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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The sufficient conditions are established for the existence of solutions for a class of boundary value problems for fractional differential equations involving the Caputo fractional derivative.

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1. Introduction

This paper deals with the existence and uniqueness of solutions for boundary value problems (BVP for short), for fractional order differential equations

$$^cD^\alpha y(t) = f(t, y), \text{ for each } t \in J = [0, T], \quad 2 < \alpha \leq 3, \tag{1}$$

$$y(0) = y_0, \quad y'(0) = y_0^*, \quad y''(T) = y_T, \tag{2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and y_0, y_0^*, y_T are real constants. Differential equations of fractional order have been recently proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [5, 11, 12, 14, 21, 22, 26]). There has been a significant progress in the investigation of fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al* [17], Miller and Ross [23], Samko *et al* [30] and the papers of Delbosco and Rodino [4], Diethelm *et al* [5, 6, 7], El-Sayed [8, 9, 10], Kaufmann and Mboumi [15], Kilbas and Marzan [16], Mainardi [21], Momani and Hadid [24], Momani *et al* [25], Podlubny *et al* [29], Yu and Gao [31] and Zhang [32] and the references therein. Very recently some basic theory for the initial boundary value problems of fractional differential equations involving a Riemann–Liouville differential operator of order $0 < \alpha \leq 1$ has been discussed by Lakshmikantham and Vatsala [18, 19, 20]. In a series of papers (see [1, 2, 3]) the authors considered some classes of initial value problems for functional differential equations involving Riemann–Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$.

Applied problems require the definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc., the same requirements come from boundary conditions. Caputo's fractional derivative satisfies these demands. For more details concerning the

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geometric and physical interpretation of fractional derivatives of Riemann–Liouville and Caputo types see [28].

In this paper, we present the existence results for the problem (1)–(2). We give three results, one based on the Banach fixed point theorem (Theorem 3.5), and another one based on Schaefer's fixed point theorem (Theorem 3.6) and the third one on a Leray–Schauder type nonlinear alternative (Theorem 3.7). Finally, we present an example illustrating the applicability of the imposed conditions.

2. Preliminaries

In this section, we introduce the notation, definitions, and preliminary facts which are used throughout the paper. We denote by $C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Definition 2.1 ([17, 27])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 ([17, 27])**.** For a function h given on the interval $[a,b]$ the αth Riemann–Liouville fractional-order derivative of h is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([16])**.** For a function h given on the interval $[a,b]$ the Caputo fractional-order derivative of h is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds.$$

3. Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)–(2).

Definition 3.1. A function $y \in C^2(J,\mathbb{R})$ with its α-derivative existing on J is said to be a solution of (1)–(2) if y satisfies the equation ${}^cD^\alpha y(t) = f(t,y(t))$ on J and the conditions $y(0) = y_0,\ \ y'(0) = y_0^*,\ \ y''(T) = y_T$.

For the existence of solutions for the problem (1)–(2), we need the following lemmas:

Lemma 3.2 ([32]). *Let $\alpha > 0$, then the differential equation*

$$^cD^{\alpha}h(t) = 0$$

has solutions $h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_nt^n, c_i \in \mathbb{R},\ \ i = 0, 1, 2, \ldots, n,\ n = [\alpha] + 1.$

Lemma 3.3 ([32]). *Let $\alpha > 0$, then*

$$I^{\alpha c}D^{\alpha}h(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_nt^n$$

for some $c_i \in \mathbb{R},\ \ i = 0, 1, 2, \ldots, n,\ n = [\alpha] + 1.$

As a consequence of Lemmas 3.2 and 3.3 we have the following result which is useful in what follows.

Lemma 3.4. *Let $2 < \alpha \le 3$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ & - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} h(s)ds \\ & + y_0 + y_0^* t + \frac{y_T}{2} t^2 \end{aligned} \tag{3}$$

if and only if y is a solution of the fractional BVP

$$^cD^{\alpha}y(t) = h(t),\ \ t \in J, \tag{4}$$

$$y(0) = y_0,\ \ y'(0) = y_0^*,\ \ y''(T) = y_T. \tag{5}$$

Proof. Assume y satisfies (4), then Lemma 3.3 implies that

$$y(t) = c_0 + c_1t + c_2t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds.$$

By (5), a simple calculation gives

$$c_0 = y_0,\ c_1 = y_0^*,$$

and

$$y''(T) = 2c_2 + \frac{1}{\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} h(s)ds = y_T.$$

Hence we get equation (3). Conversely, it is clear that if y satisfies equation (3), then equations (4)–(5) hold. □

Our first result is based on the Banach fixed point theorem.

Theorem 3.5. *Assume that*

(H1) *There exists a constant* $k > 0$ *such that*

$$|f(t,u) - f(t,\overline{u})| \leq k|u - \overline{u}|, \text{ for each } t \in J, \text{ and all } u, \ \overline{u} \in \mathbb{R}.$$

If

$$kT^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] < 1, \tag{6}$$

then the BVP (1)–(2) *has a unique solution on* J.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the operator

$$F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

defined by

$$\begin{aligned} F(y)(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \\ & - \frac{t^2}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3} f(s,y(s))ds \\ & + y_0 + y_0^* t + \frac{y_T}{2}t^2. \end{aligned}$$

Clearly, the fixed points of the operator F are solutions of the problem (1)–(2). We shall use the Banach contraction principle to prove that F has a fixed point. We shall show that F *is a contraction.*

Let $x, y \in C(J,\mathbb{R})$. Then for each $t \in J$ we have

$$\begin{aligned} |F(x)(t) - F(y)(t)| \leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|ds \\ & + \frac{T^2}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}|f(s,x(s)) - f(s,y(s))|ds \\ \leq & \frac{k\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ & + \frac{T^2 k\|x-y\|_\infty}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-1}ds \\ \leq & kT^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right]\|x-y\|_\infty. \end{aligned}$$

Thus

$$\|F(x)-F(y)\|_\infty \leq kT^\alpha\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right]\|x-y\|_\infty.$$

Consequently, by (6) F is a contraction. As a consequence of the Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)–(2). □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.6. *Assume that:*

(H2) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(H3) *There exists a constant $M > 0$ such that*

$$|f(t,u)| \leq M \text{ for each } t \in J \text{ and all } u \in \mathbb{R}.$$

Then the BVP (1)–(2) *has at least one solution on J.*

Proof. We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

Step 1: *F is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then for each $t \in J$

$$\begin{aligned}|F(y_n)(t)-F(y)(t)| \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds\\ &+\frac{T^2}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}|f(s,y_n(s))-f(s,y(s))|ds.\end{aligned}$$

Since f is a continuous function, we have

$$\|F(y_n)-F(y)\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Step 2: *F maps the bounded sets into the bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $y \in B_{\eta^*} = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leq \eta^*\}$ we have $\|F(y)\|_\infty \leq \ell$. By (H3) we have for each $t \in J$,

$$\begin{aligned}|F(y)(t)| \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds\\ &+\frac{T^2}{\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}|f(s,y(s))|ds\\ &+|y_0|+|y_0^*|T+\frac{|y_T|}{2}|T^2\end{aligned}$$

$$\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{T^2 M}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}ds$$
$$+|y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2$$
$$\leq \frac{M}{\Gamma(\alpha+1)}T^\alpha + \frac{M}{\Gamma(\alpha-1)}T^\alpha + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2.$$

Thus

$$\|F(y)\|_\infty \leq \frac{M}{\Gamma(\alpha+1)}T^\alpha + \frac{M}{\Gamma(\alpha-1)}T^\alpha + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2 := \ell.$$

Step 3: *F maps the bounded sets into the equicontinuous sets of $C(J,\mathbb{R})$.*

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ like in Step 2, and let $y \in B_{\eta^*}$. Then

$$|F(y)(t_2) - F(y)(t_1)| = \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]f(s,y(s))ds$$
$$+\frac{(t_2-t_1)^2}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-3}f(s,y(s))ds\Big|$$
$$+|y_0| + |y_0^*|(t_2-t_1) + \frac{|y_T|}{2}(t_2-t_1)^2$$
$$\leq \frac{M}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds$$
$$+\frac{M(t_2-t_1)}{2\Gamma(\alpha-2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds + |y_0|$$
$$+|y_0^*|(t_2-t_1) + \frac{|y_T|}{2}(t_2-t_1)^2$$
$$\leq \frac{M}{\Gamma(\alpha+1)}[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha] + \frac{M}{2\Gamma(\alpha-1)}(t_2-t_1)^\alpha$$
$$+|y_0| + |y_0^*|(t_2-t_1) + \frac{|y_T|}{2}(t_2-t_1)^2.$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $F : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \{y \in C(J, \mathbb{R}) : y = \lambda F(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$ we have

$$\begin{aligned} y(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \\ &\quad - \frac{\lambda t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s, y(s)) ds \\ &\quad + \lambda y_0 + \lambda y_0^* t + \lambda \frac{y_T}{2} t^2. \end{aligned}$$

This implies by (H3) that for each $t \in J$ we have

$$\begin{aligned} |y(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\quad + \frac{MT^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} ds + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2 \\ &\leq \frac{M}{\alpha\Gamma(\alpha)} T^\alpha + \frac{M}{(\alpha-2)\Gamma(\alpha-2)} T^\alpha + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2. \end{aligned}$$

Thus for every $t \in J$ we have

$$\|y\|_\infty \leq \frac{M}{\Gamma(\alpha+1)} T^\alpha + \frac{M}{\Gamma(\alpha-1)} T^\alpha + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2 := R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)–(2). □

In the following theorem we shall give an existence result for the problem (1)–(2) by means of an application of a Leray–Schauder type nonlinear alternative, where the condition (H3) is weakened.

Theorem 3.7. *Assume that* (H2) *and the following conditions hold.*

(H4) *There exist* $\phi_f \in L^1(J, \mathbb{R}^+)$ *and continuous and nondecreasing* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$|f(t, u)| \leq \phi_f(t) \psi(|u|) \text{ for each } t \in J \text{ and all } u \in \mathbb{R}.$$

(H5) *There exists a number* $M > 0$ *such that*

$$\frac{M}{\|I^\alpha \phi_f\|_{L^1} \psi(M) + \frac{T^2}{2} (I^{\alpha-2} \phi_f)(T) \psi(M) + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2} > 1. \quad (7)$$

Then the BVP (1)–(2) *has at least one solution on* J.

Proof. Consider the operator F defined in Theorems 3.5 and 3.6. It can be easily shown that F is continuous and completely continuous. For $\lambda \in [0,1]$ let y be such that for each $t \in J$ we have $y(t) = \lambda(Fy)(t)$. Then from (H4)–(H5) we have for each $t \in J$

$$\begin{aligned}
|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_f(s)\psi(|y(s)|)ds \\
&\quad + \frac{T^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3}\phi_f(s)\psi(|y(s)|)ds \\
&\quad + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2 \\
&\leq \psi(\|y\|_\infty)\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_f(s)ds \\
&\quad + \psi(\|y\|_\infty)\frac{T^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3}\phi_f(s)ds \\
&\quad + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)\|I^\alpha\phi_f\|_{L^1} + \frac{T^2}{2}(I^{\alpha-2}\phi_f)(T)\psi(\|y\|_\infty) + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2} \leq 1.$$

Then, by condition (7), there exists M such that $\|y\|_\infty \neq M$.

Let

$$U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M\}.$$

The operator $F : \overline{U} \to C(J,\mathbb{R})$ is continuous and completely continuous. By the choice of U, there exists no $y \in \partial U$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [13], we deduce that F has a fixed point y in $\overline{U}$, which is a solution of the problem (1)–(2). This completes the proof. □

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional BVP

$$^cD^\alpha y(t) = \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}, \quad t \in J := [0,1], \quad 2 < \alpha \leq 3, \tag{8}$$

$$y(0) = 0, \; y'(0) = 1, \; y''(1) = 0. \tag{9}$$

Set

$$f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}, \quad (t,x) \in J \times [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-t}}{(9+e^t)}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| \\ &= \frac{e^{-t}|x-y|}{(9+e^t)(1+x)(1+y)} \\ &\le \frac{e^{-t}}{(9+e^t)}|x-y| \\ &\le \frac{1}{10}|x-y|. \end{aligned}$$

Hence the condition $(H1)$ holds with $k = \dfrac{1}{10}$. We shall check that condition (6) is satisfied with $T = 1$. Indeed,

$$kT^\alpha\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] < 1 \Leftrightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} < 10. \tag{10}$$

We have

$$\frac{1}{6} \le \frac{1}{\Gamma(\alpha+1)} < \frac{1}{2}, \tag{11}$$

and

$$\frac{1}{2} \le \frac{1}{2\Gamma(\alpha-1)} < c, \tag{12}$$

for an appropriately chosen constant c that will be specified. (10)–(12) imply that

$$\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} < \frac{1}{2} + c < 10, \tag{13}$$

thus from (13) the positive constant c must satisfy

$$c < \frac{19}{2}.$$

From (12) we get

$$\Gamma(\alpha-1) > \frac{1}{19} \simeq 0.0526, \tag{14}$$

which is satisfied for some $\alpha \in (2,3]$. Then by Theorem 3.5 the problem (8)–(9) has a unique solution on $[0,1]$ for the values of α satisfying (14).

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