \right| < \varepsilon.$$

We conclude that the limit $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$ and it's equal to ℓ .

(★★) : We consider the case when $\ell = +\infty$. Let $A > 0$. We have

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = +\infty \Rightarrow \exists \delta_A > 0 : \forall x \in]a, a + \delta_A[, \quad \frac{f'(x)}{g'(x)} > A.$$

So for any segment $[a, x] \subset [a, a + \delta_\varepsilon]$, there exists $c \in]a, x[$ such that

$$\frac{f(x)}{g(x)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{f'(c)}{g'(c)} > A.$$

So $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = +\infty$.

The case $\ell = -\infty$ is treated in an analogous way by replacing $\frac{f'(c)}{g'(c)} > A$ by $\frac{f'(c)}{g'(c)} < -A$. $\square$

Remark 6.1. It is useful to note that in general the existence of the limit $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$ does not necessarily imply the existence of the limit $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$, for example, for the functions

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{and} \quad g(x) = x \quad (x \in]0, 1[);$$

we have

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0^+} \left[x \sin \left(\frac{1}{x} \right) \right] = 0.$$

But

$$\lim_{x \rightarrow 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0^+} \left(2x \sin \left(\frac{1}{x} \right) - \cos \left(\frac{1}{x} \right) \right) = \text{does not exist}.$$

Theorem 6.9. [Hospital's Rule I] Let $a < b$ and let f, g be differentiable on $I =]a, b[$ such that $g'(x) \neq 0$ for all $x \in I$. Suppose that

$$\lim_{x \rightarrow b^-} f(x) = 0 = \lim_{x \rightarrow b^-} g(x).$$

If

$$\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty, +\infty\},$$

then

$$\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}.$$

Proof :

The proof is treated in an analogous way that the previous theorem. □

Theorem 6.10. [Hospital's Rule II] Let $a < b$ and let f, g be differentiable on $I =]a, b[$ such that $g'(x) \neq 0$ for all $x \in I$. Suppose that

$$\lim_{x \rightarrow b^-} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow b^-} g(x) = \pm\infty.$$

If

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty, +\infty\},$$

then

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

Proof :

First, we consider the case

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}.$$

Let $\varepsilon > 0$. We can write

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \Rightarrow \exists \delta_\varepsilon > 0 : \forall x \in]a, a + \delta_\varepsilon[, \left| \frac{f'(x)}{g'(x)} - \ell \right| < \frac{\varepsilon}{2}. \quad (6.1)$$

Let the real number y such that $y \in]a, a + \delta_\varepsilon[$, so for all $x \in]a, y[$, we suppose that

$$\psi(x, y) = \frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}}.$$

From the hypotheses we get

$$\lim_{x \rightarrow a^+} \psi(x, y) = 1.$$

So, there exists $\delta_\varepsilon > 0$ such that

$$x \in]a, a + \delta_\varepsilon[\Rightarrow |\psi(x, y) - 1| < \frac{\varepsilon}{2(|\ell| + \frac{2}{\varepsilon})}.$$

From Lagrange's theorem, we can write

$$\frac{f(x)}{g(x)} = \left(\frac{f(x) - f(y)}{g(x) - g(y)} \right) \psi(x, y) = \frac{f'(c)}{g'(c)} \psi(x, y), \quad c \in]x, y[\subset]a; y[.$$

Using (6.1), we obtain

$$\left| \frac{f'(c)}{g'(c)} \right| < |\ell| + \frac{\varepsilon}{2}.$$

So, we get

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \ell \right| &= \left| \frac{f'(c)}{g'(c)} \psi(x, y) - \ell \right| \\ &= \left| \frac{f'(c)}{g'(c)} (\psi(x, y) - 1) + \frac{f'(c)}{g'(c)} - \ell \right| \\ &\leq \left| \frac{f'(c)}{g'(c)} (\psi(x, y) - 1) \right| + \left| \frac{f'(c)}{g'(c)} - \ell \right| \\ &= \left| \frac{f'(c)}{g'(c)} \right| |\psi(x, y) - 1| + \left| \frac{f'(c)}{g'(c)} - \ell \right| \\ &\leq \left(|\ell| + \frac{\varepsilon}{2} \right) |\psi(x, y) - 1| + \frac{\varepsilon}{2} \\ &< \left(|\ell| + \frac{\varepsilon}{2} \right) \left(\frac{\varepsilon}{2(|\ell| + \frac{\varepsilon}{2})} \right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We conclude that

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \ell.$$

For the case, $\ell = \infty$, we proceed in a similar way to the first rule with some minimal adaptations. $\square$

Theorem 6.11. [Hospital's Rule II] Let $a < b$ and let f, g be differentiable on $I =]a, b[$ such that $g'(x) \neq 0$ for all $x \in I$. Suppose that

$$\lim_{x \rightarrow b^-} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow b^-} g(x) = \pm\infty.$$

If

$$\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty, +\infty\},$$

then

$$\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}.$$

Remark 6.2. In the last four theorems, we can take $a = -\infty$ or $b = +\infty$.

Exercise 6.3. Use hospital rule to calculate the following limits

- (1.) $\lim_{x \rightarrow 0^+} \frac{\sin(x)}{\sqrt{x}},$ (2.) $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2},$ (3.) $\lim_{x \rightarrow 1} \frac{\ln(x)}{x - 1}$ (4.) $\lim_{x \rightarrow 1} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$
 (5.) $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\arctan(x)} \right),$ (6.) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x},$ (7.) $\lim_{x \rightarrow 0^+} x^3 \ln(x)$ (8.) $\lim_{x \rightarrow 0^+} \left(\frac{1}{x(\ln x)^2} \right)$
 (9.) $\lim_{x \rightarrow +\infty} \left(\frac{x + \ln x}{x \ln x} \right).$

7

Elementary functions

7.1 Exponential and Logarithmic functions

7.1.1 Natural Exponential function

Theorem and Definition 7.1. *There exists a unique function f differentiable on $\mathbb{R}$ such that*

$$\begin{cases} f' = f \\ f(0) = 1. \end{cases} \quad (7.1)$$

This function f is called the exponential function denoted by

$$x \mapsto f(x) = \exp(x) \quad \text{or} \quad x \mapsto e^x.$$

e is an irrational number, approximately by $e \simeq 2.71828183$.

Proof :

We consider the function ψ defined on $\mathbb{R}$ by

$$\psi(x) = f(x)f(-x).$$

This function is constant on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \psi(x) = 1.$$

Indeed, ψ is differentiable on $\mathbb{R}$ by definition and for all x in $\mathbb{R}$ we have

$$\psi'(x) = f'(x)f(-x) - f(x)f'(-x) = f(x)f(-x) - f(x)f(-x) = 0.$$

Hence,

$$\forall x \in \mathbb{R}, \psi(x) = \psi(0) = [f(0)]^2 = 1.$$

Moreover, f is non-zero function on $\mathbb{R}$ because if f is the null function on $\mathbb{R}$ i.e.

$$\forall x \in \mathbb{R}, f(x) = 0,$$

we obtain

$$\begin{cases} f(x) = 0 \\ f(-x) = 0 \end{cases} \Rightarrow f(x)f(-x) = 0.$$

Which contradicts the fact that

$$\forall x \in \mathbb{R}, \psi(x) = 1.$$

We suppose now that there exists an another function g differentiable on $\mathbb{R}$ solution of (7.1) i.e.

$$\begin{cases} g' = g \\ g(0) = 1. \end{cases}$$

According to the above

$$\forall x \in \mathbb{R}, f(x) \neq 0.$$

We show that $f = g$. To establish this equality, we need to define on $\mathbb{R}$ the function ρ by

$$\forall x \in \mathbb{R}, \rho(x) = \frac{g(x)}{f(x)}.$$

ρ is differentiable on $\mathbb{R}$ and for all x belongs to $\mathbb{R}$ we have

$$\rho'(x) = \frac{g'(x)f(x) - f'(x)g(x)}{[f(x)]^2} = \frac{g(x)f(x) - f(x)g(x)}{[f(x)]^2} = 0.$$

We deduce that the function ρ is constant on $\mathbb{R}$, so

$$\forall x \in \mathbb{R}, \rho(x) = \frac{g(0)}{f(0)} = 1.$$

Thus

$$\forall x \in \mathbb{R}, f(x) = g(x).$$

Hence the uniqueness of the function f .

□

Property 7.1. [The basic property of an exponential function]

$$\forall x, y \in \mathbb{R}, \exp(x + y) = \exp(x) \exp(y) \text{ or } e^{x+y} = e^x e^y.$$

Proof :

Let ψ the function defined on $\mathbb{R}$ by :

$$\psi(x) = \frac{\exp(x + y)}{\exp(x)},$$

where $y \in \mathbb{R}$. This function is differentiable on $\mathbb{R}$ and for all $x \in \mathbb{R}$, we have

$$\psi'(x) = \frac{\exp'(x + y) \exp(x) - \exp'(x) \exp(x + y)}{[\exp(x)]^2} = \frac{\exp(x + y) \exp(x) - \exp(x) \exp(x + y)}{[\exp(x)]^2} = 0.$$

Thus, ψ is a constant function, so

$$\forall x \in \mathbb{R}, \psi(x) = \psi(0) = \frac{\exp(y)}{\exp(0)} = \exp(y).$$

We deduce that

$$\forall x, y \in \mathbb{R}, \exp(x + y) = \exp(x) \exp(y).$$

Hence the property. □

Proposition 7.1. *For all x, y in $\mathbb{R}$, then*

1. $\exp(0) = 1$, $\exp(x) \neq 0$, and $\exp'(x) = \exp(x)$.
2. $\exp(-x) = \frac{1}{\exp(x)}$.
3. $\exp(x - y) = \frac{\exp(x)}{\exp(y)}$.
4. $\forall n \in \mathbb{Z}$, $\exp(nx) = [\exp(x)]^n$.
5. $\exp(x) > 0$.

Proof :

(1.) Direct result of the definition..

(2) We apply the property (7.1) with $x = -y$ and we have

$$\begin{aligned} 1 &= \exp(0) = \exp(x - x) = \exp(x + (-x)) \\ &= \exp(x) \exp(-x) \Rightarrow \exp(-x) = \frac{1}{\exp(x)}. \end{aligned}$$

(3) We have

$$\begin{aligned} \exp(x - y) &= \exp(x + (-y)) = \exp(x) \exp(-y) \\ &= \exp(x) \frac{1}{\exp(y)} = \frac{\exp(x)}{\exp(y)}. \end{aligned}$$

(4) Let $n \in \mathbb{Z}$. We suppose that

$$f(x) = \frac{\exp(nx)}{[\exp(x)]^n}.$$

This function is constant on $\mathbb{R}$. In fact f is differentiable on $\mathbb{R}$ and for all x belongs to $\mathbb{R}$, we have

$$\begin{aligned} f'(x) &= \frac{n \exp'(nx) [\exp(x)]^n - n \exp'(x) [\exp(x)]^{n-1} \exp(nx)}{[\exp(x)]^{2n}} \\ &= \frac{n \exp(nx) [\exp(x)]^n - n \exp(x) [\exp(x)]^{n-1} \exp(nx)}{[\exp(x)]^{2n}} \\ &= \frac{n \exp(nx) [\exp(x)]^n - n [\exp(x)]^n \exp(nx)}{[\exp(x)]^{2n}} = 0. \end{aligned}$$

We deduce that

$$f(x) = f(0) = 1 \Rightarrow \exp(nx) = [\exp(x)]^n.$$

(5) For all x belongs to $\mathbb{R}$ we have

$$\exp(x) = \exp\left(2\left(\frac{x}{2}\right)\right) = \left[\exp\left(\frac{x}{2}\right)\right]^2 > 0.$$

Corollary 7.1. *The function $\exp$ is strictly increasing function on $\mathbb{R}$.*

Proof :

The function $\exp$ is differentiable on $\mathbb{R}$ and it's derivative function is $\exp$. Therefore,

$$\forall x \in \mathbb{R}, e^x > 0,$$

we deduce that $\exp$ is strictly increasing function on $\mathbb{R}$. □

Corollary 7.2. *Let a, b be two real numbers.*

1. $a > b \Leftrightarrow e^a > e^b$.
2. $a = b \Leftrightarrow e^a = e^b$.
3. $a \geq 0 \Leftrightarrow e^a \geq 1$.
3. $a \leq 0 \Leftrightarrow 0 < e^a \leq 1$.

7.2 Limits of the exponential function

Proposition 7.2.

$$\lim_{x \rightarrow +\infty} e^x = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} e^x = 0^+.$$

Proof :

We consider the function f defined on $]0, +\infty[$ by

$$f(x) = e^x - x.$$

f is differentiable on $]0, +\infty[$ and for all $x \in]0, +\infty[$, we have

$$f'(x) = e^x - 1.$$

So f is strictly increasing function on $[0, +\infty[$ and $f(0) = 0$. We deduce that

$$\forall x \in [0, +\infty[, f(x) \geq 0.$$

That is

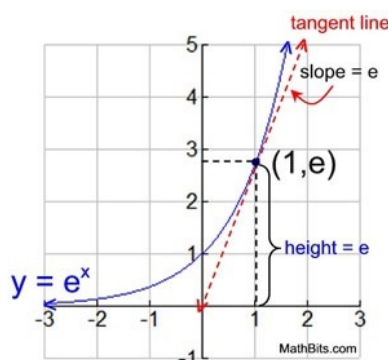
$$\forall x \in [0, +\infty[, e^x \geq x.$$

We have $\lim_{x \rightarrow +\infty} x = +\infty$ thus $\lim_{x \rightarrow +\infty} e^x = +\infty$.

Furthermore

$$\lim_{x \rightarrow -\infty} e^x = \lim_{x \rightarrow -\infty} \left(\frac{1}{e^{-x}}\right) = 0^+.$$

Graph natural exponential function : □



7.3 Functions of the form $\exp \circ f$

Proposition 7.3. *If the function f is differentiable on an open interval I of $\mathbb{R}$, then the function $\exp \circ f$ is differentiable on I and*

$$\forall x \in I, (\exp \circ f)'(x) = f'(x)e^{f(x)}.$$

Proof :

Simply apply the result of the derivative of the composite of two functions.

Examples 7.1. (1) *The derivative function of the function $x \mapsto f(x) = e^{x^3-4x}$ on $\mathbb{R}$ is*
 $x \mapsto f'(x) = (3x^2 - 4)e^{x^3-4x}.$

(2) *The derivative function of the function $x \mapsto g(x) = e^{\frac{1}{x^3-4x}}$ defined on $\mathbb{R} - \{0, -2, 2\}$ is*
 $x \mapsto g'(x) = \left(-\frac{3x^2-4}{(x^3-4x)^2}\right)e^{\frac{1}{x^3-4x}}.$

7.4 Natural logarithm function $\ln$

The natural exponential function is continuous and strictly increasing function for $\mathbb{R}$ to $\mathbb{R}_+^*$. The inverse function is also continuous and strictly increasing function from $\mathbb{R}_+^*$ to $\mathbb{R}$.

Definition 7.1. *The inverse function of the function $y \mapsto \exp(y)$ is called the natural logarithm function, denoted by $x \mapsto \ln(x)$ and satisfies*

$$(y \in \mathbb{R}), \quad x = \exp(y) \Leftrightarrow y = \ln x, \quad (x \in \mathbb{R}_+^*).$$

From the definition we can have the following result

Property 7.2. 1. $\forall x \in \mathbb{R}, \ln(e^x) = x.$

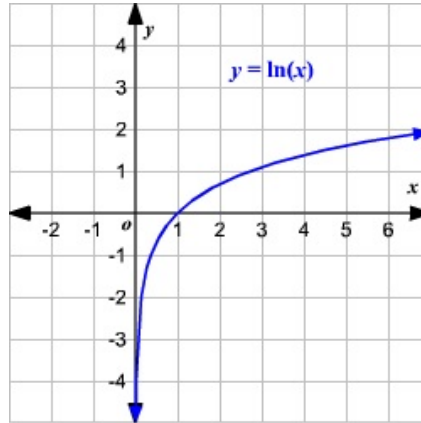
2. $\forall x \in]0, +\infty[, \quad e^{\ln x} = x.$

3. $\ln 1 = 0$ and $\ln e = 1.$

4. $\forall a, b \in \mathbb{R}_+^*, \quad a > b \Leftrightarrow \ln a > \ln b.$

5. $\ln a \geq 0 \Leftrightarrow a \in [1, +\infty[$ and $\ln a \leq 0 \Leftrightarrow a \in]0, 1]$.
6. The graph of the natural logarithm function is deduced from that of the inverse gg function natural exponential function by symmetry around the first bisector axes for equation $y = x$.

Graph natural exponential function :



7.5 Derivative of natural logarithm function

Property 7.3. The natural logarithm function is continuous and differentiable function on $]0, +\infty[$ and

$$\forall x \in]0, +\infty[, \quad (\ln)'(x) = \frac{1}{x}.$$

Proof :

It follows from the definition and the differentiation theorem of the inverse function that the natural logarithm function is continuous and differentiable function on $]0, +\infty[$. Furthermore, for all $x \in]0, +\infty[$ and for all $y \in \mathbb{R}$ such that $\ln x = y$, we have

$$(\ln)'(x) = (\ln)'(\exp(y)) = \frac{1}{\exp'(y)} = \frac{1}{\exp(y)} = \frac{1}{x}.$$

□

7.6 Limits of the natural logarithm function

From the definition of the inverse function we can have the following result

Proposition 7.4.

$$\lim_{x \rightarrow +\infty} \ln x = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \ln x = -\infty.$$

7.7 Algebraic properties of the natural logarithm function

Proposition 7.5. *For all a and b be a strictly positif real numbers, we have*

1. $\ln(a.b) = \ln a + \ln b$.
2. $\ln\left(\frac{1}{b}\right) = -\ln b$.
3. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.
4. $\forall n \in \mathbb{Q}, \ln(a)^n = n \ln a$.

Proof : Let a and b be in $]0, +\infty[$.

1. We pose $A = \ln(a.b)$ and $B = \ln a + \ln b$. Then we have

$$B = \ln(e^B) = \ln(e^{\ln a + \ln b}) = \ln(e^{\ln a} e^{\ln b}) = \ln(a.b) = \ln A.$$

- 2.

$$0 = \ln 1 = \ln\left(\frac{b}{b}\right) = \ln\left(b \cdot \left[\frac{1}{b}\right]\right) = \ln b + \ln\left(\frac{1}{b}\right).$$

We deduce that

$$\ln\left(\frac{1}{b}\right) = -\ln b.$$

- 3.

$$\ln\left(\frac{a}{b}\right) = \ln\left(a \times \frac{1}{b}\right) = \ln a + \ln\left(\frac{1}{b}\right) = \ln a - \ln b.$$

Let n be rational number. We have tree cases

- 1.) n is a natural number.
- 2.) n is integer number.
- 3.) n is a rational number.

Case 1 : In this case, we use the induction to prove that

$$\forall n \in \mathbb{N}, \ln(a)^n = n \ln a.$$

When $n = 0$, we have

$$\ln(a)^0 = \ln 1 = 0 = 0 \times \ln a.$$

So the statement is true for $n = 0$.

We assume that the statement $\ln(a)^n = n \ln a$ is true for the natural number n and we prove that is true for the natural number $n + 1$ i.e. we prove that

$$\ln(a)^{n+1} = (n + 1) \ln a.$$

We have

$$\ln(a)^{n+1} = \ln(a^n \times a) = \ln(a)^n + \ln a = n \ln a + \ln a = (n + 1) \ln a.$$

So the statement is true for the natural number $n + 1$. We deduce by the induction principle that

$$\forall n \in \mathbb{N}, \ln(a)^{n+1} = (n+1) \ln a.$$

Case 2 : We prove that,

$$\forall n \in \mathbb{Z}, \ln(a)^n = n \ln a. \quad (7.2)$$

In this case, n is an integer number thus $n \in \mathbb{Z}_+$ or $n \in \mathbb{Z}_-$. $n \in \mathbb{Z}_+ = \mathbb{N}$ is proved. It remains for us to show the statement for $n \in \mathbb{Z}_-$. So we can write

$$n \in \mathbb{Z}_- \Leftrightarrow \exists \alpha \in \mathbb{N} : n = -\alpha.$$

Furthermore,

$$\ln a^n = \ln a^{-\alpha} = \ln \left(\frac{1}{a^\alpha} \right) = -\ln a^\alpha = -\alpha \ln a = n \ln a.$$

Whence (7.2). **Case 2 :** We prove that the statement is also true for all n in $\mathbb{Q}$. We have

$$n \in \mathbb{Q} \Leftrightarrow \exists p \in \mathbb{Z} : \exists q \in \mathbb{N}^* : n = \frac{p}{q}.$$

We pose that

$$\begin{aligned} a^n = b &\Leftrightarrow a^{\frac{p}{q}} = b \\ &\Leftrightarrow a^p = b^q \\ &\Leftrightarrow \ln a^p = \ln b^q \\ &\Leftrightarrow p \ln a = q \ln b \\ &\Leftrightarrow \ln b = \frac{p}{q} \ln a \\ &\Leftrightarrow \ln a^n = n \ln a. \end{aligned}$$

So the statement is proved for all $n \in \mathbb{Q}$. □

7.8 Functions of the form $\ln \circ f$

Proposition 7.6. *If the function f is differentiable and strictly positif on an open interval I then the function $\ln \circ f$ is differentiable on I . Furthermore*

$$\forall x \in I, (\ln \circ f)'(x) = \frac{f'(x)}{f(x)}.$$

7.9 Basic logarithmic function

Exercise 7.1. *Let's look for all the functions f differentiable on $\mathbb{R}_+^*$ and which verify*

$$\forall x, y \in \mathbb{R}_+^*, f(xy) = f(x) + f(y). \quad (7.3)$$

Solution :

For $x = y = 1$ we obtain $f(1) = 0$. We note that the function $x \mapsto \ln x$ is one solution of the equation (??). Let's look for other solutions

By deriving the function $y \mapsto f(xy)$, we obtain

$$xf'(yx) = f'(y).$$

Replace y by 1 and set $f'(1) = k$, we obtain

$$xf'(x) = f'(1) = k.$$

Thus

$$\exists c \in \mathbb{R}, f(x) = k \ln x + c \text{ and } f(1) = 0.$$

So

$$f(x) = k \ln x.$$

The function f is strictly monotone from $\mathbb{R}_+^*$ to $\mathbb{R}$, so there exists $a \in \mathbb{R}_+^* - \{1\}$ such that $k \ln a = 1$. Thus $k = \frac{1}{\ln a}$. Then f is the function defined on $\mathbb{R}_+^*$ by

$$x \mapsto f(x) = \frac{\ln x}{\ln a}.$$

□

Then it follows from what precedes the following definition.

Definition 7.2. Let $a \in \mathbb{R}_+^* - \{1\}$. We called Logarithmic function with base a denoted by $\log_a$ the function defined from $]0, +\infty[$ to $\mathbb{R}$ by

$$x \mapsto \log_a x = \frac{\ln x}{\ln a}.$$

$\log_a x$ can be read as log basis a of x .

Remarks 7.1. Let $a \in \mathbb{R}_+^* - \{1\}$.

1. $\log_a 1 = 0$.
2. $\forall x \in \mathbb{R}_+^*, \log_e x = \ln x$.
3. $\forall x, y \in \mathbb{R}_+^*, \log_a (x.y) = \log_a x + \log_a y$.
4. The logarithmic function with base 10 is called the common logarithmic function and it is denoted by $\log_{10}$ or simply $\log$.
5. The logarithmic function to the base e is the natural logarithmic function and it is denoted by $\log_e = \ln$.
6. The logarithmic function to the base 2 is called binary logarithm function.

Study of the logarithmic function with base a :

Let $a \in \mathbb{R}_+^* - \{1\}$. The function $\log_a$ is differentiable on $\mathbb{R}_+^*$ and it's derivative function is

$$\log'_a x = \frac{1}{x \ln a}.$$

Therefore,

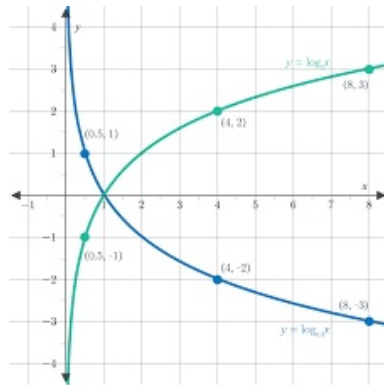
$$\begin{aligned} a > 1 &\implies \log'_a > 0 \implies \log_a \text{ is strictly increasing,} \\ 0 < a < 1 &\implies \log'_a < 0 \implies \log_a \text{ is strictly decreasing.} \end{aligned}$$

Remark 7.1. Let $a \in \mathbb{R}_+^* - \{1\}$. We remark that the function $a \mapsto \frac{1}{a}$ it is bijective from $]1, +\infty[$ to $]0, 1[$. Furthermore,

$$\ln\left(\frac{1}{a}\right) = -\ln a,$$

we can deduce

$$\forall x \in \mathbb{R}, \log_{\frac{1}{a}} x = -\log_a x.$$

Graph of the logarithmic function with base a :**7.9.1 Change of basis**

Let a and b be two distinct bases. We have

$$\forall x \in \mathbb{R}_+^*, \log_a x = \frac{\ln x}{\ln a} = \frac{\ln b}{\ln a} \times \frac{\ln x}{\ln b} = \log_a b \times \log_b x.$$

Hence the formula for changing the base of logarithms :

$$\forall x \in \mathbb{R}_+^*, \log_a x = \log_a b \times \log_b x.$$

7.10 Exponential function

Let $a \in \mathbb{R}_+^* - \{1\}$. The function $\log_a$ is continuous and differentiable function on $\mathbb{R}_+^*$ so it's bijective from $\mathbb{R}_+^*$ to $\mathbb{R}$ then admits an inverse function defined from $\mathbb{R}$ to $\mathbb{R}_+^*$.

Definition 7.3. Let $a \in \mathbb{R}_+^* - \{1\}$. The inverse function of the function $y \rightarrow \log_a y$ is called the exponential function with base a or simply the exponential function denoted by $x \mapsto a^x$.

Remark 7.2. Let $a \in \mathbb{R}_+^* - \{1\}$. We thus have by definition,

$$\forall y \in \mathbb{R}_+^*, x = \log_a y \Leftrightarrow y = a^x, \quad x \in \mathbb{R}.$$

Study of the exponential function with base a :

Let $a \in \mathbb{R}_+^* - \{1\}$. From the previous remark, for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}_+^*$, we deduce that

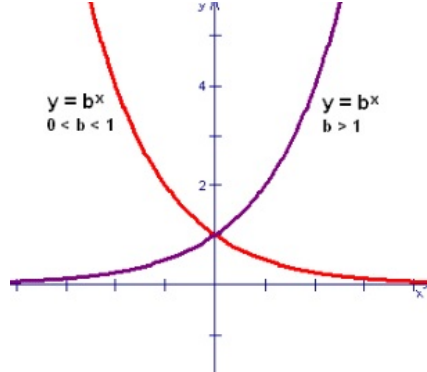
$$y = a^x \Leftrightarrow x = \frac{\ln y}{\ln a} \Leftrightarrow x \ln a = \ln y \Leftrightarrow y = e^{x \ln a}.$$

Thus, the derivative function of the function $\psi : x \mapsto \psi(x) = a^x$ is

$$\psi'(x) = (\ln a) e^{x \ln a}.$$

So, the function ψ is strictly increasing if $a > 1$ and is strictly decreasing if $0 < a < 1$.

Graph of the exponential function with base a :



7.11 Power Function

Definition 7.4. Let α be real number. The function defined on $\mathbb{R}_+^*$ by

$$x \mapsto x^\alpha$$

is called the power function of exponent α .

Remark 7.3. Let $\alpha \in \mathbb{R}$. We thus have by definition,

$$\forall y \in \mathbb{R}_+^*, y = x^\alpha \Leftrightarrow \ln y = \alpha \ln x \Leftrightarrow y = e^{\alpha \ln x}.$$

Study of the power function of exponent α

Let $\alpha \neq 0$. The function $\psi : x \mapsto x^\alpha$ is differentiable on $]0, +\infty[$ and for all $x \in]0, +\infty[$, we can write

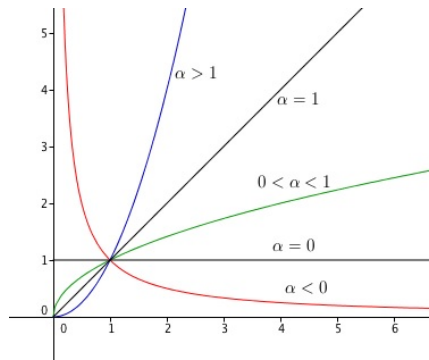
$$\psi'(x) = \alpha \left(\frac{1}{x} \right) e^{\alpha \ln x} = \frac{\alpha}{x} x^\alpha = \alpha x^{\alpha-1}.$$

Then the function $x \mapsto x^\alpha$ is strictly increasing if $\alpha > 0$ and is strictly decreasing if $\alpha < 0$. If $\alpha > 0$, we have

$$\lim_{x \rightarrow 0^+} x^\alpha = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} x^\alpha = +\infty.$$

If $\alpha < 0$, we have

$$\lim_{x \rightarrow 0^+} x^\alpha = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} x^\alpha = 0^+.$$

Graph of the power function of exponent α :

Exercise 7.2. Let α and β be two strictly positive real numbers. Find the following limits

1. $\lim_{x \rightarrow +\infty} \frac{(\ln x)^\beta}{x^\alpha}$.
2. $\lim_{x \rightarrow 0^+} x^\alpha |\ln x|^\beta$.

7.12 Hyperbolic Functions**7.12.1 Hyperbolic cosine and sine**

Definition 7.5. 1. We called hyperbolic cosine function, denoted $x \mapsto \cosh x$, the function defined on $\mathbb{R}$ as follows :

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

2. We called hyperbolic sine function, denoted $x \mapsto \sinh x$, the function defined on $\mathbb{R}$ as follows :

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

7.12.2 Properties

1. The function $\cosh$ is an even function and the function $\sinh$ is an odd function. Furthermore,

$$\forall x \in \mathbb{R}, \cosh(-x) = \cosh x \quad \text{and} \quad \sinh(-x) = -\sinh x.$$

2. For all $x \in \mathbb{R}$ we have

$$\cosh x + \sinh x = e^x \quad \text{and} \quad \cosh x - \sinh x = e^{-x}.$$

3. For all $x \in \mathbb{R}$ we have

$$\cosh^2 x - \sinh^2 x = 1.$$

4. For all $x \in \mathbb{R}$ we have

$$\cosh x > \sinh x.$$

7.12.3 Study of the Hyperbolic cosine and sine

1. **Derivative function of the Hyperbolic cosine and sine :** The two functions $\cosh$ and $\sinh$ are differentiable on $\mathbb{R}$. Furthermore, for all $x \in \mathbb{R}$, we have

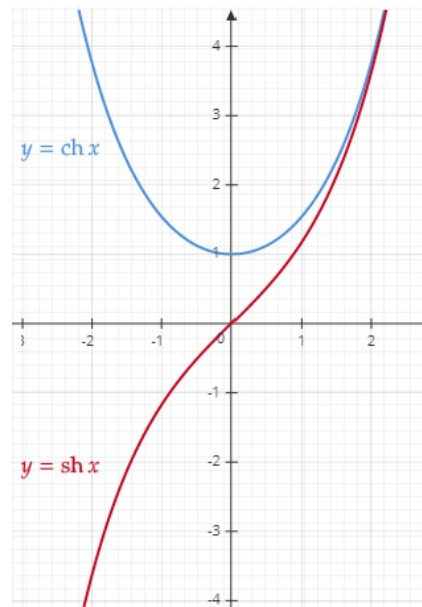
$$\cosh' x = \frac{e^x - e^{-x}}{2} = \sinh x \quad \text{and} \quad \sinh' x = \frac{e^x + e^{-x}}{2} = \cosh x.$$

2. **Limits :**

$$\lim_{x \rightarrow +\infty} \cosh x = +\infty = \lim_{x \rightarrow +\infty} \sinh x.$$

3. The function $\sinh$ is strictly increasing function on $\mathbb{R}$ and $\cosh$ is strictly increasing function on $[0, +\infty[$, strictly decreasing function on $] -\infty, 0]$.

4. **Graphs :**



7.12.4 Hyperbolic trigonometric

Exercise 7.3. *For all $a, b \in \mathbb{R}$, show that*

1. $\cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b.$
2. $\sinh(a + b) = \sinh a \cosh b + \cosh a \sinh b.$
1. $\cosh(a - b) = \cosh a \cosh b - \sinh a \sinh b.$
2. $\sinh(a - b) = \sinh a \cosh b - \cosh a \sinh b.$

Bibliographie

- [1] Berrabah Bendoukha, *INTRODUCTION A L'ANALYSE MATHEMATIQUE*, Poly-copié, centre universitaire de Naama (Algérie), 2016.
- [2] J.-M. Monier, *Analyse PCSI-PTSI*, Dunod, Paris 2003.
- [3] Y. Bougrov et S. Nikolski, *Cours de Mathématiques Supérieures*, Editions Mir, Moscou, 1983.
- [4] N. Piskounov, *Calcul différentiel et intégral*, Tome 1, Editions Mir, Moscou, 1980.
- [5] K. Allab, *Eléments d'Analyse*, OPU, Alger, 1984.
- [6] B. Calvo, J. Doyen, A. Calvo, F. Boschet, *Cours d'analyse*, Librairie Armand Colin, Paris, 1976.
- [7] J. Lelong-Ferrand et J. M. Arnaudiés, *Cours de mathématiques*, tome 2, Edition Dunod, 1978.
- [8] Michael J. Schramm, *Introduction to real analysis*, ISBN-13 : 978-0-486-46913-3, Englewood Cliffs, N.J. : Prentice Hall, 1996.
- [9] Robert G. Bartle Donald R. Sherbert, *Introduction to Real Analysis*, 4th ed, Library of Congress Cataloging-in-Publication Data, ISBN 978-0-471-43331-6 (hardback), 2011.