The ADM method for solving the modified kawahara equation with fractional spatial and temporal derivatives

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Abstract

In this paper, by introducing the fractional derivative in the sense of Caputo, we apply the Adomian decomposition method (ADM) to solve the modified Kawahara equation with fractional derivatives. Two cases of special interest such as the temporal-fractional modified Kawahara eqaution and the spatial-fractional modified Kawahara equation are discussed in details. Further, we compare the obtained results with those obtained for integer derivatives.

1. Introduction

Since Adomian firstly proposed the decomposition method [4, 5] at the begin of 1980s, the algorithm has been widely used for obtaining analytic solutions of physical significant equations [6, 16, 26, 27, 28, 29, 30, 32]. With this method, we can easily obtain approximate solutions in the form of rapidly convergent infinite series with each term computed conveniently[1, 2, 9, 14]. As we know, for the nonlinear equations with derivatives of integer order, many methods are used to derive approximate solutions [3, 4, 5, 8, 12, 19, 24, 25]. However, for the fractional differential equations, there are only limited approaches, such as Laplace transform method [21], Fourier transform method [17], the iteration method [23] and the operational method [20].

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In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the area of physics and engineering [10, 15, 33], like phenomena in electromagnetic theory, acoustics, electrochemistry and material science [11, 13, 21, 23, 33]. We introduce Caputo fractional derivative and we apply the ADM to derive numerical solutions of the modified Kawahara equation with temporal and spatial fractional derivatives. This equation is a mathematical model of a wide range in important physical phenomena; it arises in the context of shallow water waves [7]. Also, this equation describes water waves with surface tension [31].

The purpose of the present paper is to establish approximate solutions of the modified Kawahara equation with temporal and spatial fractional derivatives and to compare these solutions with those obtained for integer derivatives. The equation studied is given by

$$D_t^{\alpha} u + (1+u^2) D_x^{\beta} u + p u_{xxxx} + q u_{xxxxx} = 0, 0 < \alpha, \beta < 1 \tag{1}$$

where p and q are nonzero real constants [7], $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $x > 0$.

When $\alpha = \beta = 1$, the fractional equation reduces to the modified Kawahara equation of the form

$$u_t + u_x + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, 0 < \alpha, \beta < 1, p \in \mathbb{R}, q \in \mathbb{R} \tag{2}$$

The paper is organized as follows: In Sec. II, some necessary details on the fractional calculus are provided. In Sec. III, the modified Kawahara equation with temporal-and-spatial-fractional derivatives are studied with the Adomian method. In Sec IV and Sec V numerical applications are given and figures are used to show the efficiency as well as the accuracy of the approximate results which are achieved. Finally, conclusion is followed.

2. Preliminaries and notations

For the concept of fractional derivatives, there are multiple definitions [11]. Here, we will adopt Caputo's definition which is a modification of Riemann-Liouville definition and has the advantage of dealing properly with initial condition assumption and boundary conditions.

Definition 1: *A real valued function* $f(x), x > 0$ *is said to be in the space* $C_{\mu}, \mu \in \mathbb{R}$ *if there exists a real number* $p > \mu$ *such that* $f(x) = x^p f_1(x)$ *where* $f_1(x) \in C([0, \infty))$.

Definition 2: *A function* $f(x), x > 0$ *is said to be in the space* $C_{\mu}^{n}, n \in IN$*, if* $f^{(n)} \in C_{\mu}$.

Definition 3: *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *for a function* $f \in C_{\mu}, (\mu \geq -1)$ *is defined as*

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt; \ \alpha > 0, x > 0$$

$$J^{0} f(x) = f(x). \tag{3}$$

For the convenience of establishing the results for the fractional modified Kawahara equation, we give one basic property:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x). \tag{4}$$

For the expression (2), when $f(x) = x^{\beta}$ we get another expression that will be used later:

$$J^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta} \tag{5}$$

Definition 4: *The fractional derivative of* $f \in C_{-1}^{n}$ *in the Caputo's sense is defined as*

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-T)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in IN^{*}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \end{cases} \tag{6}$$

According to the Caputo's derivative, we can easily obtain the following expressions:

$$D^{\alpha} K = 0; \quad K \text{ is a constant.}$$

$$D^{\alpha} t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0. & \beta \leq \alpha - 1. \end{cases} \tag{7}$$

Remark 5 *In this work, we consider the equation (1). When 2 R+, we have:*

$$D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} =$$

$$= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n \\ \dfrac{\partial^n u(x,t)}{\partial t^n} & a = n \end{cases} \tag{8}$$

3. Proposed method

It is more natural to consider the equation (1) under the following form

$$D_t^{\alpha} u = -(1+u^2) D_x^{\beta} u - p u_{xxx} - q u_{xxxxx,} \; 0 < \alpha < 1, 0 < \beta < 1, \tag{9}$$

where the operators D_t^{α}, D_x^{B} stand for the fractional derivative and are defined as in(6).

Take the initial condition as

$$u(x,0) = f(x). \tag{10}$$

By applying the operator J^{α}, the inverse of D^{α} on the corresponding subequation of Eq. (9) and using the initial condition (10), we get:

$$u(x,t) = f(x) - J^{\alpha}\Phi_1(u(x,t) - J^{\alpha} L_1(u(x,t) + J^{\alpha} L_2(u(x,t), \tag{11}$$

where

$$\Phi_1(u) = (1+u^2) D_x^{\beta} u, \, , \; L_1(u(x,t) = p u_{xxx} \; and \; L_2(u(x,t) = q u_{xxxxx}.$$

The ADM method consists in finding a solution in the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{12}$$

The nonlinear operator Φ1(u) can be written in the decomposed form

$$\Phi_1(u) = \sum_{n=0}^{\infty} A_n, \tag{13}$$

where $A_n, n = 0, 1, 2, \ldots$, are the so called Adomian polynomials and have the form

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\Phi_1 \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right]_{\lambda=0}$$

$$= \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\left(1 + \left(\sum_{k=0}^{\infty} \lambda^k u_k \right)^2 \right) D_x^{\beta} \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right]_{\lambda=0} \tag{14}$$

We give the first four components of these polynomials:

$$\begin{aligned} A_0 &= (1+u_0^2)D_x^{\beta}u_0,\\ A_1 &= 2u_0\,u_1D_x^{\beta}\,u_0+(1+u_0^2)\,D_x^{\beta}\,u_1,\\ A_2 &= (u_1^2+2u_0\,u_2)\,D_x^{\beta}\,u_0+u_0\,u_1\,D_x^{\beta}\,u_1+(1+u_0^2)\,D_x^{\beta}\,u_2,\\ A_3 &= \Big(\frac{5}{6}\,u_1\,u_2+2u_0\,u_3\Big)\,D_x^{\beta}\,u_0+\Big(\frac{2}{3}u_1^2+\frac{4}{3}\,u_0\,u_2\Big)D_x^{\beta}\,u_1\\ &\qquad+\frac{4}{3}u_0\,u_1\,D_x^{\beta}\,u_2+(u_0^2-1)D_x^{\beta}\,u_3. \end{aligned} \tag{15}$$

Substituting (12) and (13) into Eq.(11), yields the following recursive formula:

$$\begin{aligned} u_0(x,t) &= f(x),\\ u_n+1(x,t) &= -J^{\alpha}(A_n)-J^{\alpha}L_1(u_n(x,t)+J^{\alpha}L_2(u_n(x,t),n\geq 0. \end{aligned} \tag{16}$$

According to the above steps, we will derive numerical solutions for the Eq.(1) in details.

4. Fractional temporal numerical solution

Consider the following form of the temporal-fractional equation

$$D_t^{\alpha}\,u=-(1+u^2)u_x-pu_{xxx}-qu_{xxxxx}, \tag{17}$$

with the initial equation

$$u(x,0)=\frac{3p}{\sqrt{-10p}}\,\mathrm{sech}^2\,(kx)), \tag{18}$$

where $k=\frac{1}{2}\sqrt{\frac{-p}{5q}}$ [7].

The recursive formula (16) permits to give the successive terms:

$$\begin{aligned} u_0 &= \frac{3p}{\sqrt{-10p}}\,sec\,h^2\,(kx)), \end{aligned} \tag{19}$$

$$u_1=\left(-(1+u_0^2)\,\frac{du_0}{dx}-p\frac{d^3u0}{dx^3}-q\frac{d^5u_0}{dx^5}\right)\frac{t^{\alpha}}{\Gamma(a+1)},$$

$$u_2=\left(2u_0\,u_1\frac{du_0}{dx}+(1+u_0^2)\frac{du_1}{dx}-p\,\frac{d^3u_1}{dx^3}-q\frac{d^5u_1}{dx^5}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3 = \left(\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}u_1^2 + 2u_0\,u_2\right)\frac{du_0}{dx} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}u_0\,u_1\frac{du_1}{dx}\right.$$
$$\left. + \left(1+u_0^2\right)\frac{du_2}{dx} - p\,\frac{d^3u_2}{dx^3} - q\,\frac{d^5u_2}{dx^5}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

As an illustration, we compute the terms $(u_i)_{i\in\mathbb{N}}$ for $p=1; q=-1$ and $k=\frac{1}{2}\sqrt{\frac{1}{5}}$, we get:

$$u_0 = \frac{3}{10}\,\frac{\sqrt{10}}{\cosh^2 0.22361x},$$

$$u_1 = \frac{1}{4000\cosh^7 0.22361x}\,h(x)\,\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2 = \frac{-1}{16000000}\,\frac{\sqrt{10}}{\cosh^{12} 0.22361x}\,(f_1(x)+f_2(x)+f_3(x))$$
$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_2 = \frac{-1}{16000000}\,\frac{\sqrt{10}}{\cosh^{12} 0.22361x}\,(f_1(x)+f_2(x)+f_3(x))$$
$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3 = \frac{6.25\times 10^{-27}}{\cosh^{17} 0.22361x}\,(g_1(x)+g_1(x)+g_1(x)+g_1(x))$$
$$\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

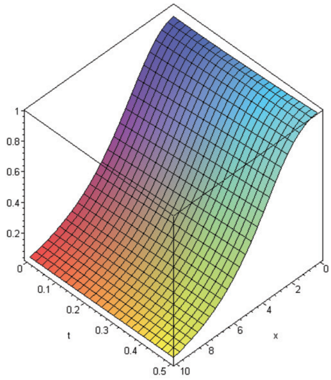
where

$$h(x) = 87\sqrt{2}\sinh 0.22361\,x + 174\sqrt{2}\sinh 0.22361x$$
$$+261\sqrt{2}\sinh 0.67083x,$$

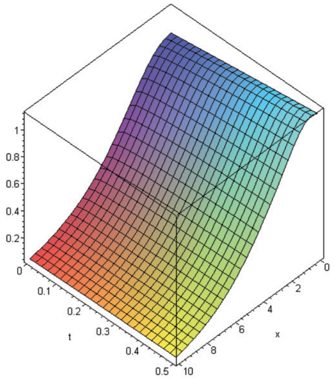
$$f_1(x) = 1827\cosh\sqrt{5}\,x - 919242\cosh\frac{1}{5}\sqrt{5}\,x,$$

$$f_2(x) = 162\,864\cosh\frac{2}{5}\sqrt{5}\,x + 182\,439\cosh\frac{3}{5}\sqrt{5}\,x,$$

$$f_3(x) = 7308\cosh\frac{4}{5}\sqrt{5}\,x - 905\,148,$$



Solution obtained for $\alpha = \beta = 1$



Solution obtained for $\alpha = \beta/2 = \frac{1}{2}$

$$g_1(x) = 6.7824\times 10^{20} \sinh 2.9069x - 6.1641\times 10^{26} \sinh 0.67082x,$$

$$g_2(x) = +1.1057\times 10^{26} \sinh 1.5652x + 6.7824\times 10^{20} \sinh 3.3541x,$$

$$g_3(x) = -5.6676\times 10^{26} \sinh 0.22361x - 1.8247\times 10^{25} \sinh 2.0125x,$$

$$g_4(x) = +7.9838\times 10^{25} \sinh 1.118x + 6.6926\times 10^{23} \sinh 2.4597x,$$

The approximate solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \tag{21}$$

we conclude by drawing figures for numerical solution (21) with $\alpha = \beta/2 = \frac{1}{2}$ as well as the solution obtained when $\alpha = \beta = 1$.

5. Fractional spatial numerical solution

In this section, we present a numerical example to demonstrate the behavior of the solution of the fractional spatial of modified Kawahara equation.

Considering the operator form of the spatial-fractional equation

$$u_t = -(1+u^2)D_x^{\beta} u - pu_{xxx} - qu_{xxxxx} \quad 0 < \beta < 1, \tag{22}$$

Assuming the initial condition as

$$u(x,0) = f(x) = x^2, \tag{23}$$

Initial condition has been taken as the above polynomial to avoid heavy calculation of fractional differentiation.

In order to estimate the numerical solution of equation (22), substituting (13), (14) and the initial condition (23) into (16), we get the Adomian solution.

Here, we give the first few terms of the series solution:

$$u_0 = f(x) = x^2,$$

$$u_1 = -J(A_0) - JL_1(u_0(x,t) + JL_2(u_0(x,t),$$

$$= -J\big((1+u_0^2)\, D_x^{\beta}\, u_0\big) - JL_1(u_0(x,t) + JL_2(u_0(x,t),$$

$$= \frac{\Gamma(3)}{\Gamma(3-\beta)}\,(x^{6-\beta} + x^{2-\beta})t,$$

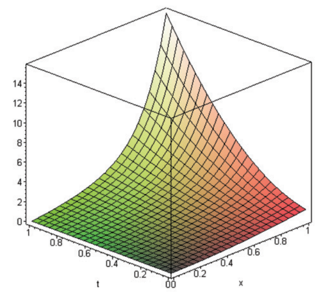
$$u_2 = -J(2u_0\, u_1\, D_x^{\beta}\, u_0 + (1+u_0^2)\, D_x^{\beta}\, u_1) - JL_1(u_1(x,t) + JL_2(u_1(x,t),$$

$$= (g_1(x) + g_2(x))\,\frac{t^2}{2},$$

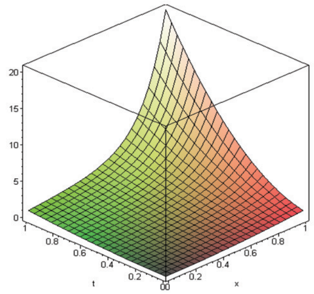
such that

$$g_1(x) = \frac{2}{\Gamma(3-\beta)}\,(1+x^2)\left(\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)}\,x^{2-2\beta} + \frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)}\,x^{6-2\beta}\right),$$

$$g_2(x) = \frac{8}{\Gamma(3-\beta)}\,(x^{6-2\beta} + x^{10-2\beta}),$$



Figure(a): Solution of Eq.(16) obtained by ADM method for $\beta = \frac{1}{2}$.



Figure(b): Solution of Eq.(16) obtained by ADM method for $\beta = 1$.

The numerical solution of spatial-fractional equation (22)derivative in series form is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ... \tag{25}$$

Figures (a,b) show respectively the numerical solutions given by expression (25) for the equation (16) using the expressions (24) with $\beta = \frac{1}{2}$ and $\beta = 1$.

6. Conclusion

In this paper, the ADM has been successfully applied to derive explicit numerical solutions for the modified Kawahara equation with temporal-and-spatial fractional derivatives. The above procedure shows that the ADM method is efficient and powerful in solving wide classes of equations in particular evolution fractional order equations.

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